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Bifurcation of nontrivial periodic solutions for an impulsively controlled pest management model

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ABSTRACT

This paper investigates the onset of nontrivial periodic solutions for an integrated pest management model which is subject to pulsed biological and chemical controls. The biological control consists in the periodic release of infective individuals, while the chemical control consists in periodic pesticide spraying. It is assumed that both controls are used with the same periodicity, although not simultaneously. To model the spread of the disease which is propagated through the release of infective individuals, an unspecified force of infection is employed.

The problem of finding nontrivial periodic solutions is reduced to showing the existence of nontrivial fixed points for the associated stroboscopic mapping of time snapshot equal to the common period of controls. The latter problem is in turn treated via a projection method. It is then shown that once a threshold condition is reached, a stable nontrivial periodic solution emerges via a supercritical bifurcation.

1. Introduction

Synthetic pesticides were initially viewed as a miraculous way of controlling pest populations. However, it has been quickly noticed that heavy pesticide use creates on a long run more problems than it solves. In some situations, due to the survival of pest individuals which are genetically predisposed to pesticide resistance and to a rapid reproductive rate, repeated pesticide use selects the resistant pest individuals, and the entire pest population becomes resistant in a short time. Moreover, when pesticides are used to control a given pest species, its natural predators may be removed from the environment as well as a side effect; that may actually cause for a long term an increase in the size of the pest population, instead of the expected reduction. If the pest is living out of reach or just hiding, then pesticides may simply have no effect on the pest population. Finally, many pesticides are known to cause environmental problems and to damage human health.

An integrated pest management (IPM) strategy is considered to be more effective and less damaging to the environment than using pesticides alone. This approach involves the use of a wide array of controls, which includes mechanical, biological and chemical controls. The emphasis is put on the control of the pest population, not on its eradication, as the later might be unfeasible or counterproductive. Generally, an IPM strategy is considered successful when the pest population is stabilized under the economic injury level (EIL), defined by Stern et al. [16] as "the amount of pest injury which justifies the cost of using controls or the lowest pest density which causes economic damage".

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In general, biological controls are of much less environmental concern and of lower cost than pesticides. Also, they might be more effective if are applied correctly, and are self-regulating up to some extent. To use biological controls effectively, detailed knowledge of the pest and of its natural enemies is needed. One approach to biological control consists in the periodic release of parasitoids, pathogens or natural predators of target pests. Another possible approach is to release pests which are infected in laboratories, with the purpose of spreading a disease in the targeted pest population, on the grounds that infective pests usually cause less environmental damage and are less likely to reproduce. This is the approach to biological control which we consider in the present paper.

Regarding the disease which is spread through the periodic release of infective pests, it has been observed [11,8,7] that the dependence on the size of the infective population I plays a more prominent role than the dependence on size of the susceptible pest population S, as far as the incidence rate of the infection is concerned. Consequently, an incidence rate of type $g(I)S$ may be an appropriate choice in many situations. (See also [2,5,15,17], in which particular rates of this type are employed.) In the following, we shall use a general incidence rate of type $g(I)S$ to model the transmission of the disease, under a few natural assumptions on the nonlinear force of infection g.

As far as chemical controls are concerned, the synthetic pesticides are used in IPM strategies only as the last resort, when deemed an absolute necessity, and are specifically chosen to target the pest species to be controlled.

A central problem for IPM strategies is to choose the appropriate moment for using each type of control. To account for the fact that pesticides cannot be sprayed continuously, we use a model introduced in Georgescu and Moroşanu [4], where the biological and chemical controls are employed in an impulsive and periodic fashion, with the same periodicity but not simultaneously. The choice of using impulsive controls is, in our opinion, justified since for certain pesticides the effect follows shortly after application and also since the size of the infective pest population grows immediately after the release of infective individuals. Consequently, such changes can be modeled as immediate jumps in the population sizes. In this regard, a general account of the theory of impulsive ordinary differential equations can be found in Bainov and Simeonov [1].

An unified approach of dealing with the existence of nontrivial periodic solutions for a large class of two dimensional systems of differential equations, which are impulsively perturbed in a periodic fashion by means of possibly nonlinear controls, has been devised in Lakmeche and Arino [9]. Their approach consists in reformulating the problem of finding nontrivial periodic solutions as a fixed point problem for the associated stroboscopic mapping and solving the latter by the methods of bifurcation theory. Specifically, a projection method is employed. The subsequent theoretical findings were applied to the study of a particular model arising from the chemotherapeutic treatment of tumors, originally introduced by Panetta [14], that features nonlinearities of logistic type and linear impulses.

In this paper, we employ the method and some of the notations introduced in [9], although our model is structurally different from Panetta's in the sense that it is not a competitive model (it is actually neither competitive nor cooperative). Notably, we establish the bifurcation of nontrivial periodic solutions for a nonlinear force of infection expressed in a general form and employ two distinct types of impulsive controls, biological and chemical. The approach devised by Lakmeche and Arino is also employed, among others, by Lu et al. [12] for a predator–pest model that is a subject to pulsed use of insecticides, and by the same authors in [13] for a SIR epidemic model with horizontal and vertical transmission which is subject to pulsed vaccination. See also [10], where the bifurcation of nontrivial periodic solutions for a Kolmogorov-type system arising from heterogeneous tumor therapy by several drugs with instantaneous effects administered one at a time is studied by this method.

This paper is organized as follows: in Section 2, we formulate our impulsive control model and state its stability and persistence properties. A basic reproduction number R_0 is then constructed and it is shown that R_0 is a threshold parameter for this model, as far as the stability of the trivial periodic solution is concerned. In Section 3, we introduce a few definitions and notations and reformulate the problem of finding nontrivial periodic solutions as a fixed point problem. The latter problem is then treated through the use of a projection method and the onset of nontrivial periodic solutions is consequently established on condition that $R_0 = 1$. Our findings are then discussed in Section 4. Finally, some more technical computations used to prove the above results are deferred to Appendices A–E.

2. The model and its stability properties

In the following, we consider the model which has been studied in [4] from the viewpoint of finding sufficient conditions for the global stability of the susceptible pest-eradication solution and for the persistence of the disease, respectively. We also attempt to establish a certain bifurcation result which complements those already obtained in [4]. We denote by $S(t)$ and $I(t)$ the sizes of the susceptible and infective pest population, respectively, at time t, and suppose that all pests are either susceptible or infective.

In [4], the following impulsively controlled system has been formulated to describe the variation of S and I:

$$\begin{aligned}
&I'(t) = g(I(t))S(t) - wI(t), \quad t \neq (n+l-1)T,\ t \neq nT,\\
&S'(t) = S(t)h(S(t)) - g(I(t))S(t), \quad t \neq (n+l-1)T,\ t \neq nT,\\
&\Delta I(t) = -\delta_2 I(t), \quad t = (n+l-1)T,\\
&\Delta S(t) = -\delta_1 S(t), \quad t = (n+l-1)T,\\
&\Delta I(t) = \mu, \quad t = nT,\\
&\Delta S(t) = 0, \quad t = nT.
\end{aligned} \tag{2.1}$$

Here, $T > 0$, $0 < l < 1$, $\Delta\varphi(t) = \varphi(t+) - \varphi(t)$ for $\varphi \in \{S, I\}$, $0 \leqslant \delta_1, \delta_2 < 1$, $n \in \mathbb{N}^*$, $w > 0$ and $h(0) = r > 0$. Assume that the functions h and g satisfy the following hypotheses:

(H1) h is decreasing on $[0,\infty)$, $\lim_{S\to\infty} h(S) < -w$, $S \mapsto Sh(S)$ locally Lipschitz on $(0,\infty)$,
(H2) $g(I)$ is increasing and globally Lipschitz on $[0,\infty)$, and $g(0) = 0$.

Under these assumptions, it has been shown in [4] that the initial value problem for the system (2.1) is biologically well-posed in the sense that to any positive initial data $(I(0), S(0))$ there corresponds a positive solution $(I(t), S(t))$ which is globally defined, and if the initial data is strictly positive component-wise, then the solution is also strictly positive component-wise as well. It has also been shown in [4, Lemma 3.3] that all solutions of (2.1) are bounded.

We now introduce a few stability properties of the subsystem:

$$\begin{aligned} &I'(t) = -wI(t), \quad t \neq nT,\ (n+l-1)T, \\ &\Delta I(t) = -\delta_2 I(t), \quad t = (n+l-1)T, \\ &\Delta I(t) = \mu, \quad t = nT, \\ &I(0+) = I_0, \end{aligned} \tag{2.2}$$

which is used to describe the dynamics of the susceptible pest-eradication state. It has been seen in [4] that the system formed with the first three equations of (2.2) has a periodic solution I^* such that all the solutions of (2.2) tend to I^* as $t \to \infty$. More precisely, I^* is given by

$$\begin{aligned} &I^*(t) = e^{-wt} I^*(0+) \quad \text{for } t \in (0, lT], \\ &I^*(t) = e^{-wt} I^*(0+)(1-\delta_2) \quad \text{for } t \in (lT, T], \end{aligned}$$

where, by the T-periodicity requirement,

$$I^*(0+) = \frac{\mu}{1 - e^{-wT}(1-\delta_2)}. \tag{2.3}$$

It has also been shown in [4, Theorems 4.1 and 5.1], that the susceptible pest-eradication periodic solution $(I^*, 0)$, called also in the following *the trivial periodic solution*, is globally asymptotically stable provided that:

$$\int_0^T g(I^*(t))\mathrm{d}t - \ln(1-\delta_1) > rT,$$

while if the opposite inequality is satisfied, then the susceptible pest-eradication solution loses its stability and the system (2.1) becomes uniformly persistent. We shall now be concerned with the threshold situation, that is, the case when:

$$\int_0^T g(I^*(t))\mathrm{d}t - \ln(1-\delta_1) = rT. \tag{2.4}$$

Let us briefly discuss biological meaning of (2.4). Suppose that $(I(t), S(t))$ approaches the trivial periodic solution $(I^*, 0)$. Then, as the incidence rate of the infection is of the form $g(I)S$, the integral $\int_0^T g(I^*(t))\mathrm{d}t$ approximates the normalized loss of susceptible pests for a period due to their movement to the infective class, while since the production of newborn susceptible pests is given by $Sh(S)$ and $h(0) = r$, rT approximates the normalized gain of susceptible pests for a period. A correction term $-\ln(1-\delta_1)$ accounts for the loss of the susceptible pests due to pesticide spraying. Then the threshold condition represents the fact that the total normalized loss of susceptible pests for a period due to the infection or pesticide spraying balances the total normalized gain of newborn susceptible pests for a period.

Let us define the basic reproduction number R_0 associated with (2.1) as

$$R_0 = \frac{\int_0^T g(I^*(s))\mathrm{d}s - \ln(1-\delta_1)}{rT}.$$

Note that the above-defined R_0, although being, as usual, a measure for the virulence of infection, cannot be interpreted in the usual sense. This happens since in the classical situation the survival of the susceptible pest population is usually unquestioned, the alternative endings being either an infection-free state or an endemic state where the infective pest population persists alongside the susceptible pest population. Moreover, in the classical situation, the basic reproduction number is defined as the average number of new infections caused by a single infective individual which is introduced in a totally susceptible population; that is, it is defined based on the dynamics of the system near the infective pest-eradication equilibrium. For the impulsively controlled system (2.1), the outcome is different. Due to the pulsed supply of infective pests at $t = nT$, the survival of the infective population is unquestioned. Therefore, the alternative endings now are either a susceptible pest-eradication state, or an endemic state. Moreover, as seen in the discussion above, our definition of the basic reproduction number R_0 is based on the dynamics of the system near the susceptible pest-eradication periodic solution, as opposed to what happens in the classical case. This explains why the dynamics of (2.1) is structurally different from that of the unperturbed system, composed of the first two equations in (2.1).

With this notation, the threshold condition (2.4) can simply be rewritten as $R_0 = 1$. According to the above discussion, if $R_0 > 1$, then the susceptibles are depleted too fast, due to infection and to pesticide spraying, and the system tends to the susceptible pest-eradication periodic solution. If $R_0 < 1$, then the system becomes uniformly persistent. See [4] for details.

3. The fixed point approach

We now proceed to study bifurcation, which occurs at $R_0 = 1$. To this purpose, we shall employ a fixed point argument. We denote by $\Phi(t; U_0)$ the solution of the (unperturbed) system consisting of the first two equations of (2.1) for the initial data $U_0 = (u_0^1, u_0^2)$; also, $\Phi = (\Phi_1, \Phi_2)$. We define the mappings $I_1, I_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$I_1(x_1, x_2) = ((1-\delta_2)x_1, (1-\delta_1)x_2), \quad I_2(x_1, x_2) = (x_1 + \mu, x_2)$$

and the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x_1, x_2) = (g(x_1)x_2 - wx_1, x_2h(x_2) - g(x_1)x_2).$$

Furthermore, let us define $\Psi : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(T, U_0) = I_2(\Phi((1-l)T; I_1(\Phi(lT; U_0)))), \quad \Psi(T, U_0) = (\Psi_1(T; U_0), \Psi_2(T; U_0)).$$

It is easy to see that Ψ is actually the stroboscopic mapping associated to the system (2.1), which puts in correspondence the initial data U_0 at 0+ with the subsequent state of the system $\Psi(T, U_0)$ at T+, where T is the stroboscopic time snapshot. The idea of using a stroboscopic mapping is motivated not by the periodicity of the functional coefficients which appear in the first two equations of (2.1) (which are in this particular situation not periodic), as usual, but by the periodicity of the impulsive controls.

We reduce the problem of finding a periodic solution of (2.1) to a fixed point problem. Here, U is a periodic solution of period T for (2.1) if and only if its initial data $U(0) = U_0$ is a fixed point for $\Psi(T, \cdot)$. Consequently, to establish the existence of nontrivial periodic solutions of (2.1), one needs to prove the existence of nontrivial fixed points of Ψ.

By the chain rule, we note that:

$$D_X\Psi(T, X) = D_X\Phi((1-l)T; I_1(\Phi(lT; X)))\begin{pmatrix} 1-\delta_2 & 0 \\ 0 & 1-\delta_1 \end{pmatrix} D_X\Phi(lT; X).$$

We are interested in the bifurcation of nontrivial periodic solutions near $(I^*, 0)$. Assume that $X_0 = (x_0, 0)$ is the starting point for the trivial periodic solution $(I^*, 0)$, where $x_0 = I^*(0+)$, $I^*(0+)$ being given by (2.3). To find a nontrivial periodic solution of period τ with initial data X, we need to solve the fixed point problem $X = \Psi(\tau, X)$, or, denoting $\tau = T + \bar{\tau}$, $X = X_0 + \overline{X}$,

$$X_0 + \overline{X} = \Psi(T + \bar{\tau}, X_0 + \overline{X}).$$

Let us define:

$$N(\bar{\tau}, \overline{X}) = X_0 + \overline{X} - \Psi(T + \bar{\tau}, X_0 + \overline{X}) = (N_1(\bar{\tau}, \overline{X}), N_2(\bar{\tau}, \overline{X})). \tag{3.1}$$

At the fixed point, $N(\bar{\tau}, \overline{X}) = 0$. Let us denote:

$$D_XN(0, (0, 0)) = \begin{pmatrix} a_0' & b_0' \\ c_0' & d_0' \end{pmatrix}.$$

It follows that:

$$a_0' = 1 - (1-\delta_2)e^{-wT}, \tag{3.2}$$

$$b_0' = -e^{-wT}\left[(1-\delta_2)\int_0^{lT} g(I^*(s))e^{(r+w)s - \int_0^s g(I^*(\tau))d\tau} ds + (1-\delta_1)\int_{lT}^{T} g(I^*(s))e^{(r+w)s - \int_0^s g(I^*(\tau))d\tau} ds\right], \tag{3.3}$$

$$c_0' = 0, \tag{3.4}$$

$$d_0' = 1 - (1-\delta_1)e^{rT - \int_0^T g(I^*(s))ds}. \tag{3.5}$$

(See Appendix A for details.) A necessary condition for the bifurcation of nontrivial periodic solutions near $(I^*, 0)$ is then

$$\det[D_XN(0, (0, 0))] = 0. \tag{3.6}$$

Since $D_XN(0, (0, 0))$ is an upper triangular matrix and $a_0' = 1 - (1-\delta_2)e^{-wT} > 0$ always, it consequently follows that $d_0' = 0$ is necessary for the bifurcation. It is easy to see that $d_0' = 0$ is equivalent to (2.4) or to $R_0 = 1$. It now remains to show that this necessary condition is sufficient as well. This assertion represents the statement of the following theorem, which is our main result.

Theorem 1. *A supercritical bifurcation occurs at $R_0 = 1$, in the sense that there is $\varepsilon > 0$ such that for all $0 < \tilde{\varepsilon} < \varepsilon$ there is a stable positive nontrivial periodic solution of (2.1) with period $T + \tilde{\varepsilon}$.*

Proof 1. With the above notations, it is seen that:

$$\dim(\mathrm{Ker}[D_X N(0,(0,0))]) = 1,$$

and a basis in $\mathrm{Ker}[D_X N(0,(0,0))]$ is $\left(-\frac{b_0'}{a_0'},1\right)$. Then the equation $N(\bar{\tau},\overline{X}) = 0$ is equivalent to:

$$N_1(\bar{\tau},\alpha Y_0 + zE_0) = 0, \quad N_2(\bar{\tau},\alpha Y_0 + zE_0) = 0,$$

where

$$E_0 = (1,0), \quad Y_0 = \left(-\frac{b_0'}{a_0'},1\right)$$

and $\overline{X} = \alpha Y_0 + zE_0$ represents the direct sum decomposition of $\overline{X}$ using the projections onto $\mathrm{Ker}[D_X N(0,(0,0))]$ (the central manifold) and $\mathrm{Im}[D_X N(0,(0,0))]$ (the stable manifold). See [3, Section 2.4], or [6], for details.

Let us denote:

$$f_1(\bar{\tau},\alpha,z) = N_1(\bar{\tau},\alpha Y_0 + zE_0), \quad f_2(\bar{\tau},\alpha,z) = N_2(\bar{\tau},\alpha Y_0 + zE_0). \tag{3.7}$$

Firstly, we see that:

$$\frac{\partial f_1}{\partial z}(0,0,0) = \frac{\partial N_1}{\partial x_1}(0,(0,0)) = a_0' \neq 0.$$

Therefore, by the implicit function theorem, one may solve the equation $f_1(\bar{\tau},\alpha,z) = 0$ near $(0,0,0)$ with respect to z as a function of $\bar{\tau}$ and α, and find $z = z(\bar{\tau},\alpha)$ such that $z(0,0) = 0$ and

$$f_1(\bar{\tau},\alpha,z(\bar{\tau},\alpha)) = N_1(\bar{\tau},\alpha Y_0 + z(\bar{\tau},\alpha)E_0) = 0.$$

Moreover,

$$\frac{\partial z}{\partial \alpha}(0,0) = 0, \quad \frac{\partial z}{\partial \bar{\tau}}(0,0) = -\frac{w}{a_0'} I^*(T).$$

(See Appendix B for details.)

It now remains to study the solvability of the equation:

$$f_2(\bar{\tau},\alpha,z(\bar{\tau},\alpha)) = 0, \tag{3.8}$$

or the equivalent equation:

$$N_2(\bar{\tau},\alpha Y_0 + z(\bar{\tau},\alpha)E_0) = 0. \tag{3.9}$$

Eq. (3.9) is called the "determining equation" and the number of its solutions equals the number of periodic solutions of (2.1). We now proceed to solving (3.8) (or, equivalently, (3.9)). Let us denote:

$$f(\bar{\tau},\alpha) = f_2(\bar{\tau},\alpha,z(\bar{\tau},\alpha)). \tag{3.10}$$

First, it is easy to see that:

$$f(0,0) = N(0,(0,0)) = 0.$$

We determine the Taylor expansion of f around $(0,0)$. For this, we compute the first order partial derivatives $\frac{\partial f}{\partial \bar{\tau}}(0,0)$ and $\frac{\partial f}{\partial \alpha}(0,0)$ and observe that:

$$\frac{\partial f}{\partial \bar{\tau}}(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = 0.$$

(See Appendix C for the proof of this fact.)

Furthermore, it is observed in Appendix E that:

$$A = \frac{\partial^2 f}{\partial \alpha^2}(0,0) = 0, \quad B = \frac{\partial^2 f}{\partial \alpha \partial \bar{\tau}}(0,0) < 0, \quad C = \frac{\partial^2 f}{\partial \bar{\tau}^2}(0,0) > 0,$$

and hence,

$$f(\bar{\tau},\alpha) = B\alpha\bar{\tau} + C\frac{\alpha^2}{2} + \mathrm{o}(\bar{\tau},\alpha)(\bar{\tau}^2 + \alpha^2).$$

By denoting $\bar{\tau} = k\alpha$ (where $k = k(\alpha)$), we obtain that (3.8) is equivalent to:

$$Bk + C\frac{k^2}{2} + \mathrm{o}(\alpha,k\alpha)(1 + k^2) = 0.$$

Since $B < 0$ and $C > 0$, this equation is solvable with respect to k as a function of α. Moreover, here $k \approx -\frac{2B}{C} > 0$.

This implies that there is a supercritical bifurcation to a nontrivial periodic solution near a period T which satisfies the sufficient condition for the bifurcation (2.4). It is noteworthy that since this periodic solution appears via a supercritical bifurcation, the nontrivial periodic solution is stable. That is, there is $\varepsilon > 0$ such that for all $0 < \alpha < \varepsilon$ there is a stable positive nontrivial periodic solution of (2.1) with period $T + \bar{\tau}(\alpha)$ which starts in $X_0 + \alpha Y_0 + z(\bar{\tau}(\alpha), \alpha)E_0$. Here, X_0, Y_0, E_0, z, $\bar{\tau}$ are as defined above. □

4. Conclusion

The focus of this paper is the behavior of an impulsively controlled integrated pest management model. To limit the damaging potential of the pest population, a biological control, consisting in the release of infective pests, and a chemical control, consisting in pesticide spraying, are applied in a periodic fashion, with the same period, but not simultaneously. An unspecified nonlinear force of infection is assumed to describe the transmission of the disease which is spread through the release of infective individuals, and it is assumed that the infective pest population neither damages the crops, nor reproduces.

Our model is then investigated from the viewpoint of bifurcation theory. We investigate the existence of nontrivial periodic solutions by introducing the corresponding stroboscopic mapping and investigating its nontrivial fixed points. It is shown that once a threshold condition is reached, then the trivial periodic solution loses its stability. This stability is transferred to a newly emerging nontrivial periodic solution which appears via a supercritical bifurcation. This threshold condition may be expressed in terms of a balance condition for the susceptible class, or in terms of a basic reproduction number associated to the model. In precise terms, a nontrivial periodic solution corresponds to a persistent susceptible pest population, while a nontrivial periodic solution with small amplitude, below the economic injury level, indicates that the pest management strategy is still successful.

We have to add a remark upon the significance of the threshold condition (2.4). In the case in which $g(I)$ is a linear function, $g(I) = \beta I$, then the threshold condition is

$$\frac{1}{T}\int_0^T I^*(s)\mathrm{d}s = \frac{r + (1/T)\ln(1-\delta_1)}{\beta}.$$

We define $I_C = \frac{r+(1/T)\ln(1-\delta_1)}{\beta}$ as an "epidemic threshold". It is then seen from the above and Theorem 1 that nontrivial periodic solutions (I, S) appear when the average of the susceptible pest-eradication periodic solution over a period reaches the epidemic threshold I_C. As mentioned above, if the average of I^* is greater than I_C, then the susceptible pest-eradication periodic solution is globally stable, while if the average of I^* is less than I_C, then the system (2.1) is uniformly persistent.

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Appendix A. The first order partial derivatives of Φ_1, Φ_2

By formally deriving the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t;X_0)) = F(\Phi(t;X_0)),$$

which characterizes the dynamics of the unperturbed flow associated to the first two equations in (2.1), one obtains that:

$$\frac{\mathrm{d}}{\mathrm{d}t}[D_X\Phi(t;X_0)] = D_XF(\Phi(t;X_0))D_X\Phi(t;X_0). \tag{4.1}$$

This relation will be integrated in what follows in order to compute the components of $D_X\Phi(t;X_0)$ explicitly. Firstly, it is clear that:

$$\Phi(t;X_0) = (\Phi_1(t;X_0), 0).$$

We then deduce that (4.1) takes the particular form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \frac{\partial\Phi_1}{\partial x_1} & \frac{\partial\Phi_1}{\partial x_2} \\ \frac{\partial\Phi_2}{\partial x_1} & \frac{\partial\Phi_2}{\partial x_2} \end{pmatrix}(t;X_0) = \begin{pmatrix} -w & g(\Phi_1(t;X_0)) \\ 0 & r - g(\Phi_1(t;X_0)) \end{pmatrix}\begin{pmatrix} \frac{\partial\Phi_1}{\partial x_1} & \frac{\partial\Phi_1}{\partial x_2} \\ \frac{\partial\Phi_2}{\partial x_1} & \frac{\partial\Phi_2}{\partial x_2} \end{pmatrix}(t;X_0), \tag{4.2}$$

the initial condition for (4.2) at $t = 0$ being:

$$D_X\Phi(0;X_0) = I_2. \tag{4.3}$$

Here, I_2 is the identity matrix in $M_2(\mathbb{R})$. It follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi_2}{\partial x_1}(t;X_0)\right) = (r - g(\Phi_1(t;X_0)))\frac{\partial \Phi_2}{\partial x_1}(t;X_0),$$

and consequently,

$$\frac{\partial \Phi_2}{\partial x_1}(t;X_0) = \mathrm{e}^{\int_0^t (r-g(\Phi_1(s;X_0)))\mathrm{d}s}\frac{\partial \Phi_2}{\partial x_1}(0;X_0).$$

This implies, using the initial condition (4.3), that:

$$\frac{\partial \Phi_2}{\partial x_1}(t;X_0) = 0 \quad \text{for } t \geqslant 0. \tag{4.4}$$

To compute $\frac{\partial \Phi_1}{\partial x_1}(t;X_0)$, $\frac{\partial \Phi_1}{\partial x_2}(t;X_0)$ and $\frac{\partial \Phi_2}{\partial x_2}(t;X_0)$, one sees from (4.2) that:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi_1}{\partial x_1}(t;X_0)\right) = -w\frac{\partial \Phi_1}{\partial x_1}(t;X_0),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi_1}{\partial x_2}(t;X_0)\right) = -w\frac{\partial \Phi_1}{\partial x_2}(t;X_0) + g(\Phi_1(t;X_0))\frac{\partial \Phi_2}{\partial x_2}(t;X_0),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi_2}{\partial x_2}(t;X_0)\right) = (r - g(\Phi_1(t;X_0)))\frac{\partial \Phi_2}{\partial x_2}(t;X_0).
\end{aligned} \tag{4.5}$$

Using one more time the initial condition (4.3), one deduces that:

$$\begin{aligned}
&\frac{\partial \Phi_1}{\partial x_1}(t;X_0) = \mathrm{e}^{-wt},\\
&\frac{\partial \Phi_1}{\partial x_2}(t;X_0) = \mathrm{e}^{-wt}\int_0^t g(\Phi_1(s;X_0))\mathrm{e}^{(r+w)s-\int_0^s g(\Phi_1(\tau;X_0))\mathrm{d}\tau}\mathrm{d}s,\\
&\frac{\partial \Phi_2}{\partial x_2}(t;X_0) = \mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}.
\end{aligned} \tag{4.6}$$

From (3.1), one obtains that:

$$D_X N(0,(0,0)) = I_2 - D_X\Psi(T, X_0),$$

which implies:

$$D_X N(0,(0,0)) = \begin{pmatrix} a_0' & b_0' \\ 0 & d_0' \end{pmatrix},$$

with a_0', b_0', d_0' given by

$$a_0' = 1 - (1-\delta_2)\frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(lT;X_0)))\frac{\partial \Phi_1}{\partial x_1}(lT;X_0), \tag{4.7}$$

$$b_0' = -\left[(1-\delta_2)\frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(lT;X_0)))\frac{\partial \Phi_1}{\partial x_2}(lT;X_0) + (1-\delta_1)\frac{\partial \Phi_1}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0)))\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)\right], \tag{4.8}$$

$$d_0' = 1 - (1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0)))\frac{\partial \Phi_2}{\partial x_2}(lT;X_0). \tag{4.9}$$

Consequently, one may explicitly determine a_0', b_0', d_0' using (4.6) and obtain that:

$$a_0' = 1 - (1-\delta_2)\mathrm{e}^{-wT}, \tag{4.10}$$

$$\begin{aligned}
b_0' &= -\Bigg[(1-\delta_2)\mathrm{e}^{-w(1-l)T}\mathrm{e}^{-wlT}\int_0^{lT} g(I^*(s))\mathrm{e}^{(r+w)s-\int_0^s g(I^*(\tau))\mathrm{d}\tau}\mathrm{d}s\\
&\quad + (1-\delta_1)\mathrm{e}^{-w(1-l)T}\int_0^{(1-l)T} g(\Phi_1(s; I_1(\Phi(lT;X_0))))\mathrm{e}^{(r+w)s-\int_0^s g(\Phi_1(\tau;I_1(\Phi(lT;X_0))))\mathrm{d}\tau}\mathrm{d}s\cdot \mathrm{e}^{rlT-\int_0^{lT} g(I^*(s))\mathrm{d}s}\Bigg]\\
&= -\Bigg[(1-\delta_2)\mathrm{e}^{-wT}\int_0^{T} g(I^*(s))\mathrm{e}^{(r+w)s-\int_0^s g(I^*(\tau))\mathrm{d}\tau}\mathrm{d}s\\
&\quad + (1-\delta_1)\mathrm{e}^{-w(1-l)T}\int_0^{(1-l)T} g(I^*(s+lT))\mathrm{e}^{(r+w)s-\int_0^s g(I^*(\tau+lT))\mathrm{d}\tau}\mathrm{d}s\cdot \mathrm{e}^{rlT-\int_0^{lT} g(I^*(s))\mathrm{d}s}\Bigg]\\
&= -\mathrm{e}^{-wT}\left[(1-\delta_2)\int_0^{lT} g(I^*(s))\mathrm{e}^{(r+w)s-\int_0^s g(I^*(\tau))\mathrm{d}\tau}\mathrm{d}s + (1-\delta_1)\int_{lT}^{T} g(I^*(s))\mathrm{e}^{(r+w)s-\int_0^s g(I^*(\tau))\mathrm{d}\tau}\mathrm{d}s\right],
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
d'_0 &= 1-(1-\delta_1)\mathrm{e}^{r(1-l)T-\int_0^{(1-l)T} g(\Phi_1(s;I_1(\Phi(lT;X_0))))\mathrm{d}s}\mathrm{e}^{rlT-\int_0^{lT} g(I^*(s))\mathrm{d}s} = 1-(1-\delta_1)\mathrm{e}^{rT-\int_0^{(1-l)T} g(I^*(s+lT))\mathrm{d}s-\int_0^{lT} g(I^*(s))\mathrm{d}s} \\
&= 1-(1-\delta_1)\mathrm{e}^{rT-\int_0^{T} g(I^*(s))\mathrm{d}s}.
\end{aligned} \tag{4.12}
$$

Appendix B. The partial derivatives of z at $(\mathbf{0},\mathbf{0})$

From the implicit function theorem, it follows that:

$$
\frac{\partial N_1}{\partial x_1}(0,(0,0))\left(-\frac{b'_0}{a'_0}\right)+\frac{\partial N_1}{\partial x_2}(0,(0,0))+\frac{\partial N_1}{\partial x_1}(0,(0,0))\frac{\partial z}{\partial \alpha}(0,0)=0
$$

and consequently,

$$
a'_0\left(-\frac{b'_0}{a'_0}\right)+b'_0+a'_0\frac{\partial z}{\partial \alpha}(0,0)=0,
$$

and hence we obtain that:

$$
\frac{\partial z}{\partial \alpha}(0,0)=0. \tag{4.13}
$$

The computations required to find $\frac{\partial z}{\partial \bar\tau}(0,0)$ are somewhat more complicated, as $\frac{\partial N_1}{\partial \bar\tau}(0,(0,0))$ is not known beforehand, unlike $\frac{\partial N_1}{\partial x_1}(0,(0,0))$ and $\frac{\partial N_1}{\partial x_2}(0,(0,0))$. Again, by the implicit function theorem, it follows from (3.7) that:

$$
\begin{aligned}
\frac{\partial z}{\partial \bar\tau}(0,0) &= \frac{\partial \Phi_1}{\partial \bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))(1-l)+\frac{\partial \Phi_1}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\left(\frac{\partial \Phi_1}{\partial \bar\tau}(lT;X_0)\cdot l+\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar\tau}(0,0)\right) \\
&\quad +\frac{\partial \Phi_1}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\left(\frac{\partial \Phi_2}{\partial \bar\tau}(lT;X_0)\cdot l+\frac{\partial \Phi_2}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar\tau}(0,0)\right).
\end{aligned}
$$

Since

$$
\frac{\partial \Phi_2}{\partial x_1}(lT;X_0)=0, \tag{4.14}
$$

$$
\frac{\partial \Phi_2}{\partial \bar\tau}(lT;X_0)=0, \tag{4.15}
$$

it follows that:

$$
\frac{\partial z}{\partial \bar\tau}(0,0) = \frac{\partial \Phi_1}{\partial \bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))(1-l)+\frac{\partial \Phi_1}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\left(\frac{\partial \Phi_1}{\partial \bar\tau}(lT;X_0)\cdot l+\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar\tau}(0,0)\right),
$$

and consequently,

$$
\begin{aligned}
&\frac{\partial z}{\partial \bar\tau}(0,0)\left(1-\frac{\partial \Phi_1}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\right) \\
&= \frac{\partial \Phi_1}{\partial \bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))(1-l)+\frac{\partial \Phi_1}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\frac{\partial \Phi_1}{\partial \bar\tau}(lT;X_0)\cdot l.
\end{aligned}
$$

From (4.7), it now follows that:

$$
\frac{\partial z}{\partial \bar\tau}(0,0)=\frac{1}{a'_0}\left[\frac{\partial \Phi_1}{\partial \bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))(1-l)+\frac{\partial \Phi_1}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\frac{\partial \Phi_1}{\partial \bar\tau}(lT;X_0)\cdot l\right].
$$

Consequently, one may obtain that:

$$
\begin{aligned}
\frac{\partial z}{\partial \bar\tau}(0,0) &= \frac{1}{a'_0}[-wI^*(T)(1-l)+(1-\delta_2)\mathrm{e}^{-w(1-l)T}(-wI^*(lT))\cdot l] = -\frac{w}{a'_0}[I^*(T)(1-l)+\mathrm{e}^{-w(1-l)T}I^*(lT+)\cdot l] \\
&= -\frac{w}{a'_0}[I^*(T)(1-l)+I^*(T)\cdot l] = -\frac{w}{a'_0}I^*(T).
\end{aligned}
$$

Appendix C. The first order partial derivatives of f at $(\mathbf{0},\mathbf{0})$

By (3.1), (3.7) and (3.10), it is easy to see that:

$$
\begin{aligned}
\frac{\partial f}{\partial \alpha}(\bar\tau,\alpha) &= \frac{\partial}{\partial \alpha}[\alpha-\Psi_2(T+\bar\tau,X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)] = 1-\frac{\partial}{\partial \alpha}[\Phi_2((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))] \\
&= 1-\frac{\partial \Phi_2}{\partial x_1}((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))\cdot(1-\delta_2)
\end{aligned}
$$

$$\times\left(\frac{\partial\Phi_1}{\partial x_1}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial\alpha}(\bar\tau,\alpha)\right)+\frac{\partial\Phi_1}{\partial x_2}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\right)$$
$$-\frac{\partial\Phi_2}{\partial x_2}((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))\cdot(1-\delta_1)$$
$$\times\left(\frac{\partial\Phi_2}{\partial x_1}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial\alpha}(\bar\tau,\alpha)\right)+\frac{\partial\Phi_2}{\partial x_2}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\right).$$

It then follows that:

$$\frac{\partial f}{\partial\alpha}(0,0)=1-\frac{\partial\Phi_2}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_2)\left(\frac{\partial\Phi_1}{\partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial\alpha}(0,0)\right)+\frac{\partial\Phi_1}{\partial x_2}(lT;X_0)\right)$$
$$-\frac{\partial\Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\left(\frac{\partial\Phi_2}{\partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial\alpha}(0,0)\right)+\frac{\partial\Phi_2}{\partial x_2}(lT;X_0)\right).$$

From (4.14) and

$$\frac{\partial\Phi_2}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))=0,\tag{4.16}$$

it is seen that:

$$\frac{\partial f}{\partial\alpha}(0,0)=1-\frac{\partial\Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial\Phi_2}{\partial x_2}(lT;X_0)=d_0'=0.$$

Using one more time (3.1), (3.7) and (3.10), it is seen that:

$$\frac{\partial f}{\partial\bar\tau}(\bar\tau,\alpha)=\frac{\partial}{\partial\bar\tau}[\alpha-\Psi_2(T+\bar\tau,X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)]=-\frac{\partial}{\partial\bar\tau}[\Phi_2((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))]$$
$$=-\frac{\partial\Phi_2}{\partial\bar\tau}((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))(1-l)-\frac{\partial\Phi_2}{\partial x_1}((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0$$
$$+z(\bar\tau,\alpha)E_0)))\cdot(1-\delta_2)\left(\frac{\partial\Phi_1}{\partial\bar\tau}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\cdot l+\frac{\partial\Phi_1}{\partial x_1}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\frac{\partial z}{\partial\bar\tau}(\bar\tau,\alpha)\right)$$
$$-\frac{\partial\Phi_2}{\partial x_2}((1-l)(T+\bar\tau);I_1(\Phi(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)))\cdot(1-\delta_1)$$
$$\times\left(\frac{\partial\Phi_2}{\partial\bar\tau}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\cdot l+\frac{\partial\Phi_2}{\partial x_1}(l(T+\bar\tau);X_0+\alpha Y_0+z(\bar\tau,\alpha)E_0)\frac{\partial z}{\partial\bar\tau}(\bar\tau,\alpha)\right).$$

Therefore,

$$\frac{\partial f}{\partial\bar\tau}(0,0)=-\frac{\partial\Phi_2}{\partial\bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))(1-l)-\frac{\partial\Phi_2}{\partial x_1}((1-l)T;I_1(\Phi(lT;X_0)))\cdot(1-\delta_2)$$
$$\times\left(\frac{\partial\Phi_1}{\partial\bar\tau}(lT;X_0)\cdot l+\frac{\partial\Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial\bar\tau}(0,0)\right)-\frac{\partial\Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))\cdot(1-\delta_1)$$
$$\times\left(\frac{\partial\Phi_2}{\partial\bar\tau}(lT;X_0)\cdot l+\frac{\partial\Phi_2}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial\bar\tau}(0,0)\right).$$

From (4.14)–(4.16) and

$$\frac{\partial\Phi_2}{\partial\bar\tau}((1-l)T;I_1(\Phi(lT;X_0)))=0,\tag{4.17}$$

it follows that:

$$\frac{\partial f}{\partial\bar\tau}(0,0)=0.$$

Appendix D. The second order partial derivatives of Φ_2

Again, by formally deriving:

$$\frac{d}{dt}(\Phi(t;X_0))=F(\Phi(t;X_0)),$$

as done in Appendix A, one may obtain $\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)$, $\frac{\partial^2\Phi_2}{\partial x_2^2}(t;X_0)$, $\frac{\partial^2\Phi_2}{\partial x_1\partial x_2}(t;X_0)$ as the solutions of certain initial value problems. One sees that:

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)\right)=(r-g(\Phi_1(t;X_0)))\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)-g'(\Phi_1(t;X_0))\frac{\partial\Phi_1}{\partial x_1}(t;X_0)\frac{\partial\Phi_2}{\partial x_1}(t;X_0)$$

and since

$$\frac{\partial \Phi_2}{\partial x_1}(t;X_0)=0 \quad \text{for } t \geqslant 0.$$

It then follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 \Phi_2}{\partial x_1^2}(t;X_0)\right)=(r-g(\Phi_1(t;X_0)))\frac{\partial^2 \Phi_2}{\partial x_1^2}(t;X_0),$$

and consequently,

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}(t;X_0)=\mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}\frac{\partial^2 \Phi_2}{\partial x_1^2}(0;X_0).$$

Since $\frac{\partial^2 \Phi_2}{\partial x_1^2}(0;X_0)=0$, this implies that:

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}(t;X_0)=0 \quad \text{for } t \geqslant 0.$$

Also, by a similar argument,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 \Phi_2}{\partial x_2^2}(t;X_0)\right)=(r-g(\Phi_1(t;X_0)))\frac{\partial^2 \Phi_2}{\partial x_2^2}(t;X_0)-g'(\Phi_1(t;X_0))\frac{\partial \Phi_1}{\partial x_2}(t;X_0)\frac{\partial \Phi_2}{\partial x_2}(t;X_0),$$

and since

$$\frac{\partial^2 \Phi_2}{\partial x_2^2}(0;X_0)=0,$$

one may deduce that:

$$\begin{aligned}\frac{\partial^2 \Phi_2}{\partial x_2^2}(t;X_0)&=-\mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}\int_0^t g'(\Phi_1(s;X_0))\frac{\partial \Phi_1}{\partial x_2}(s;X_0)\frac{\partial \Phi_2}{\partial x_2}(s;X_0)\mathrm{e}^{-\left(rs-\int_0^s g(\Phi_1(\tau;X_0))\mathrm{d}\tau\right)}\mathrm{d}s\\&=-\mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}\int_0^t g'(\Phi_1(s;X_0))\frac{\partial \Phi_1}{\partial x_2}(s;X_0)\mathrm{d}s.\end{aligned}\tag{4.18}$$

Likewise,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 \Phi_2}{\partial x_1\partial x_2}(t;X_0)\right)=(r-g(\Phi_1(t;X_0)))\frac{\partial^2 \Phi_2}{\partial x_1\partial x_2}(t;X_0)-g'(\Phi_1(t;X_0))\frac{\partial \Phi_1}{\partial x_1}(t;X_0)\frac{\partial \Phi_2}{\partial x_2}(t;X_0),$$

and since

$$\frac{\partial^2 \Phi_2}{\partial x_1\partial x_2}(0;X_0)=0,$$

one obtains that:

$$\begin{aligned}\frac{\partial^2 \Phi_2}{\partial x_1\partial x_2}(t;X_0)&=-\mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}\int_0^t g'(\Phi_1(s;X_0))\frac{\partial \Phi_1}{\partial x_1}(s;X_0)\frac{\partial \Phi_2}{\partial x_2}(s;X_0)\mathrm{e}^{-\left(rs-\int_0^s g(\Phi_1(\tau;X_0))\mathrm{d}\tau\right)}\mathrm{d}s\\&=-\mathrm{e}^{rt-\int_0^t g(\Phi_1(s;X_0))\mathrm{d}s}\int_0^t g'(\Phi_1(s;X_0))\frac{\partial \Phi_1}{\partial x_1}(s;X_0)\mathrm{d}s.\end{aligned}\tag{4.19}$$

Appendix E. The second order partial derivatives of *f*

One notes that:

$$\frac{\partial^2 \Phi_2}{\partial x_1\partial \bar{\tau}}((1-l)T;I_1(\Phi(lT;X_0)))=0,\tag{4.20}$$

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}((1-l)T;I_1(\Phi(lT;X_0)))=0,\tag{4.21}$$

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}(lT;X_0)=0.\tag{4.22}$$

Combining (4.20)–(4.22) with (4.14)–(4.17), we obtain:

$$\frac{\partial^2 f}{\partial \bar{\tau}^2}(0,0) = -\frac{\partial^2 \Phi_2}{\partial \tau^2}((1-l)T; I_1(\Phi(lT;X_0)))(1-l)^2.$$

Since,

$$\frac{\partial^2 \Phi}{\partial \bar{\tau}^2}((1-l)T; I_1(\Phi(lT;X_0))) = 0, \tag{4.23}$$

it is then concluded that:

$$\frac{\partial^2 f}{\partial \bar{\tau}^2}(0,0) = 0.$$

We then compute $\frac{\partial^2 f}{\partial \alpha^2}(0,0)$. By (4.14) and (4.16), it follows that:

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha^2}(0,0) = &-\frac{\partial}{\partial \alpha}\left[\frac{\partial \Phi_2}{\partial x_1}((1-l)(T+\bar{\tau}); I_1(\Phi(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)))\right]\Big|_{(\bar{\tau},\alpha)=(0,0)} \cdot (1-\delta_2) \\
&\times \left(\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial \alpha}(0,0)\right) + \frac{\partial \Phi_1}{\partial x_2}(lT;X_0)\right) \\
&-\frac{\partial}{\partial \alpha}\left[\frac{\partial \Phi_2}{\partial x_2}((1-l)(T+\bar{\tau}); I_1(\Phi(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)))\right]\Big|_{(\bar{\tau},\alpha)=(0,0)} \cdot (1-\delta_1) \\
&\times \left(\frac{\partial \Phi_2}{\partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial \alpha}(0,0)\right) + \frac{\partial \Phi_2}{\partial x_2}(lT;X_0)\right) - \frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0))) \\
&\cdot \frac{\partial}{\partial \alpha}\left[(1-\delta_1)\left(\frac{\partial \Phi_2}{\partial x_1}(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial \alpha}(\bar{\tau},\alpha)\right) + \frac{\partial \Phi_2}{\partial x_2}(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)\right)\right]\Big|_{(\bar{\tau},\alpha)=(0,0)}.
\end{aligned}$$

Using again (4.21) and (4.13), it follows that:

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha^2}(0,0) = &-2\frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)(1-\delta_2) \cdot \left(\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'}\right) + \frac{\partial \Phi_1}{\partial x_2}(lT;X_0)\right)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0) \\
&-\frac{\partial^2 \Phi_2}{\partial x_2^2}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)^2\left(\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)\right)^2 \\
&-\frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1) \cdot \left[2\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(lT;X_0)\left(-\frac{b_0'}{a_0'}\right) + \frac{\partial^2 \Phi_2}{\partial x_2^2}(lT;X_0)\right].
\end{aligned}$$

Consequently, from (4.18), (4.20), (3.2) and (3.3) one easily gets that:

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) > 0.$$

From (4.14)–(4.16), one may see that:

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha \partial \bar{\tau}}(0,0) = &-\frac{\partial}{\partial \alpha}\left[\frac{\partial \Phi_2}{\partial \bar{\tau}}((1-l)(T+\bar{\tau}); I_1(\Phi(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)))\right]\Big|_{(\bar{\tau},\alpha)=(0,0)} \cdot (1-l) \\
&-\frac{\partial}{\partial \alpha}\left[\frac{\partial \Phi_2}{\partial x_1}((1-l)(T+\bar{\tau}); I_1(\Phi(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)))\right]\Big|_{(\bar{\tau},\alpha)=(0,0)} \cdot (1-\delta_2) \\
&\times \left(\frac{\partial \Phi_1}{\partial \bar{\tau}}(lT;X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right) - \frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0))) \cdot (1-\delta_1) \\
&\times \frac{\partial}{\partial \alpha}\left[\frac{\partial \Phi_2}{\partial \bar{\tau}}(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0) \cdot l + \frac{\partial \Phi_2}{\partial x_1}(l(T+\bar{\tau}); X_0+\alpha Y_0+z(\bar{\tau},\alpha)E_0)\frac{\partial z}{\partial \bar{\tau}}(\bar{\tau},\alpha)\right]\Big|_{(\bar{\tau},\alpha)=(0,0)}.
\end{aligned}$$

Using again (4.20) and (4.22), one sees that:

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha \partial \bar{\tau}}(0,0) = &-\frac{\partial^2 \Phi_2}{\partial x_2 \partial \bar{\tau}}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)(1-l) \\
&-\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0) \cdot (1-\delta_2)\left(\frac{\partial \Phi_1}{\partial \bar{\tau}}(lT;X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right) \\
&-\frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT;X_0))) \cdot (1-\delta_1)\left(\frac{\partial^2 \Phi_2}{\partial x_2 \partial \bar{\tau}}(lT;X_0) \cdot l + \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right).
\end{aligned}$$

We now determine the sign of $\frac{\partial^2 f}{\partial\alpha\partial\bar{\tau}}(0,0)$. It is seen that:

$$
\begin{aligned}
-\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0) &= e^{r(1-l)T-\int_0^{(1-l)T} g(\Phi_1(s;I_1(\Phi(lT;X_0))))ds} \\
&\quad \cdot\left(\int_0^{(1-l)T} g'(\Phi_1(s;I_1(\Phi(lT;X_0))))e^{-ws}\,ds\right)\cdot(1-\delta_1)e^{rlT-\int_0^{lT} g(\Phi_1(s;X_0))ds} \\
&= e^{rT-\int_0^{(1-l)T} g(I^*(s+lT))ds-\int_0^{lT} g(I^*(s))ds}(1-\delta_1)\left(\int_0^{(1-l)T} g'(I^*(s+lT))e^{-ws}\,ds\right) \\
&= e^{rT-\int_0^{T} g(I^*(s))ds}(1-\delta_1)\left(\int_0^{(1-l)T} g'(I^*(s+lT))e^{-ws}\,ds\right).
\end{aligned}
$$

From (2.4), it follows that:

$$
-\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0) = \int_0^{(1-l)T} g'(I^*(s+lT))e^{-ws}\,ds.
$$

Likewise,

$$
\begin{aligned}
-\frac{\partial^2 \Phi_2}{\partial x_2 \partial \bar{\tau}}((1-l)T; I_1(\Phi(lT;X_0)))(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)(1-l) &= -(r-g(\Phi_1((1-l)T;I_1(\Phi(lT;X_0)))))\frac{\partial \Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0))) \\
&\quad \cdot(1-\delta_1)\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)(1-l) \\
&= -(r-g(I^*(T)))(1-d_0')(1-l) = -(r-g(I^*(T)))(1-l).
\end{aligned}
$$

Using the results in Appendices A and B, one may deduce that:

$$
\begin{aligned}
(1-\delta_2)\left(\frac{\partial \Phi_1}{\partial \bar{\tau}}(lT;X_0)\cdot l+\frac{\partial \Phi_1}{\partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right) &= (1-\delta_2)\left(-wI^*(lT)\cdot l+e^{-wlT}\left(\left(-\frac{1}{a_0'}\right)wI^*(T)\right)\right) \\
&= -w(1-\delta_2)e^{-wlT}\left(I^*(0+)\cdot l+\frac{1}{a_0'}I^*(T)\right).
\end{aligned}
$$

It is seen that:

$$
\begin{aligned}
&-\frac{\partial \Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\left[\frac{\partial^2 \Phi_2}{\partial x_2\partial \bar{\tau}}(lT;X_0)\cdot l+\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right] \\
&\quad = -\frac{\partial \Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\left[(r-g(\Phi_1(lT;X_0)))\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)\cdot l-\left(\frac{\partial \Phi_2}{\partial x_2}(lT;X_0)\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right].
\end{aligned}
$$

Since $d_0'=0$, it follows that:

$$
\begin{aligned}
&-\frac{\partial \Phi_2}{\partial x_2}((1-l)T;I_1(\Phi(lT;X_0)))(1-\delta_1)\left[\frac{\partial^2 \Phi_2}{\partial x_2\partial \bar{\tau}}(lT;X_0)\cdot l+\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(lT;X_0)\frac{\partial z}{\partial \bar{\tau}}(0,0)\right] \\
&\quad = -(r-g(I^*(lT)))\cdot l+\left(\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)\left(-\frac{1}{a_0'}wI^*(T)\right) = -\left[(r-g(I^*(lT)))\cdot l+\frac{w}{a_0'}\left(\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)I^*(T)\right].
\end{aligned}
$$

It is then deduced that:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial\alpha\partial\bar{\tau}}(0,0) &= -(r-g(I^*(T)))(1-l)+\left(\int_0^{(1-l)T} g'(I^*(s+lT))e^{-ws}\,ds\right)\left(-w(1-\delta_2)e^{-wlT}\left(I^*(0+)\cdot l+\frac{1}{a_0'}I^*(T)\right)\right) \\
&\quad -\left[(r-g(I^*(lT)))\cdot l+\frac{w}{a_0'}\left(\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)I^*(T)\right] \\
&= -[r-lg(I^*(lT))-(1-l)g(I^*(T))]-w\left(\int_0^{(1-l)T} g'(I^*(s+lT))e^{-w(s+lT)}\,ds\right)(1-\delta_2)\left(I^*(0+)\cdot l+\frac{1}{a_0'}I^*(T)\right) \\
&\quad -\frac{w}{a_0'}\left(\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)I^*(T).
\end{aligned}
$$

This implies that:

$$\frac{\partial^2 f}{\partial\alpha\partial\bar{\tau}}(0,0) = -[r - lg(I^*(lT)) - (1-l)g(I^*(T))] - w\left(\int_{lT}^{T} g'(I^*(s))e^{-ws}\,ds\right)(1-\delta_2)\left(I^*(0+)l + \frac{1}{a_0'}I^*(T)\right) - \frac{w}{a_0'}\left(\int_0^{lT} g'(I^*(s))e^{-ws}\,ds\right)I^*(T). \quad (4.24)$$

We note that:

$$rT - \int_0^T g(I^*(s))ds = -\ln(1-\delta_1) > 0$$

and also, since I^* is decreasing on $(0,T]$,

$$\int_0^T g(I^*(s))ds = \int_0^{lT} g(I^*(s))ds + \int_{lT}^T g(I^*(s))ds > lTg(I^*(lT)) + (1-l)Tg(I^*(T)).$$

Consequently, the first term in the right-hand side of (4.24) is negative. Since g is increasing and I^* is positive, the other terms are negative as well and consequently,

$$\frac{\partial^2 f}{\partial\alpha\partial\tau}(0,0) < 0.$$

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