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THE FOURIER METHOD FOR ABSTRACT DIFFERENTIAL EQUATIONS AND APPLICATIONS

Gh. Moroşanu, P. Georgescu and V. Grădinaru
Faculty of Mathematics, "Al.I. Cuza" University,
Iaşi, 6600, Romania

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ABSTRACT: The Fourier method is used to investigate the existence, uniqueness and regularity of solutions to first and second order abstract differential equations. Also, the same method is applied to derive the null-controlability of a second order abstract boundary control problem.

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider a linear, symmetric and coercive operator $B : D(B) \subset H \to H$ and suppose that $D(B)$ has an infinite dimension.

We shall study abstract differential equations of the first order

$$y'(t) + p(t)By(t) = f(t), \quad 0 < t < T \tag{1}$$

and of the second order

$$y''(t) + q(t)y'(t) + r(t)By(t) = g(t), \quad 0 < t < T. \tag{2}$$

We introduce on $D(B)$ the energetic inner product

$$(u,v)_E = (Bu,v), \quad (\forall)u,v \in D(B)$$

and the energetic norm

$$\|u\|_E = (u,u)_E^{1/2}, \quad (\forall)u \in D(B).$$

By H_E we denote the energetic space consisting of all the vectors $u \in H$ with the following properties:

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there exists a sequence $(u_n)_n$ of $D(B)$ such that $u_n \to u$ in H; (3)

$(u_n)_n$ is a Cauchy sequence with respect to the energetic norm $\|\cdot\|_E$. (4)

A sequence $(u_n)_n$ with the properties (3) and (4) will be called an admissible sequence for u. For $u, v \in H_E$ we define the scalar product

$$(u, v)_E = \lim(u_n, v_n)_E$$

for each $(u_n)_n$ and $(v_n)_n$ admissible sequences for u and v respectively. It follows that the scalar product $(\cdot,\cdot)_E$ is well-defined, H_E is the completion of $D(B)$ with respect to the energetic norm and, if we identify H with its dual H^*, we have:

$$H_E \subset H \subset H_E^*,$$

where the embeddings $H_E \subset H$ and $H \subset H_E^*$ are continuous.

One can easily see that the duality mapping $B_E : H_E \to H_E^*$ of the energetic space H_E is an extension of B, called the energetic extension. The Friedrichs extension $A : D(A) \subset H \to H$ of B is defined by

$$Au = B_E u, \quad (\forall) u \in D(A),$$

where

$$D(A) = \{u \in H_E;\ B_E u \in H\}.$$

A is coercive and selfadjoint and

$$D(B) \subset D(A) \subset H_E \subset H.$$

For all the things above see, e.g., Zeidler [5, p.280].

Remark. The Friedrichs extension is in fact the maximal monotone extension of B in H since $D(A)$ is dense in H and A is closed selfadjoint and positive (see, e.g., Haraux [3, p.48]).

In order to use an extended notion of solution we introduce the following "extended" equations

$$y'(t) + p(t)B_E y(t) = f(t), \quad 0 < t < T, \tag{5}$$

$$y''(t) + q(t)y'(t) + r(t)B_E y(t) = g(t), \quad 0 < t < T. \tag{6}$$

We intend to use the Fourier method so we suppose that the embedding $H_E \subset H$ is compact. In this case, we have the following result (see Moroşanu and Sburlan [5])

Theorem 1. (Multiple orthogonal sequence theorem). *If all the above conditions hold then there exist the sequences $(e_n)_{n\geq 1} \subset D(A)$ and $(\lambda_n)_{n\geq 1} \subset (0,\infty)$ of eigenvectors and corresponding eigenvalues of B_E such that*

(i) $(e_n)_{n\geq 1}$ is an orthonormal basis in H_E;

(ii) $(\sqrt{\lambda_n} e_n)_{n\geq 1}$ is an orthonormal basis in H;

(iii) $(\lambda_n e_n)_{n\geq 1}$ *is an orthonormal basis in* H_E^**;*

(iv) $(\lambda_n)_{n\geq 1}$ *is increasingly divergent to* $+\infty$.

2. CAUCHY'S PROBLEM FOR THE FIRST ORDER DIFFERENTIAL EQUATION

2.1. The existence theorem

We now study the Eq. (5) with the initial condition

$$y(0) = y_0. \tag{7}$$

We shall obtain some results more or less classical, but based on the Fourier method.

Let us suppose that

$$p \in L^\infty(0,T),\quad p(t) \geq p_0 > 0 \quad a.e.\ t \in (0,T). \tag{8}$$

We can prove the following

Theorem 2. *Suppose that all the above hypotheses hold. If* $y_0 \in H$ *and* $f \in L^2(0,T;H_E^*)$ *then problem (5), (7) has a unique solution* $y \in C([0,T];H) \cap L^2(0,T;H_E)$ *with* $y' \in L^2(0,T;H_E^*)$ *in the sense that* y *verifies (7) and*

$$y'(t)(v) + p(t)B_E y(t)(v) = f(t)(v),\quad (\forall) v \in H_E,\quad a.e.\ t \in (0,T).$$

Proof. We search for a solution of the form

$$y(t) = \sum_{n=1}^{\infty} b_n(t) e_n, \tag{9}$$

where $(e_n)_{n\geq 1}$ is the sequence given by Thm. 1. So we can formally deduce that the coefficients $b_n(t)$ satisfy

$$\begin{cases} b_n'(t) + \lambda_n p(t) b_n(t) = f_n(t), & a.e.\ t \in (0,T) \\ b_n(0) = y_{0n} \end{cases} \tag{10}$$

where

$$y_{0n} = \lambda_n (y_0, e_n),\quad f_n(t) = \lambda_n (B_E^{-1} f(t), e_n)_E.$$

Since $y_0 \in H$ and $f \in L^2(0,T;H_E^*)$ we have

$$\sum_{n=1}^{\infty} \lambda_n^{-1} y_{0n}^2 = \|y_0\|^2 \tag{11}$$

and

$$\sum_{n=1}^{\infty} \lambda_n^{-2} f_n^2(t) = \|f(t)\|_{H_E^*}^2,\quad a.e.\ t \in (0,T). \tag{12}$$

It is obvious that the problems (10) have unique solutions.

By multiplying (10.1) by $b_n(t)$ and integrating on $[0,t]$ we deduce (see also (8))

$$\frac{1}{2}b_n^2(t) + \lambda_n p_0 \int_0^t b_n^2(s)ds \le \frac{1}{2}y_{0n}^2 + \int_0^t | f_n(s) || b_n(s) | ds, \quad t \in [0,T]. \tag{13}$$

So, for $\varepsilon > 0$ small enough we have

$$\frac{1}{2\lambda_n}b_n^2(t) + p_0 \int_0^t b_n^2(s)ds \le \frac{1}{2\lambda_n}y_{0n}^2 + \frac{\varepsilon}{2}\int_0^t b_n^2(s)ds + \frac{1}{2\varepsilon}\int_0^t \frac{f_n^2(s)}{\lambda_n^2}ds, \quad t \in [0,T]. \tag{14}$$

It follows that

$$\int_0^T b_n^2(s)ds \le C\left(\frac{y_{0n}^2}{\lambda_n} + \int_0^T \frac{f_n^2(s)}{\lambda_n^2}ds\right) \tag{15}$$

which implies the convergence of the series (9) in $L^2(0,T;H_E)$ and its sum y belongs to $L^2(0,T;H_E)$.

Using again (14) we obtain the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} b_n^2(t)$ and this implies $y \in C([0,T];H)$.

By (10.1) we have the convergence of the series $\sum_{n=1}^{\infty} \lambda_n^{-2}[b_n'(t)]^2$ in $L^1(0,T)$, so the series $\sum_{n=1}^{\infty} b_n'(t)e_n$ is convergent in $L^2(0,T;H_E^*)$ and its sum y' belongs to $L^2(0,T;H_E^*)$.

It is obvious that $y(t)$ given by (9), (10) is a solution for (5), (7) and (9) assures the uniqueness.

Remark. (14) implies the following estimation (the continuous dependence of the solution with respect to the initial value y_0 and to the function f):

$$\|y(t)\|^2 + \int_0^T \|y(s)\|_E^2 ds \le C(\|y_0\|^2 + \int_0^T \|f(s)\|_{H_E^*}^2 ds), \quad 0 \le t \le T. \tag{16}$$

In particular, (16) leads to the uniqueness of the solution.

2.2. The regularity of the solution

Let us make the same assumptions as above. Then we can prove the following

Theorem 3. *If $y_0 \in H$ and $f \in L^2(0,T;H)$ then y belongs to $C((0,T];H_E)$.*

Proof. By (10) we deduce

$$b_n(t) = e^{-\lambda_n \int_0^t p(s)ds} y_{0n} + \int_0^t e^{-\lambda_n \int_s^t p(\tau)d\tau} \cdot f_n(s)ds \tag{17}$$

and consequently

$$b_n^2(t) \le 2y_{0n}^2 e^{-2\lambda_n p_0 t} + \frac{1}{\lambda_n p_0}\int_0^t f_n^2(s)ds, \quad t \in [0,T]. \tag{18}$$

Let $\delta \in (0,T)$ be fixed. Then (18) implies

$$b_n^2(t) \le \frac{y_{0n}^2}{\lambda_n p_0 \delta} + \frac{1}{\lambda_n p_0}\int_0^t f_n^2(s)ds \text{ for } \delta \le t \le T \tag{19}$$

so that $y \in C([\delta, T]; H_E)$, $(\forall)\delta \in (0,T)$. Thus, $y \in C((0,T]; H_E)$.

Using (18) we obtain that for $y_0 \in H_E$ and $f \in L^2(0,T;H)$ y belongs to $C([0,T];H_E)$.

Remark. If $p(t) \equiv p_0 > 0$ one can continue the discussion concerning the regularity as in Grădinaru [2].

Let us study into further details the regularity of the solution. We denote

$$V_s = D(A^{s/2}) = \{u \in H;\ \sum_{n=1}^{\infty} \lambda_n^{1-s}\ (u, e_n)^2 < \infty\} \tag{20}$$

for every $s > 0$, where $A^{s/2} : D(A^{s/2}) \subset H \to H$ is defined by

$$A^{s/2}u = \sum_{n=1}^{\infty} \lambda_n^{1-s/2}(u, e_n)e_n$$

for every $u \in H$. We set

$$V_0 = H \tag{21}$$

and

$$V_s = (V_{-s})^* \text{ for every } s < 0. \tag{22}$$

It is obvious that V_s is a Hilbert space equipped with the inner product

$$(u,v)_s = (A^{s/2}u, A^{s/2}v).$$

For $s = 1$, $V_1 = H_E$ and $(u,v)_1 = (u,v)_E$.

A straightforward computation shows that $(\lambda_n^{(-s+1)/2} e_n)_{n \geq 1}$ is a Hilbertian basis in V_s and

$$\|y_0\|_s^2 = \sum_{n=1}^{\infty} \lambda_n^{s+1}(y_0, e_n)_H^2 \tag{23}$$

for every $y_0 \in V_s$.

Finally we obtain a nonincreasing family of subspaces V_s, $s \in \mathbb{R}$. Obviously, V_{s_1} is dense into V_{s_2} if $s_2 < s_1$, the injection of V_{s_1} into V_{s_2} being compact.

The duality mapping $J_s : V_s \to V_s^*$ is defined by

$$(J_s(u), v) = (u,v)_s, \text{ for all } u, v \in V_s.$$

It is easy to see that for every $n \in \mathbb{N}^*$ and $s > 0$

$$J_s(\lambda_n^{(-s+1)/2} e_n) = \lambda_n^{(s+1)/2} e_n. \tag{24}$$

Theorem 4. *Under the above hypotheses we have the following:*

1) $y_0 \in H$, $f \in L^2(0,T;H)$ *imply* $y \in C((0,T]; V_s) \cap C^1((0,T]; V_{s-2})$ for every $s \in \mathbb{R}$;

2) $y_0 \in V_s$, $f \in L^2(0,T;V_{s-1})$ *imply* $y \in C([0,T]; V_s) \cap L^2(0,T;V_{s+1}) \cap$ $\cap H^1(0,T;V_{s-1}) \cap C^1([0,T];V_{s-2})$ for every $s \in \mathbb{R}$.

Proof. Let $s \in \mathbb{R}$ be fixed. By multiplying (18) with λ_n^{s-1} we get

$$\lambda_n^{s-1} b_n^2(t) \leq 2e^{-2\lambda_n p_0 t} \lambda_n^{s-1} y_{0n}^2 + \frac{1}{p_0} \lambda_n^{s-2} \int_0^t f_n^2(s) ds.$$

For $\delta \in (0, T]$ fixed we have

$$2e^{-\lambda_n p_0 t} \cdot \lambda_n^{s-1} < \frac{1}{\lambda_n} \quad (\forall) t \in (\delta, T]$$

for $n \geq n_0 \in \mathbb{N}$ sufficiently large. Thus,

$$\lambda_n^{s-1} b_n^2(t) \leq \frac{y_{0n}^2}{\lambda_n} + \frac{1}{p_0} \lambda_n^{s-2} \int_0^t f_n^2(s) ds, \text{ for } n \geq n_0(\varepsilon). \tag{25}$$

Thanks to (25) we obtain

$$y \in C((0, T]; V_s). \tag{26}$$

Then

$$[b_n'(t)]^2 \leq 2(\lambda_n^2 p^2(t) b_n^2(t) + f_n^2(t))$$

and so

$$\lambda_n^{s-3} [b_n'(t)]^2 \leq 2M^2 \lambda_n^{s-1} b_n^2(t) + 2\lambda_n^{s-3} f_n^2(t), \tag{27}$$

where $p(t) \leq M$ for a.e. $t \in (0, T)$. From (27) one deduces

$$y' \in C((0, T]; V_{s-2}). \tag{28}$$

To prove the second statement, we observe that

$$\int_0^T b_n^2(s) ds \leq C \left(\frac{y_{0n}^2}{\lambda_n} + \int_0^T \frac{f_n^2(s)}{\lambda_n^2} ds \right) \tag{29}$$

and, for the same reasons, $y \in L^2(0, T; V_{s+1})$.

Obviously, $y \in C([0, T]; V_s)$ and in the same manner we can prove that $y' \in L^2(0, T; V_{s-1}) \cap C((0, T]; V_{s-2})$.

We study now the case $f \equiv 0$.

Corollary 1. *If $y_0 \in H$ then*

$$\|y(t)\| \leq e^{-2\lambda_1 p_0 t} \|y_0\|. \tag{30}$$

Proof. $\|y(t)\| = \|\sum_{n=1}^{\infty} e^{-\lambda_n \int_0^t p(s) ds} y_{0n} e_n\| = \sum_{n=1}^{\infty} e^{-2\lambda_n \int_0^t p(s) ds} \frac{y_{0n}^2}{\lambda_n} \leq \sum_{n=1}^{\infty} e^{-2\lambda_n p_0 t} \frac{y_{0n}^2}{\lambda_n}$ $\leq e^{-2\lambda_1 p_0 t} \sum_{n=1}^{\infty} \frac{y_{0n}^2}{\lambda_n} = e^{-2\lambda_1 p_0 t} \|y_0\|$. Therefore $\lim y(t) = 0$ in H for $t \to \infty$.

Remark. In fact, $y(t) \to 0$ *in* $\|\cdot\|_s$ for every $s > 0$.

We have

$$\|y(t)\|_s^2 = \|\sum_{n=1}^{\infty} e^{-\lambda_n \int_0^t p(s) ds} y_{0n} e_n\|_s^2 \leq \sum_{n=1}^{\infty} e^{-2\lambda_n \int_0^t p(s) ds} \cdot \frac{y_{0n}^2}{\lambda_n^{1-s}}$$

$$\le \sum_{n=1}^{\infty} e^{-2\lambda_n p_0 t} \cdot \lambda_n^s \cdot \frac{y_{0n}^2}{\lambda_n} \le e^{-\lambda_1 p_0 t} \sum_{n=1}^{\infty} e^{-\lambda_n p_0 t} \lambda_n^s \frac{y_{0n}^2}{\lambda_n}.$$

We observe that for $t > \dfrac{s}{p_0}$ we have $e^{-xp_0 t} \cdot x^s < 1$ for every $x > 0$, and therefore $\|y(t)\|_s \le e^{-\lambda_n p_0 t}\|y_0\|$ for t sufficiently large.

So, one can obtain $y(t) \to 0$ in $\|\cdot\|_s$ for $t \to \infty$.

2.3. The existence of periodic solutions to Eq. 5

Assume that $f \in L^2(0,T,H)$. We search for a T-periodic solution of the form (9). From $y(0) = y(T)$ we get formally

$$b_n(0) = b_n(T).$$

Hence

$$b_n(t) = e^{-\lambda_n \int_0^t p(s)ds} C_n + \int_0^t e^{-\lambda_n \int_s^t p(\tau)d\tau} f_n(s)ds,$$

where

$$C_n = \frac{1}{1 - e^{-\lambda_n \int_0^T p(s)ds}} \int_0^T e^{-\lambda_n \int_s^T p(\tau)d\tau} f_n(s)ds.$$

It is easy to see that (8) implies

$$C_n^2 \le \left(\frac{1}{1 - e^{-\lambda_n p_0 T}}\right)^2 \left(\int_0^T e^{-\lambda_n p_0 (T-s)} \mid f_n(s) \mid ds\right)^2 \le Const. \frac{1}{\lambda_n} \int_0^T f_n(s)^2 ds.$$

Therefore

$$b_n(t)^2 \le Const. \frac{1}{\lambda_n} \int_0^T f_n(s)^2 ds. \tag{31}$$

So series (9) converges in H_E, uniformly with respect to $t \in [0,T]$ and its sum belongs to $C([0,T]; H_E)$. Of course, y is a weak solution, satisfying $y(0) = y(T)$.

In fact, we can assume that $f \in L^2(0,T;H_E^*)$. From (31) we can deduce that $y \in C([0,T];H)$. So, by Thm.2, $y \in L^2(0,T;H_E)$ with $y' \in L^2(0,T;H_E^*)$. If $f \in L^2(0,T;V_s)$ then (31) implies $y \in C([0,T];V_{s+1})$.

2.4. An example

Let $\Omega \subset \mathbb{R}^N$ be a nonvoid bounded and open set, and consider $H = L^2(\Omega)$ equipped with the usual inner product, $D(B) = C_0^\infty(\Omega)$, $B = -\Delta$.
Then $H_E = H_0^1(\Omega)$, the Friedrichs extension of B is $A = (-\Delta)_E$, where $(-\Delta_E u)(v) = \int_\Omega \nabla u \nabla v dx$ and $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Consider the problem:

$$y_t + (-\Delta)_E y = f(x,t) \text{ for } x \in \Omega, \ 0 < t < T, \tag{32}$$

$$y = 0 \text{ on } \partial\Omega \times (0,T), \tag{33}$$

$$y(x,0) = y_0(x) \ \text{ for } \ x \in \Omega. \tag{34}$$

If $y_0 \in H$ and $f \in L^2(\Omega \times (0,T))$, then (32), (33), (34) has a unique solution $y \in C((0,T); H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ with $y_t' \in L^2(0,T; H^{-1}(\Omega))$.

It is easy to see that y is a generalized solution for

$$y_t - \Delta y = f(x,t) \ \text{ for } \ x \in \Omega, \ \ 0 < t < T.$$

Indeed

$$\int_Q y(-\varphi_t - \Delta\varphi)dxdt = \int_Q f\varphi dxdt, \ \ (\forall)\varphi \in C_0^\infty(Q)$$

where $Q = \Omega \times (0,T)$. Moreover for $f \in C_0^\infty(Q)$, $y_0 \in C_0^\infty(\Omega)$ and $\partial\Omega$ sufficiently smooth, $y \in C_0^\infty(Q)$.

This example clearly shows the connection between (1) and (5).

3. CAUCHY'S PROBLEM FOR THE SECOND ORDER DIFFERENTIAL EQUATIONS

We study now (6) with the initial values

$$y(0) = y_0; \ \ y'(0) = y_1. \tag{35}$$

Let us assume that

$$q, r \in L^\infty(0,T); \ \ 0 \le q \ \ a.e. \ \ t \in (0,T); \tag{36}$$

$$r(t) \ge r_0 > 0 \ \ a.e. \ \ t \in (0,T), \ \ y_0 \in H_E, \ \ y_1 \in H, \ \ g \in L^2(0,T;H).$$

We search for $y(t)$ of the form

$$y(t) = \sum_{n=1}^\infty b_n(t) e_n. \tag{37}$$

So, we can formally deduce

$$b_n''(t) + q(t)b_n'(t) + \lambda_n r(t) b_n(t) = g_n(t) \tag{38}$$

with

$$b_n(0) = y_{0n}, \ \ b_n'(0) = y_{1n}, \tag{39}$$

where

$$y_{0n} = (y_0, e_n)_E = \lambda_n(y_0, e_n), \ \ y_{1n} = \lambda_n(y_1, e_n), \tag{40}$$

$$g_n(t) = \lambda_n(g(t), e_n) \ \text{ for a.e. } \ t \in (0,T).$$

We suppose in addition that $r(t) \equiv r_0 > 0$. By multiplying (38) with $b_n'(t)$ one can obtain

$$\left(\frac{1}{2} b_n'(t)^2\right)' + q(t) b_n'(t)^2 + \frac{\lambda_n r_0}{2}[b_n(t)^2]' = g_n(t) b_n'(t) \ \ a.e. \ \ 0 < t < T.$$

It follows that

$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n}{2}r_0 b_n^2(t) \le \frac{1}{2}y_{1n}^2 + \frac{\lambda_n r_0}{2}y_{0n}^2 + \int_0^t |\, g_n(s) \,| \cdot |\, b_n'(s) \,|\, ds, \quad 0 \le t \le T. \quad (41)$$

Therefore

$$|\, b_n'(t) \,| \le \sqrt{y_{1n}^2 + \lambda_n r_0 y_{0n}^2} + \int_0^t |\, g_n(s) \,|\, ds$$

and hence

$$(b_n'(t))^2 \le C(y_{1n}^2 + \lambda_n y_{0n}^2 + \int_0^T g_n^2(s)ds), \quad 0 \le t \le T. \quad (42)$$

From (41) and (42) one can obtain

$$b_n^2(t) \le C\left(y_{0n}^2 + \frac{y_{1n}^2}{\lambda_n} + \int_0^T \frac{g_n^2(s)}{\lambda_n}ds\right), \quad 0 \le t \le T. \quad (43)$$

By (42), (43) and (38) it follows that

$$\left(\frac{b_n''}{\lambda_n}\right)^2 \le C\left(y_{0n}^2 + \frac{y_{1n}^2}{\lambda_n} + \int_0^T \frac{g_n(s)^2}{\lambda_n}ds + \frac{g_n(t)^2}{\lambda_n}\right). \quad (44)$$

Now, (42), (43), (44) imply that

$$y \in C([0,T]; H_E) \cap C^1([0,T]; H) \cap H^2(0,T; H_E^*) \quad (45)$$

The proof can be continued as in Moroşanu and Sburlan [5] thus obtaining that y is a weak solution of (38), (39) in the sense that

$$y''(t)(\varphi) + q(t)y'(t)(\varphi) + r_0 B_E y(t)(\varphi) = g(t)(\varphi), \quad (\forall)\varphi \in H_E, \quad a.e.\ t \in (0,T) \quad (46)$$

and

$$y(0) = y_0, \quad y'(0) = y_1. \quad (47)$$

Remark 1. If $q(t)$ is also a constant function, $q(t) \equiv q_0 > 0$, then $b_n(t)$ can be expressed by a formula (the roots of the corresponding characteristic equation are complex numbers, for n large enough, because $\lambda_n \to \infty$). In this case we could do as in Moroşanu and Sburlan [5] and obtain new results.

Remark 2. If $r(t) \equiv r_i$ for $t_i < t < t_{i+1}$ where $0 = t_1 < \ldots < t_N = T$ is a partition of $[0,T]$ we can obtain again the estimates (41), (42) and (43) and so the previous existence result still holds. Moreover, if $r_1 \ge r_2 \ge \ldots \ge r_{N-1}$ the constant C does not depend on N. In fact, we have

Proposition. *If in addition to the above hypotheses we assume that r is a nonincreasing function, then the problem* (6), (5) *has a unique weak solution* y *satisfying* (45).

Proof. As already done (by multiplying (38) with $b_n'(t)$) we have:

$$\left(\frac{1}{2}b_n'(t)^2\right)' + q(t)b_n'(t)^2 + \frac{\lambda_n}{2}r(t)[b_n(t)^2]' = g_n(t)b_n'(t)$$

so
$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n}{2}\int_0^t r(s)[b_n(s)^2]'ds \le \frac{1}{2}y_{1n}^2 + \int_0^t \mid g_n(s) \mid \cdot \mid b_n'(s) \mid ds.$$
For each $m \in \mathbb{N}^*$ we define $r_m : [0,T] \to \mathbb{R}$ by
$$r_m(t) = \begin{cases} \frac{i-1}{2^m}, \text{ if } \frac{i-1}{2^m} \le r(t) < \frac{i}{2^m} & \text{for } i = 1,2,\dots,m\cdot 2^m \\ m, \text{ if } m \le r(t) \end{cases}$$

It is clear that r_m converges to r in $L^\infty(0,T)$. The monotony of r makes that r_m is defined and constant on intervals.

We rewrite the last inequality as:
$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n}{2}\int_0^t [r(s) - r_m(s)][b_n(s)^2]'ds + \frac{\lambda_n}{2}\int_0^t r_m(s)[b_n(s)^2]'ds \le$$
$$\le \frac{1}{2}y_{1n}^2 + \int_0^t \mid g(s) \mid \cdot \mid b_n'(s) \mid ds.$$
Also:
$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n}{2}\int_0^t [r(s) - r_m(s)]\cdot[b_n(s)^2]'ds + \frac{\lambda_n}{2}(c_1 b_n^2(t_1) - c_1 b_n^2(t_0) +$$
$$+ c_2 b_n^2(t_2) - c_2 b_n^2(t_1) + \dots) \le \frac{1}{2}y_{1n}^2 + \int_0^t \mid g(s) \mid \cdot \mid b_n'(s) \mid ds.$$
We denote $r_0 = \inf_{[0,T]} r(t)$ and $r_1 = \sup_{[0,T]} r(t)$ and using the fact that r is decreasing we obtain:
$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n}{2}\int_0^t [r(s) - r_m(s)]\cdot[b_n(s)^2]'ds + \frac{\lambda_n r_0}{2}b_n(t)^2 \le$$
$$\le \frac{1}{2}y_{1n}^2 + \frac{\lambda_n}{2}r_1 y_{0n}^2 + \int_0^t \mid g(s) \mid\mid b_n'(s) \mid ds.$$
Making that m tends to infinity we get
$$\frac{1}{2}b_n'(t)^2 + \frac{\lambda_n r_0}{2}b_n(t)^2 \le \frac{1}{2}y_{1n}^2 + \frac{\lambda_n}{2}r_1 y_{0n}^2 + \int_0^t \mid y(s) \mid \cdot \mid b_n'(s) \mid ds.$$

From now on, one can obtain estimations of kind of (42), (43), (44), as usually.
Remark 3. For the regularity of the solution one can see the discussion of the first order case. To see the connection between (2) and (6) one can construct an example as in the first order case.

3.2. The abstract boundary control problem

We consider a vibrating system whose motion is governed by the equation (2) with $g \equiv 0$ and with the boundary condition
$$Zy(t) = \varphi(t), \quad 0 \le t \le T \tag{48}$$

and has the initial states as in (35), where $Z : D(Z) \subset H \to H_Z$, and $B : D(B) \subset H \to H$ are densely defined linear operators with $D(B) \subseteq D(Z)$, H_Z being a second real Hilbert space.

We make the assumptions of (36) and let r be nonincreasing monotone a.e.

The control function φ is allowed to vary in the Sobolev space

$$H_0^2(0,T;H_Z) = \{\varphi \in H^2(0,T,H_Z);\ \varphi(0) = \varphi(T) = \varphi'(0) = \varphi'(T) = 0\}$$

Let us denote

$$D_0 = \{z \in D(B);\ Zz = 0\} \tag{49}$$

and $\tilde{B} = B\,|_{D_0}$.

We suppose $\tilde{B} : D_0 \to H$ as being selfadjoint and coercive, $E = D(\tilde{B}^{1/2}) \subseteq D(Z)$ and the embedding $D(\tilde{B}) \subseteq H$ as being compact. We also assume that $y_0 \in E$ and $y_0' \in H$. In the following we shall use the idea from Krabs [4].

Assumption 1. For every $\psi \in H_Z$ there exists exactly one element $w_\psi \in D(B)$ such that

$$Bw_\psi = 0 \ \text{ and } \ Zw_\psi = \psi$$

and the linear operator $G : H_Z \to D(B)$ defined by $G(\psi) = w_\psi$ is continuous.

For $\varphi \in H_0^2(0,T;H_Z)$ being given we define $\bar{r}(t) = G(\varphi(t))$, $(\forall)t \in [0,T]$ and we infer that

$$\bar{r} \in H_0^2(0,T;D(B)) = \{\bar{r} \in H^2(0,T;D(B));\ \bar{r}(0) = \bar{r}(T) = \bar{r}'(0) = \bar{r}'(T) = 0\}.$$

Let $y \in C([0,T];D(B)) \cap C^1([0,T];H)$ be a solution of (2) with $g = 0$, (48), (35) for $\varphi \in H_0^2(0,T;D(B))$ being given.

Then we define $\bar{y} = y - \bar{r}$, and conclude that $Z\bar{y}(t) = Zy(t) - Z\bar{r}(t) = 0$, and so

$$\bar{y} \in C([0,T];D_0) \cap C^1([0,T];H).$$

Finally we deduce

$$\bar{y}''(t) + q(t)\bar{y}'(t) + r(t)\tilde{B}\bar{y}(t) = -\bar{r}''(t) - q(t)\bar{r}'(t)$$

and

$$\bar{y}(0) = y_0,\ \ \bar{y}'(0) = y_1.$$

Thanks to multiple orthogonal sequence theorem (see Moroşanu and Sburlan [5]) we obtain the sequences $(e_n)_{n\geq 1} \subset D_0$ and $(\lambda_n)_n \subset (0,\infty)$ such that

1. $(e_n)_n$ is an orthonormal basis in E;
2. $(\sqrt{\lambda_n}e_n)_n$ is an orthonormal basis in H;
3. $(\lambda_n e_n)_n$ is an orthonormal basis in E^*;
4. $(\lambda_n)_n$ is increasingly divergent to $+\infty$ and $\tilde{B}e_n = \lambda_n e_n$ $(\forall)n \in \mathbb{N}^*$.

Let $\varphi_n = \sqrt{\lambda_n} e_n$. We search for a weak solution $\bar{y}(t)$ of the form:

$$\bar{y} = \sum_{n=1}^{\infty} y_n(t)\varphi_n. \tag{50}$$

We can formally deduce:

$$y_n''(t) + q(t)y_n'(t) + \lambda_n r(t) y_n(t) = -(\bar{r}''(t), \varphi_n) - q(t)(\bar{r}'(t), \varphi_n) \ \ (\forall) n \in \mathbb{N}^*.$$

Let x_n^1 and x_n^2 be functions satisfying:

$$\begin{cases} (x_n^1)''(t) + q(t)(x_n^1)'(t) + \lambda_n r(t) x_n^1(t) = 0 \\ x_n^1(0) = 0, \ \ (x_n^1)'(0) = \sqrt{\lambda_n} \end{cases} \tag{51}$$

respectively

$$\begin{cases} (x_n^2)''(t) + q(t)(x_n^2)'(t) + \lambda_n r(t) x_n^2(t) = 0 \\ x_n^2(0) = 1, \ \ (x_n^2)'(0) = 0. \end{cases} \tag{52}$$

We will denote

$$f_n(s) = -(\bar{r}''(s), \ \varphi_n) - q(s)(\bar{r}'(s), \ \varphi_n). \tag{53}$$

Then

$$y_n(t) = \frac{y_{1n}}{\sqrt{\lambda_n}} x_n^1(t) + y_{0n} x_n^2(t) + \int_0^t [x_n^1(s)x_n^2(t) - x_n^1(t)x_n^2(s)] \frac{f_n(s)}{W_n(s)} ds \tag{54}$$

where

$$y_{1n} = (y_1, \varphi_n), \ \ y_{0n} = (y_0, \varphi_n), \tag{55}$$

$$W_n(s) = x_n^1(s){x_n^2}'(s) - x_n^2(s){x_n^1}'(s). \tag{56}$$

We therefore define a weak solution of (2) with $g = 0$, (48) and (35) by $y = \bar{y} + \bar{r}$, with $\bar{r} = G(\varphi)$ and $\bar{y}$ given by (50) and (54).

According to this definition, the weak solution y can be represented in the form:

$$y(t) = \sum_{n=1}^{\infty} \left[\frac{y_{1n}}{\sqrt{\lambda_n}} x_n^1(t) + y_{0n} x_n^2(t) \right] \varphi_n + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left(\int_0^t r(s) \cdot \right. \tag{57}$$

$$\left. \cdot e^{\int_0^s q(\tau)d\tau} [x_n^1(t)x_n^2(s) - x_n^1(s)x_n^2(t)](\bar{r}(s), \ \varphi_n) ds \right) \varphi_n, \ \ \text{for } t \in [0, T]$$

and its time derivative is given by:

$$y'(t) = \sum_{n=1}^{\infty} \left[\frac{y_{1n}}{\sqrt{\lambda_n}} {x_n^1}'(t) + y_{0n} {x_n^2}'(t) \right] \varphi_n + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left(\int_0^t r(s) \cdot \right. \tag{58}$$

$$\left. \cdot e^{\int_0^s q(\tau)d\tau} [{x_n^1}'(t)x_n^2(s) - x_n^1(s){x_n^2}'(t)](\bar{r}(s), \ \varphi_n) ds \right) \varphi_n.$$

If the control space $H_0^2(0, T; H_Z)$ is replaced by

$$H_0^1(0, T; H_Z) = \{u \in H^1(0, T; H_Z); u(0) = u(T) = 0\}$$

then $y = y(t)$ defined by (57) with $\bar{r} = G(\varphi)$ for φ chosen in $H_0^1(0,T;H_Z)$ is taken as weak solution of (2) with g=0, (48) with this φ and (35), y' given by (58) being chosen as its time derivative.

We are now in position to solve the null-controlability problem associated to (2) with $g = 0$, (48) and (35).

Null-controlability problem. Given $T > 0$, $y_0 \in E$ and $y_0' \in H$, find $\varphi \in H_0^1(0,T;H_Z)$ such that the weak solution $y = y(t)$, $t \in [0,T]$ of (2) with $g = 0$, (48) with this φ and (35) given by (57) with $\bar{r} = G(\varphi)$ satisfies the end conditions

$$y(T) = 0, \quad y'(T) = 0, \tag{59}$$

where $y' = y'(t)$, $t \in [0,T]$ is given by (58).

It is obvious that the end conditions (59) are equivalent to

$$\begin{cases} y_{1n} = -\lambda_n \displaystyle\int_0^T r(s) e^{\int_0^s q(\tau)d\tau} x_n^2(s)(\bar{r}(s),\ \varphi_n) ds \\ y_{0n} = \sqrt{\lambda_n} \displaystyle\int_0^T r(s) e^{\int_0^s q(\tau)d\tau} x_n^1(s)(\bar{r}(s),\ \varphi_n) ds, \quad (\forall) n \in \mathbb{N}^*. \end{cases} \tag{60}$$

Assumption 2. There exists a linear operator $\tilde{Z} : D_0 \to H_Z$ such that

$$(\bar{r}, \tilde{B}\varphi_j) = (Z\bar{r}, \tilde{Z}\varphi_j)_{H_Z}$$

for all $j \in \mathbb{N}^*$ and all $\bar{r} \in D(B)$.

Then (60) can be rewritten as

$$\begin{cases} y_{1n} = -\displaystyle\int_0^T r(t) e^{\int_0^t q(s)ds} x_n^2(s)(Z\bar{r},\ \tilde{Z}\varphi_n)_{H_Z} dt \\ \sqrt{\lambda_n} y_{0n} = \displaystyle\int_0^T r(t) e^{\int_0^t q(s)ds} x_n^1(t)(Z\bar{r},\ \tilde{Z}\varphi_n)_{H_Z} dt, \quad (\forall) n \in \mathbb{N}^*. \end{cases} \tag{61}$$

Let $F_1^n(t) = \displaystyle\int_0^t r(s) e^{\int_0^s q(\tau)d\tau} x_n^2(s) ds$ and $F_2^n(t) = \displaystyle\int_0^t r(s) e^{\int_0^s q(\tau)d\tau} x_n^1(s) ds$.

Integrating by parts one can deduce:

$$\begin{cases} y_{1n} = \displaystyle\int_0^t F_1^n(t)(\varphi'(t),\ \tilde{Z}\varphi_n)_{H_Z} dt \\ \sqrt{\lambda_n} y_{0n} = -\displaystyle\int_0^T F_2^n(t)(\varphi'(t),\ \tilde{Z}\varphi_n)_{H_Z} dt. \end{cases} \tag{62}$$

For every $u \in L^2(0,T;H_Z)$ let us define $\varphi(t) = \displaystyle\int_0^t u(s) ds$, $t \in [0,T]$.

Then $\varphi'(t) = u(t)$ a.e. $t \in [0,T]$, i.e. $\varphi \in H^1(0,T;H_Z)$ and $\varphi(0) = 0$.

The end condition $\varphi(T) = 0$ is now equivalent to

$$\int_0^T u(t) dt = 0. \tag{63}$$

So, in order to solve the null-controlability problem, one has to find some $u \in L^2(0,T;H_Z)$ which satisfies (63) and (62) with u instead of φ'.

According to Assumption 1, for every $u \in L^2(0,T;H_Z)$ there exists exactly one $w_u \in L^2(0,T;H)$ such that $w_u(t) \in D(Z)$ for all $t \in [0,T]$ and $Bw_u(t) = 0$, $Zw_u(t) = u(t)$ a.e. $t \in [0,T]$.

By Assumption 2 we obtain:

$$\frac{1}{\lambda_j}(u(t),\ \tilde{Z}\varphi_j)_{H_Z} = (w_u(t), \varphi_j),\quad (\forall) j \in \mathbb{N}^*.$$

Using (62) one can obtain that u has to verify

$$\begin{cases} \dfrac{y_{1n}}{\sqrt{\lambda_n}} = \displaystyle\int_0^T \sqrt{\lambda_n} F_1^n(t)(w_u(t),\ \varphi_n)dt \\ y_{0n} = -\displaystyle\int_0^T \sqrt{\lambda_n} F_2^n(t)(w_u(t),\ \varphi_n)dt,\quad n \in \mathbb{N}^*. \end{cases} \tag{64}$$

Let us define $S : L^2(0,T;H_A) \to (\mathbb{R}^2)^{\mathbb{N}}$ by $S(u) = (S_n^1(u),\ S_n^2(u))_{n\in\mathbb{N}^*}$ where

$$\begin{cases} S_n^1(u) = \displaystyle\int_0^T \sqrt{\lambda_n} F_1^n(t)(w_u(t),\ \varphi_n)dt \\ S_n^2(u) = -\displaystyle\int_0^T \sqrt{\lambda_n} F_2^n(t)(w_u(t),\ \varphi_n)dt. \end{cases} \tag{65}$$

Let us also define $c = (c_n^1,\ c_n^2)_{n\in\mathbb{N}^*} \in (\mathbb{R}^2)^{\mathbb{N}}$ by

$$c_n^1 = \frac{y_{1n}}{\sqrt{\lambda_n}};\quad c_n^2 = y_{0n}. \tag{66}$$

So, the null-controlability problem is equivalent to finding $u \in L^2(0,T;H_Z)$ such that

$$\begin{cases} S(u) = c, \\ \displaystyle\int_0^T u(t)dt = 0. \end{cases} \tag{67}$$

We also obtain

$$(S_n^1(u))^2 + (S_n^2(u))^2 \le \lambda_n \int_0^T [(F_1^n(t))^2 + (F_2^n(t))^2](w_u(t),\ \varphi_n)^2 dt$$

so that

$$\sum_{n=1}^{\infty} [(S_n^1(u))^2 + (S_n^2(u))^2] \le C\|w_u\|^2_{L^2(0,T;H)}(\|x_1\|^2_{C([0,T],E)} + \|x_2\|^2_{C([0,T],E)})$$

and hence $S(L^2(0,T;H_Z)) \subset l^2((\mathbb{R}^2)^{\mathbb{N}})$.

In addition, S is closed. Therefore, by the closed graph theorem, S is continuous. Its adjoint $S^* : l^2((\mathbb{R}^2)^{\mathbb{N}}) \to L^2(0,T;H_Z)$ is given by

$$(S^*Y)(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}}(y_n^1 F_n^1(t) - y_n^2 F_n^2(t))\tilde{Z}\varphi_n \tag{68}$$

for almost all $t \in [0,T]$ and $y = (y_n^1, y_n^2)_{n\in\mathbb{N}} \in l^2((\mathbb{R}^2)^{\mathbb{N}})$.

Let us define $\mathcal{G} : L^2(0,T;H_Z) \to L^2(0,T;H_Z)$ by

$$\mathcal{G}(u) = u - \frac{1}{T}\int_0^T u(t)dt, \text{ for } u \in L^2(0,T;H_Z).$$

It is obvious that $\mathcal{G}$ is linear, continuous, selfadjoint and $\int_0^T (\mathcal{G}(u))(t)dt = 0$, for every $u \in L^2(0,T;H_Z)$.

Therefore, the null-controlability problem is equivalent to finding $v \in L^2(0,T;H_Z)$ such that

$$S(\mathcal{G}(v)) = C. \tag{69}$$

If $u \in L^2(0,T;H_Z)$ solves (68) then $\mathcal{G}(u) = u$ solves (69) and, conversely, if $v \in L^2(0,T;H_Z)$ solves (69) then $u = \mathcal{G}(v)$ solves (67).

From now on, the approach is similar to that of Krabs [4].

To be complete, we write down the essential arguments.

Using the dual formula of norm, one can show that (69) has a solution $u \in L^2(0,T;H_Z)$ if and only if there exists a constant $\lambda > 0$ such that $\langle c, y\rangle_{l^2} \leq \lambda \|(S \circ \mathcal{G})^*(y)\|_{L^2(0,T;H_Z)}$ so one has to solve the following

Minimization problem. Minimize $\frac{1}{2}\|S^*(y) - \frac{1}{T}\int_0^T S^*(y)(t)dt\|^2_{L^2(0,T;H_Z)}$ for $y \in l^2((\mathbb{R}^2)^{\mathrm{N}})$ subject to $\langle c, y\rangle_{l^2} = 1$.

Let $\hat{y}$ be a solution of this problem. One can show that if

$$\frac{1}{2}\|S^*(\hat{y}) - \frac{1}{T}\int_0^T (S^*(\hat{y}))(t)dt\|_{L^2(0,T;H_Z)} = 0$$

then (69) has no solution, and if

$$\frac{1}{2}\|S^*(\hat{y}) - \frac{1}{T}\int_0^T (S^*(\hat{y}))(t)dt\|_{L^2(0,T;H_Z)} > 0 \tag{70}$$

then there exists $\lambda > 0$ such that

$$((\mathcal{G} \circ S^*)(\hat{y}),\ (\mathcal{G} \circ S^*)(y))_{L^2(0,T;H_Z)} = \lambda\langle c, y\rangle_{l^2}$$

for all $y \in l^2((\mathbb{R}^2)^{\mathrm{N}})$, hence, if we put $\hat{u} = \frac{1}{\lambda}(\mathcal{G} \circ S^*)(\hat{y})$ then $S(\mathcal{G}(\hat{u})) = c$ and $\mathcal{G}(\hat{u}) \in L^2(0,T;H_Z)$ and solves (67).

Let us suppose that (70) holds. If we define $y^* = \frac{\hat{y}}{\lambda}$ then $y^* \in l^2((\mathbb{R}^2)^{\mathrm{N}})$ and solves

$$(S \circ \mathcal{G} \circ S^*)(y^*) = c \tag{71}$$

that can be rewritten as

$$S(S^*(y^*) - \frac{1}{T}\int_0^T S^*(y^*)(t)dt) = c.$$

One can show that $S \circ \mathcal{G} \circ S^*$ is coercive and so (71) has a unique solution which is given by $y^* = (S \circ \mathcal{G} \circ S^*)^{-1}c$.

So, if (70) holds, the null-controlability problem has a unique solution.

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