South Eastern European Mathematical
Olympiad for University Students
Iaşi, România - March 7, 2014

Problem 1. Let n be a nonzero natural number and $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a function such that $f(2014) = 1 - f(2013)$. Let $x_1, x_2, x_3, \ldots, x_n$ be real numbers not equal to each other. If

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \ldots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \ldots & 1+f(x_n) \end{vmatrix} = 0,$$

prove that f is not continuous.

Problem 2. Consider the sequence (x_n) given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 2.$$

Prove that the sequence $y_n = \displaystyle\sum_{k=1}^{n} \frac{1}{x_k^2 - 1}, \quad n \geq 1$ is convergent and find its limit.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $a \in \mathbb{C},\ a \neq 0$ such that $A - A^* = 2aI_n$, where $A^* = (\bar{A})^t$ and $\bar{A}$ is the conjugate of the matrix A.

(a) Show that $|\det A| \geq |a|^n$

(b) Show that if $|\det A| = |a|^n$ then $A = aI_n$.

Problem 4. a) Prove that $\displaystyle\lim_{n\to\infty} n \int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\,dx = \frac{\pi}{2}$.

b) Find the limit $\displaystyle\lim_{n\to\infty} n \left(n \int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\,dx - \frac{\pi}{2} \right)$.