Matematici Speciale 1- Seminar

Integrale curbilinii.

Integrale curbilinii de speta I.

Consideram o curba in spatiu, notata γ, dat de ecuatiile parametrice:

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t), \quad t \in [a,b] \end{cases} \tag{1}$$

Pentru γ - curba neteda, lungimea sa este data de:

$$L(\gamma) = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,\mathrm{d}t,$$

unde

$$ds = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,\mathrm{d}t$$

se numeste *elementul de arc* al curbei γ.

Definitie 1 *Fie $F : D \subseteq \mathbb{R}^3 \mapsto \mathbb{R}$ continua si $\gamma \subset D$ o curba neteda data prin ecuatiile parametrice (1). Definim* **integrala curbilinie de speta I** *ca:*

$$\int_\gamma F(x,y,z)\,\mathrm{d}s = \int_a^b F(f(t),g(t),h(t))\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,\mathrm{d}t. \tag{2}$$

Propozitie 1 **i.** *Dacă γ este o curbă netedă şi $F_1, F_2 : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ sunt funcţii continue pe domeniul D ce conţine pe γ iar $\alpha \in \mathbb{R}$, atunci*

$$\int_\gamma [F_1(x,y,z) + F_2(x,y,z)]\,\mathrm{d}s = \int_\gamma F_1(x,y,z)\,\mathrm{d}s + \int_\gamma F_2(x,y,z)\,\mathrm{d}s;$$

$$\int_\gamma \alpha F_1(x,y,z)\,\mathrm{d}s = \alpha \int_\gamma F_1(x,y,z)\,\mathrm{d}s.$$

ii. *Fie $\gamma = \gamma_1 \cup \gamma_2$, cu γ_1, γ_2 curbe netede şi fie $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ o funcţie continuă pe domeniul D ce conţine pe γ. Atunci*

$$\int_\gamma F(x,y,z)\,\mathrm{d}s = \int_{\gamma_1} F(x,y,z)\,\mathrm{d}s + \int_{\gamma_2} F(x,y,z)\,\mathrm{d}s.$$

Exemplul 1 *Sa se calculeze integrala curbilinie de primul tip:*

$$\int_\gamma (x+y)\,\mathrm{d}s$$

unde γ este triunghiul cu varfurile $O(0,0)$, $A(1,0)$, $B(0,1)$.
Rezolvare:
Se observa ca

$$\gamma = [OA] \cup [AB] \cup [BO]$$

Folosim parametrizarea segmentului de dreaptă

$$\begin{cases} x = x_A + t(x_B - x_A) \\ y = y_A + t(y_B - y_A), \quad t \in [0,1] \end{cases}$$

$$[OA]:\begin{cases} x=0+t(1-0)=t \\ y=0+t(0-0)=0, \quad t\in[0,1]\end{cases} \Longrightarrow \mathrm{d}s=\sqrt{(x'(t))^2+(y'(t)^2}\,\mathrm{d}t=\sqrt{1+0}\,\mathrm{d}t=\,\mathrm{d}t$$

$$[AB]:\begin{cases} x=1+t(0-1)=1-t \\ y=0+t(1-0)=t, \quad t\in[0,1]\end{cases} \Longrightarrow \mathrm{d}s=\sqrt{(x'(t))^2+(y'(t)^2}\,\mathrm{d}t=\sqrt{1+1}\,\mathrm{d}t=\sqrt{2}\,\mathrm{d}t$$

$$[BO]:\begin{cases} x=0+t(0-0)=0, \\ y=1+t(0-1)=1-t, \quad t\in[0,1]\end{cases} \Longrightarrow \mathrm{d}s=\sqrt{(x'(t))^2+(y'(t)^2}\,\mathrm{d}t=\sqrt{0+1}\,\mathrm{d}t=\,\mathrm{d}t$$

Rezulta:

$$\begin{aligned}\int_\gamma (x+y)\,\mathrm{d}s &= \int_{[OA]}(x+y)\,\mathrm{d}s+\int_{[AB]}(x+y)\,\mathrm{d}s+\int_{[BO]}(x+y)\,\mathrm{d}s \\ &= \int_0^1 t\,\mathrm{d}t+\int_0^1\sqrt{2}\,\mathrm{d}t+\int_0^1(1-t)\,\mathrm{d}t=\left.\frac{t^2}{2}\right|_0^1+\left.\sqrt{2}t\right|_0^1+\left.t\right|_0^1-\left.\frac{t^2}{2}\right|_0^1=1+\sqrt{2}\end{aligned}$$

Exemplul 2 *Sa se calculeze integrala curbilinie de primul tip:*

$$\int_\gamma ye^{-x}\,\mathrm{d}s$$

$$\gamma:\begin{cases} x=\ln(1+t^2), \\ y=2\,\mathrm{arctg}\,t-t, t\in[0,1]\end{cases}$$

Rezolvare:
Vom calcula mai intai elementul de arc $\mathrm{d}s=\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t$*:*

$$\begin{cases} x'(t)=\dfrac{2t}{1+t^2}, \\ y'(t)=\dfrac{2}{1+t^2}-1=\dfrac{1-t^2}{1+t^2}, \end{cases}$$

deci:

$$\mathrm{d}s=\sqrt{\frac{4t^2+(1-t^2)^2}{(1+t^2)^2}}\,\mathrm{d}t=\sqrt{\frac{4t^2+1-2t^2+t^4}{(1+t^2)^2}}\,\mathrm{d}t=\sqrt{\frac{1+2t^2+t^4}{(1+t^2)^2}}\,\mathrm{d}t=\sqrt{\frac{(1+t^2)^2}{(1+t^2)^2}}\,\mathrm{d}t=\,\mathrm{d}t$$

Integrala devine, deci:

$$\begin{aligned}&\int_\gamma ye^{-x}\,\mathrm{d}s=\int_0^1(2\,\mathrm{arctg}\ t-t)e^{-\ln(1+t^2)}\,\mathrm{d}t=\int_0^1\frac{2\,\mathrm{arctg}\ t}{1+t^2}\,\mathrm{d}t-\int_0^1\frac{t}{1+t^2}\,\mathrm{d}t \\ &=\int_0^1\left(\mathrm{arctg}^{\,2}(t)\right)'\,\mathrm{d}t-\left.\frac{1}{2}\ln(1+t^2)\right|_0^1=\,\mathrm{arctg}^{\,2}(1)-\,\mathrm{arctg}^{\,2}(0)-\frac{\ln(2)}{2}=\frac{\pi^2}{16}-\frac{\ln(2)}{2}\end{aligned}$$

Observatie:

In cazul in care curba γ este o linie poligonala formata din mai multe segmente, integrala pe o astfel de curba va fi data de suma integralelor pe fiecare segment. In acest scop, reamintim ecuatiile parametrice ale unei drepte care trece prin doua puncte date $A(x_A,y_A,z_A)$, $B(x_B,y_B,z_B)$:

$$\begin{cases} x=x_A+t(x_B-x_A) \\ y=y_A+t(y_B-y_A) \\ z=z_A+t(z_B-z_A), \quad t\in[0,1]\end{cases}$$

Exemplul 3 ***Sa se calculeze***

$$\int_\gamma \frac{1}{\sqrt{x^2+y^2+4}}\,\mathrm{d}s$$

unde (γ) ***este segmentul*** $[OA]$***,*** $O(0,0)$***,*** $A(1,2)$***.*** **Rezolvare:**
Ecuatiile dreptei (OA) ***sunt date de :***

$$\begin{cases} x = t \\ y = 2t, \quad t \in [0,1]. \end{cases}$$

Cum $\mathrm{d}s = \sqrt{5}\,\mathrm{d}t$***, integrala devine:***

$$\int_\gamma \frac{1}{\sqrt{x^2+y^2+4}}\,\mathrm{d}s = \int_0^1 \frac{1}{\sqrt{t^2+4t^2+4}}\sqrt{5}\,\mathrm{d}t = \int_0^1 \frac{1}{\sqrt{5t^2+4}}\sqrt{5}\,\mathrm{d}t$$
$$= \int_0^1 \frac{1}{\sqrt{t^2+\frac{4}{5}}}\,\mathrm{d}t = \ln\left(t+\sqrt{t^2+\frac{4}{5}}\right)\Bigg|_0^1 = \ln\left(1+3/\sqrt{5}\right) - \ln(2/\sqrt{5}) = \ln\left(\frac{\sqrt{5}+3}{2}\right)$$

Exemplul 4 ***Sa se calculeze***

$$\int_\gamma xyz\,\mathrm{d}s$$

unde

$$(\gamma) : \begin{cases} x = t \\ y = \dfrac{2\sqrt{2}}{3}t\sqrt{t} \\ z = \dfrac{1}{2}t^2 \quad t \in [0,1]; \end{cases}$$

Rezolvare:

$$\begin{cases} x'(t) = 1 \\ y'(t) = \sqrt{2t} \\ z'(t) = t \end{cases}$$

Prin urmare $\mathrm{d}s = \sqrt{1+2t+t^2}\,\mathrm{d}t = (t+1)\,\mathrm{d}t$***, si integrala devine:***

$$\int_\gamma xyz\,\mathrm{d}s = \frac{\sqrt{2}}{3}\int_0^1 t^{9/2}(1+t)\,\mathrm{d}t = \frac{\sqrt{2}}{3}\left[\frac{2}{11}t^{\frac{11}{2}} + \frac{2}{13}t^{\frac{13}{2}}\right]\Bigg|_0^1 = \frac{16\sqrt{2}}{143}$$

Exercitiul 1 Calculaţi următoarele integrale curbilinii de speţa I.

1) $\int_\gamma xy\,\mathrm{d}s$***, unde*** $(\gamma) : \{(x,y) \mid |x|+|y| = a\},\ a > 0;$

2) $\int_\gamma xy\,\mathrm{d}s$***, unde*** (γ) ***este elipsa*** $\frac{x^2}{9}+\frac{y^2}{4} = 1$ ***situată în primul cadran;***

se va folosi parametrizarea elipsei de ecuatie $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$:

$$\begin{cases} x = a\cos t \\ y = b\sin t, \quad t \in [0,2\pi]; \end{cases}$$

3) $\displaystyle\int_\gamma y\,\mathrm{d}s$***, unde*** $(\gamma):\{(x,y)\mid y^2=8x,\ x\in[0,2]\}$;

4) $\displaystyle\int_\gamma (x^2+y^2)\,\mathrm{d}s$***, unde*** (γ) ***este segmentul*** $[AB]$***,*** $A(a,a),\ B(b,b),\ a<b$;

5) $\displaystyle\int_\gamma (x^2+y^2)\,\mathrm{d}s$***, unde*** $(\gamma):\begin{cases} x=\cos t+t\sin t\\ y=\sin t-t\cos t, \quad t\in[0,2\pi];\end{cases}$

6) $\displaystyle\int_\gamma ye^{-x}\,\mathrm{d}s$ ***, unde*** $(\gamma):\begin{cases} x=\ln(1+t^2)\\ y=2\,\mathrm{arctg}\,t-t+3, \quad t\in[0,1];\end{cases}$

Integrale curbilinii de speta a II a

Definitie 2 ***Fie*** $\gamma=\widehat{AB}$ ***o curbă netedă dată de***

$$(\gamma)\quad \begin{cases} x=f(t)\\ y=g(t)\\ z=h(t), \quad t\in[\,a,b\,],\end{cases} \tag{3}$$

orientată de la $A(t=a)$ ***la*** $B(t=b)$ ***în sensul de creştere al parametrului*** t ***de la*** a ***la*** b***. Fie*** $P,\,Q,\,R:D\subseteq\mathbf{R}^3\to\mathbf{R}$ ***funcţii continue pe domeniul*** D ***ce conţine curba*** γ***. Numim*** **integrala curbilinie de speţa II**

$$\int_\gamma P(x,y,z)\,\mathrm{d}x+Q(x,y,z)\,\mathrm{d}y+R(x,y,z)\,\mathrm{d}z=$$
$$=\int_a^b\left[P(f(t),g(t),h(t))\cdot f'(t)+Q(f(t),g(t),h(t))\cdot g'(t)+R(f(t),g(t),h(t))\cdot h'(t)\right]\mathrm{d}t.$$

Exemplul 5 ***Sa se calculeze urmatoarele integrale:***

1.

$$\int_\gamma (2x-y)\,\mathrm{d}x+(2y+x)\,\mathrm{d}y$$

unde $\gamma=\widehat{OABO}$***,*** $O(0,0)$***,*** $A(1,0)$***,*** $B(0,1)$ ***formată din segmentele*** $[OA]$***,*** $[BO]$ ***si arcul de cerc*** $AB:x^2+y^2=1$***.***

2.

$$\int_\gamma \frac{\mathrm{d}x}{y}+\sqrt{2x}\,\mathrm{d}y$$

unde $\gamma=\{(x,y)|x^2+y^2-2x=0,y>0\}$

Rezolvări:

5.1 ***Avem***

$$\int_\gamma (2x-y)\,\mathrm{d}x+(2y+x)\,\mathrm{d}y=\int_{OA}(2x-y)\,\mathrm{d}x+(2y+x)\,\mathrm{d}y$$
$$+\int_{\widehat{AB}}(2x-y)\,\mathrm{d}x+(2y+x)\,\mathrm{d}y+\int_{BO}(2x-y)\,\mathrm{d}x+(2y+x)\,\mathrm{d}y$$

Paramentrizam segmentul $[OA]$ ***astfel:***

$$[OA]:\begin{cases} x=t\\ y=0, \quad t\in[0,1];\end{cases}$$

$dx = dt$***,*** $dy = 0$ ***şi atunci avem***

$$I_1 = \int_{OA} (2x - y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y = \int_0^1 2t\,\mathrm{d}t = t^2\Big|_0^1 = 1.$$

Reamintim parametrizarea cercului cu centrul $(0, 0)$ ***şi rază*** $a > 0$***:***

$$(\gamma) : \begin{cases} x = a\cos t \\ y = a\sin t, \quad t \in [\,0, 2\pi\,]\,; \end{cases}$$

În cazul arcului $\widehat{AB}$ ***din cercul cu centrul*** $(0, 0)$***, rază*** 1 ***şi situat în primul cadran, avem:***

$$\widehat{AB} : \begin{cases} x = \cos t \\ y = \sin t, \quad t \in \left[0, \dfrac{\pi}{2}\right]; \end{cases}$$

$dx = -\sin t\,dt$***,*** $dy = \cos t\,dt$ ***şi atunci găsim***

$$I_2 = \int_{\widehat{AB}} (2x - y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y = \int_0^{\pi/2} [(2\cos t - \sin t)(-\sin t) + (2\sin t + \cos t)\cos t]\,\mathrm{d}t$$
$$= \int_0^{\pi/2} [-2\cos t\sin t + \sin^2 t + 2\sin t\cos t + \cos^2 t]\,\mathrm{d}t = \frac{\pi}{2}.$$

Pentru segmentul $[BO]$ ***putem scrie***

$$[BO] : \begin{cases} x = 0 + t(0 - 0) = 0 \\ y = 1 + t(0 - 1) = 1 - t, \quad t \in [0, 1]; \end{cases}$$

$dx = 0$***,*** $dy = -dt$

$$I_3 = \int_{BO} (2x - y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y = -\int_0^1 2(1 - t)\,\mathrm{d}t = (1 - t)^2\Big|_0^1 = -1$$

Prin urmare,

$$\int_\gamma (2x - y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y = \frac{\pi}{2}$$

5.2 ***Curba*** γ ***se rescrie astfel***

$$(\gamma) : (x - 1)^2 + y^2 = 1, y > 0$$

deci γ ***reprezinta semicercul situat deasupra axei*** Oy ***al cercului de centru*** $C(1, 0)$ ***si de raza*** $R = 1$***.***

În general, cercul de ecuaţie $(x - x_0)^2 + (y - y_0)^2 = a^2$ ***se parametrizează***

$$\gamma : \begin{cases} x = x_0 + a\cos t \\ y = y_0 + a\sin t, \quad t \in [0, 2\pi]. \end{cases}$$

t ***este unghiul orientat dintre semiaxa pozitivă*** Ox ***şi vectorul care uneşte centrul cercului cu punctul de coordonate*** (x, y) ***situat pe cerc.***

Avem

$$\gamma : \begin{cases} x = 1 + \cos t \\ y = \sin t, \quad t \in [0, \pi] \end{cases}$$

Cum $x'(t) = -\sin t$ ***si*** $y'(t) = \cos t$***, integrala devine:***

$$I = \int_\gamma \frac{\mathrm{d}x}{y} + \sqrt{2x}\,\mathrm{d}y = \int_0^\pi \left[\frac{-\sin t}{\sin t} + \sqrt{2(1 + \cos t)}\cos t\right]\mathrm{d}t.$$

Dar $1+\cos t = 2\cos^2\frac{t}{2}$ ***şi atunci***

$$
\begin{aligned}
I &= \int_0^\pi \left[-1+2\cos\left(\frac{t}{2}\right)\cos(t)\right]\,\mathrm{d}t = -\pi + \int_0^\pi \left[\cos\left(\frac{3t}{2}\right)+\cos\left(\frac{t}{2}\right)\right]\,\mathrm{d}t \\
&= -\pi - \left[\frac{2}{3}\sin\left(\frac{3t}{2}\right)+2\sin\left(\frac{t}{2}\right)\right]\Bigg|_0^\pi = -(4/3+\pi)
\end{aligned}
$$

Exercitiul 2 Să se calculeze următoarele integrale curbilinii de speţa II.

1) $\displaystyle\int_{AB} 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y$ unde $AB,\ A(0,0),\ B(1,1)$ este

a) segmentul de dreaptă $[AB]$,
b) arcul de pe parabola $y^2=x$,
c) arcul de pe parabola $y=x^2$,
d) arcul de pe parabola cubică $y=x^3$;

2) $\int_{AB} xy\,\mathrm{d}x + (y-x)\,\mathrm{d}y$ pe arcul de cerc $x^2+y^2=4$ din semiplanul superior parcurs în sens trigonometric;

3) $\displaystyle\int_\gamma (x-y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y$, unde γ este curba de extremităţi $O(0,0),\ A(1,1)$ indicată

a) segmentul $[OA]$,
b) curba OMA unde M este proiecţia lui A pe axa Ox,
c) curba ONA unde N este proiecţia lui A pe axa Oy;

4) $\displaystyle\int_\gamma \frac{1}{x+3}\,\mathrm{d}y$ unde γ este arcul elipsei $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ situat în semiplanul superior, cuprins între punctele $A(3,0)$ şi $B(0,2)$, parcurs în sens trigonometric.

5) $\displaystyle\int_\gamma (x^2+2xy)\,\mathrm{d}y$ unde γ este arcul elipsei $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, situat în primul cadran, parcurs în sens trigonometric;

6) $\displaystyle\int_\gamma (y+1)\,\mathrm{d}x + (x+y)\,\mathrm{d}y$ unde $\gamma=\{(x,y)\mid\ x^2+y^2=4,\ x\geq 0\}$, parcurs în sens trigonometric.

Exercitiul 2.4:

Folosim ecuatiile parametrice ale elipsei şi observăm că, deoarece γ reprezinta arcul elipsei situat între punctele $A(3,0)$ şi $B(0,2)$, atunci $t\in\left[0,\frac{\pi}{2}\right]$:

$$
\gamma : \begin{cases} x = 3\cos t \\ y = 2\sin t, \quad t\in\left[0,\dfrac{\pi}{2}\right] \end{cases}
$$

Avem, deci:

$$
I = \int_\gamma \frac{1}{x+3}\,\mathrm{d}y = \int_0^{\pi/2} \frac{1}{3\cos t+3}2\cos t\,\mathrm{d}t = \frac{2}{3}\int_0^{\pi/2}\frac{\cos t}{\cos t+1}\,\mathrm{d}t
$$

Prin schimbarea de variabila:

$$
\operatorname{tg}\frac{t}{2} = u, \quad \cos t = \frac{1-u^2}{1+u^2}, \quad t = 2\operatorname{arctg} u, \quad dt = \frac{2}{1+u^2}\,\mathrm{d}u
$$

obtinem:

$$
\begin{aligned}
I &= \frac{2}{3}\int_0^1 \frac{\frac{1-u^2}{1+u^2}}{\frac{1-u^2}{1+u^2}+1}\,\frac{2}{1+u^2}\,\mathrm{d}u = \frac{2}{3}\int_0^1 \frac{1-u^2}{1+u^2}\,\mathrm{d}u = \frac{2}{3}\int_0^1\left(-1+\frac{2}{1+u^2}\right)\mathrm{d}u \\
&= \frac{2}{3}\left[-u+2\operatorname{arctg}(u)\right]\big|_0^1 = \frac{2}{3}\left(-1+2\cdot\frac{\pi}{4}\right) = -\frac{2}{3}+\frac{\pi}{3}.
\end{aligned}
$$