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Taming obstinate spreaders: the dynamics of a rumor spreading model incorporating inhibiting mechanisms and attitude adjustment

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Abstract
This paper proposes and investigates a model for rumor spreading and control which accounts for different attitudes and for a distinct type of variable, related to the strength and effectiveness of rumor inhibiting mechanisms. The control mechanisms essentially amount to budgeting and to adjusting the attitude of spreaders. The existence and stability of the trivial (rumor-free) equilibrium and of the semi-trivial equilibrium are characterized in terms of two threshold parameters which quantify the influence of both categories of spreaders. The existence of a positive (rumor-prevailing) equilibrium is also established, its stability being discussed with the help of a bifurcation theorem. A nonstandard finite difference (NSFD) scheme is devised to construct approximate solutions while preserving their positivity, necessary conditions for the existence of the optimal discrete rumor spreading controls being then established.

Keywords Rumor spreading · Stability of equilibria · Conformable spreaders · Obstinate spreaders · Influence numbers · Nonstandard finite difference (NFSD) scheme · Backward bifurcation · Fixed horizon optimal control regime

Mathematics Subject Classification 34C23 · 34D05 · 91D30 · 92D30

1 Introduction

Rumors, unconfirmed and often unconfirmable elaborations upon matters of private or public interest for which truth is sometimes of little-to-no importance, can often distort facts and shape misinformed opinions, irreparably damaging long-standing reputations in the process. Rumors can also cause public unrest with unpredictable consequences, increase the volatility of financial markets, and create panic in times of need such as during natural disasters and disease outbreaks. In comparatively rarer occasions, rumors can also have a positive impact,

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drawing attention to deserving persons and raising public awareness of worthy causes or of specific events. There are also viral marketing campaigns in which marketing information can be spread to and by unsuspecting potential customers from person to person or shared via the ubiquitous social networks.

Rumors often spread due to lack of scientific knowledge and official information. Releasing official information may and often does contribute toward changing the attitude of rumor spreaders, even to the point of stifling the rumor they previously spread with just as much enthusiasm. Better knowledge of the mechanisms of rumor spreading and a realistic approach toward their modeling would be of major importance in containing the spread of misinformation and, on the other hand, in low-key dissemination of information meant to shape public opinions.

In their pioneering papers Daley and Kendall (1964, 1965), Daley and Kendall proposed a model of rumor spreading, now known as the Daley–Kendall (DK) model, which is functionally similar to a disease propagation model and makes use of three compartments: ignorants, spreaders, and stiflers. However, they assumed that, upon hearing a rumor, an ignorant will necessarily become a spreader. This shortcoming has been mitigated by Maki and Thompson in Maki and Thompson (1973).

With the advent of complex network theory, further enhancements of the DK model have been devised to account for network topology. Zanette (2001, 2002) proposed a rumor propagation model on a small-world network and determined a critical threshold for rumor spreading. The stochastic properties of the MT model on scale-free networks have been investigated in Moreno et al. (2004) by means of Monte Carlo simulations. Kawachi (2008) discussed an age-structured transmission model, accounting also for the variability of the rumor. Dodds and Watts (2004) explicitly incorporated the memory of past exposures into a susceptible-infected-removed (SIR) model. A comprehensive model which uses the concept of information entropy to account for the role of memory, conformity effects, differences in the subjective propensity to produce distortions, and variations in the degree of trust that people place in each other has been proposed by Wang et al. (2017). Musa and Fori (2019) distinguished between spreading by means of personal communication and spreading by means of mass-media. Piqueira et al. (2020) introduced an additional class of fact-checkers to discuss the propagation of fake news, building on an earlier work of Piqueira (2020).

Other recent developments include considering the effects of other mechanisms such as forgetting mechanisms (Zhao et al. (2003)), incubation mechanisms (Al-Tuwairqi et al. (2015)), hesitation mechanisms (Xia et al. (2015)), different probabilities for spreaders to become stiflers (Zhao et al. (2013)), different spread inhibiting and attitude adjusting mechanisms (Li et al. (2021)), influence of dissenting opinions (Bodaghi and Goliaei (2018)), use of evolutionary game theory to explore the causes of user behavior (Xiao et al. (2019)), use of fractional calculus to better capture the past history of variables (Singh (2019)), crowd classification based on personality (Chen and Wang (2020)), stochastic perturbations of white noise type (Jia and Lv (2018)), and fuzzy environments (Kumar et al. (2020)).

A model which keeps track of ignorant, spreader, and stifler populations, respectively, along with a fourth compartment quantifying the cumulative rate of awareness programs, has been employed by Huo et al. (2019) to describe the effects of behavioral changes induced by awareness campaigns via media reporting on rumor spreading. The model of concern in Huo et al. (2019) also accounts for loss of awareness that happens as a result of forgetting mechanisms or of other social reasons, allowing aware individuals to become ignorants again. Apart from a stability analysis, a control problem with media reporting as a control variable is also investigated in Huo et al. (2019), the objective being the minimization of negative effects caused by rumor dissemination.

A model for rumor spreading on social networks has been introduced in Zhu and Wang (2020), its distinguishing features being the use, in addition to the population compartments mentioned above, of a fourth one, the rumor-indifferent population, which believes the rumor but does not transmit it, and of a silence-forcing function acting as a limiting factor for rumor spreading. Also, the model allows for a rumor-indifferent or rumor-propagating individual to self-recover (i.e., to become a stifler) as a result of own judgment. Apart from a stability and bifurcation analysis, an optimal control problem for which the control variable is a certain component of the silence-forcing function is investigated, the objective being to reduce the number of rumor-propagating individuals with minimal investment.

Another model for rumor spreading on social networks allowing for both instant and delayed conversion of ignorants into spreaders has been investigated in Zhu et al. (2020), sufficient conditions for the Hopf bifurcation of the rumor-elimination equilibrium being presented alongside extensive numerical simulations. A control problem for an associated nondelayed model is also investigated, the objective being increasing the number of stiflers and decreasing the number of spreaders with minimal costs.

It is then of great importance for the concerned governmental bodies to establish rumor control mechanisms to identify and mitigate potentially disruptive rumors. However, dispelling and rebutting rumors at a societal level requires coordinate efforts and a dedicated personnel, necessitating adequate budgeting. In our model of rumor spreading, we attempt to introduce the inhibiting mechanism, which depends only upon governmental input, acting toward changing the attitude of spreaders. We then discuss the qualitative properties and optimal control regime of the inhibiting mechanism upon rumor spreading together with the influences of the design parameters on budget input and, respectively, the natural decay rate of the inhibiting mechanisms, our aim being to provide a more accurate perspective to rumor control, along with actionable insights.

Most previous studies on rumor spreading relied, as mentioned above, on models which were originally introduced as disease progression models. This is not the case with our model. Apart from using a state variable which is not related to the attitude toward rumor spreading (that is, not related to disease status, in an epidemiological setting) but to the strength and efficiency of rumor control mechanisms, there is no natural progression between the spreader compartments in our model, which consequently cannot and should not be thought as an augmented $SEIR$ model.

The remaining part of this paper is organized as follows. In Sect. 2, after detailing our modeling considerations, we introduce our rumor spreading model, which accounts for different attitudes toward rumor spreading and attempts to quantify the effects of rumor control measures taken by the concerned bodies. In Sect. 3, stability properties for the trivial (rumor-free) and semi-trivial equilibrium, respectively, are established in terms of threshold parameters which bear a certain resemblance to the basic reproduction number of mathematical epidemiology and quantify the influence of spreaders. The existence of a positive (rumor-prevailing) equilibrium is also established, its stability being discussed with the help of a bifurcation theorem. In Sect. 4, we devise an NSFD scheme for the construction of the approximate solutions, discuss its consistency, and then establish necessary conditions for the existence of the optimal discrete rumor spreading controls. In Sect. 5, we illustrate and enhance our theoretical findings via numerical simulations. Finally, several concluding remarks are given in Sect. 6.

2 Modeling considerations

2.1 Model parameters and control variables

The population which is subject to rumor control mechanisms is assumed to be of variable size (N), being subdivided into the following compartments: ignorants (I), conformable spreaders (latents) (L), obstinate spreaders (S), and stiflers (R). Also, P is a rumor control variable which quantifies the strength and effectiveness of the rumor control mechanisms. We make several assumptions upon the ways in which the rumor spreads and is controlled, as shown in Fig. 1 and then summarized below.

(i) The movement of individuals between compartments is irreversible.

(ii) The rumor spreads in a population which is subject to both immigration and emigration. All recruitment is into the ignorant class, at a constant rate Λ. The emigration rate is μ, independent of the compartment.

(iii) Upon successful contact with a spreader, an ignorant moves to the conformable spreader compartment. That is, it is assumed that not even the most obstinate spreaders are persuasive enough to turn ignorants into obstinate spreaders immediately (or perhaps not even the most gullible ignorants can be instantly persuaded to become rumor spreading zealots without further evidence). Also, it is assumed that only an obstinate spreader can turn a conformable spreader into an obstinate spreader, while another conformable spreader cannot. Specifically, when an ignorant contacts a conformable (or obstinate) spreader, the former turns into a new conformable spreader with probability β_1 (respectively, β_2). When a conformable spreader contacts an obstinate spreader, the former turns into an obstinate spreader with probability β_3.

(iv) The obstinate spreaders may cease to spread rumors only when they are subjected to rumor control, in which case they become stiflers, at a rate $\varepsilon f(P)$.

(v) The spread inhibiting function $f(P)$ depends on the strength and effectiveness of the rumor control mechanisms, satisfying:

(a) $f(P) \geq 0,\ P \in (0, +\infty)$; (b) $f'(P) \geq 0,\ P \in (0, +\infty)$.

(vi) The effects of ongoing government policies upon the strength and effectiveness of the rumor control mechanisms are described by a positive, continuous, and bounded budget input function $\psi(t)$, while a parameter ρ can be used to describing the effectiveness of the implementation of government policies. The parameter e represents the natural decay rate of the rumor control mechanisms. In addition, the impact of spreaders eluding rumor control mechanisms upon the rumor control variable can be quantified as $\theta(t) = (1 - \lambda(t))\bar{\theta}$, in which $\lambda(t) = \exp(\gamma i(t) - \delta)$. Here, γ is the validation parameter of the inhibiting mechanism and $i(t) (> 0)$ is the attitude adjusting behavioral response of the spreader population. Both ψ and i are to be understood as control functions. In addition, δ is the positive rescaling decay parameter for the spreader population and $\bar{\theta}$ is the maximum decay rate. We assume that the degradation of the inhibiting mechanism and their natural clearance in the social environment are not as rapid as to make the decay rate $\theta(t)$ negative.

Note that vertical arrows pointing upward are used in Figure 1 to indicate "mediation"(influence), rather than transition of individuals. For instance, the vertical arrow starting from P which is perpendicular on the arrow going from S to R describes the fact that the transition of $\varepsilon S f(P)$ individuals from the S class to the R class happens as a result of the control mechanisms described through the use of the P variable.

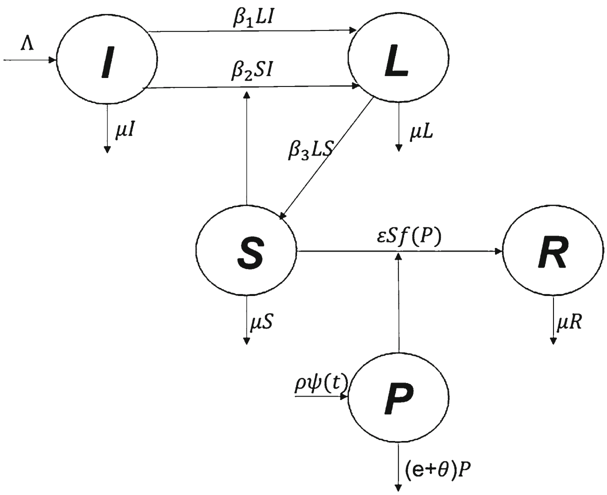

Fig. 1 A flowchart of the model

Remark 1 Examples of functions f satisfying (v.a) and (v.b) are the convex function $f(P) = aP^2$, the linear function $f(P) = aP$, the concave function $f(P) = a\sqrt{P}$, and the S-shaped function $f(P) = \frac{a}{1+be^{-P}}$, $a, b > 0$.

2.2 The model

On the basis of the above considerations, we may now introduce our rumor spreading model as seen below:

$$\begin{aligned} \frac{dI}{dt} &= \Lambda - \beta_1 LI - \beta_2 SI - \mu I, \\ \frac{dL}{dt} &= \beta_1 LI + \beta_2 SI - \beta_3 LS - \mu L, \\ \frac{dS}{dt} &= \beta_3 LS - \varepsilon Sf(P) - \mu S, \\ \frac{dR}{dt} &= \varepsilon Sf(P) - \mu R, \\ \frac{dP}{dt} &= \rho\psi(t) - (e + \theta(t))P, \end{aligned} \tag{1}$$

an overview of state variables and model parameters being given in Table 1.

Let us denote $\psi_{max} = \sup_{t\in[0,+\infty)} \psi(t)$.

By a comparison argument, it can be shown that the set:

$$\Omega = \left\{ (I, L, S, R, P) \in \mathbb{R}^5_+ : 0 \le I + L + S + R \le \frac{\Lambda}{\mu},\ 0 \le P \le \frac{\rho\psi_{max}}{e} \right\}$$

is a positively invariant set of (1).

3 The dynamics of the simplified model

3.1 The simplified model

To fully characterize the dynamical behavior of the solutions, we need detailed knowledge about the profile of the control mechanisms. In what follows, we consider fixed values for the control functions, that is, $\psi(t) \equiv \psi$ and $i(t) \equiv i$, which in turn implies that $\theta(t)$ is also a constant, that is, $\theta(t) \equiv \theta$. Since the stifler compartment does not directly influence the dynamics of the other compartments, the rumor control variable P is directly supervised by the concerned authorities and, on the long term:

$$\lim_{t\to+\infty} P(t) = \frac{\rho\psi}{e+\theta},$$

we are led to considering the simplified autonomous form below:

$$\begin{aligned}\frac{dI}{dt} &= \Lambda - \beta_1 LI - \beta_2 SI - \mu I,\\ \frac{dL}{dt} &= \beta_1 LI + \beta_2 SI - \beta_3 LS - \mu L,\\ \frac{dS}{dt} &= \beta_3 LS - \varepsilon Sf(\frac{\rho\psi}{e+\theta}) - \mu S.\end{aligned} \tag{2}$$

3.2 The conformable spreaders influence number

As previously done for the higher dimensional model (1), it can be shown that the simplified model (2) is well posed and has positivity-preserving solutions. Also, (2) has a rumor-free equilibrium E_{01}, given by:

$$E_{01} = \left(\frac{\Lambda}{\mu}, 0, 0\right).$$

To be able to establish the stability properties of (2), we now introduce a threshold parameter, which we shall call from now on the conformable spreaders influence number $\mathcal{R}_{01}$ of the model, defined *ad hoc* via the next generation method, an approach which has been proven very successful in mathematical epidemiology.

Let $X = (L, S)^T$. The second and third equations of (1) can altogether be written as:

$$\frac{dX}{dt} = \mathcal{F}(X) - \mathcal{V}(X),$$

in which:

$$\mathcal{F}(X) = \begin{pmatrix} \beta_1 LI + \beta_2 SI - \beta_3 LS \\ \beta_3 LS \end{pmatrix}, \quad \mathcal{V}(X) = \begin{pmatrix} \mu L \\ \varepsilon Sf(\frac{\rho\psi}{e+\theta}) + \mu S \end{pmatrix}.$$

We obtain:

$$F = D\mathcal{F}|_{E_{01}} = \begin{pmatrix} \frac{\beta_1\Lambda}{\mu} & \frac{\beta_2\Lambda}{\mu} \\ 0 & 0 \end{pmatrix}, \quad V = D\mathcal{V}|_{E_{01}} = \begin{pmatrix} \mu & 0 \\ 0 & \varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu \end{pmatrix}.$$

The conformable spreaders influence number $\mathcal{R}_{01}$ of (1) is then defined as the spectral radius of the matrix FV^{-1}, which leads to:

$$\mathcal{R}_{01} = \frac{\beta_1 \Lambda}{\mu^2}, \tag{3}$$

and can be interpreted as the average number of ignorants turned into conformable spreaders by a single conformable spreader when introduced into a totally ignorant population. It is to be noted that $\mathcal{R}_{01}$ does not depend upon f, since f characterizes control mechanisms which are applied to the compartment of the obstinate spreaders only.

It is easily seen that $\mathcal{R}_{01} > 1$ implies that the system (1) has yet another semi-trival equilibrium, namely an ignorant and conformable spreader-only equilibrium E_{02}, given by:

$$E_{02} = \left(\frac{\mu}{\beta_1}, \frac{\mu}{\beta_1}(\mathcal{R}_{01} - 1), 0 \right).$$

Note that the very existence of E_{02} shows the fact that our model is not a direct equivalent of a SEI model (or of a related one, for that matter), for which no such semi-trivial equilibria exist, although one could think of conformable spreaders as being "the exposed". This happens, since in our model, there is no natural progression between the conformable spreader compartment and the obstinate spreader compartment.

3.3 The obstinate spreaders influence number and the stability of the equilibria

First, it is seen that $\mathcal{R}_{01}$ is a threshold parameter for the stability of E_{01}.

Theorem 1 *The rumor-free equilibrium* $E_{01} = (\frac{\Lambda}{\mu}, 0, 0)$ *is locally asymptotically stable provided that* $\mathcal{R}_{01} < 1$ *and unstable provided that* $\mathcal{R}_{01} > 1$.

Proof The Jacobian matrix of the system (1) at E_{01} is:

$$J(E_{01}) = \begin{bmatrix} -\mu & -\frac{\beta_1 \Lambda}{\mu} & -\frac{\beta_2 \Lambda}{\mu} \\ 0 & \frac{\beta_1 \Lambda}{\mu} - \mu & \frac{\beta_2 \Lambda}{\mu} \\ 0 & 0 & -(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu) \end{bmatrix}. \tag{4}$$

We then obtain that $J(E_{01})$ has the following eigenvalues:

$$\lambda_1 = -\mu, \quad \lambda_2 = -(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu), \quad \lambda_3 = \mu(\mathcal{R}_{01} - 1).$$

While $\lambda_1, \lambda_2 < 0$, it is seen that $\mathcal{R}_{01} < 1$ if and only if $\lambda_3 < 0$. Also, $\mathcal{R}_{01} > 1$ if and only if $\lambda_3 > 0$. Hence, E_{01} is locally asymptotically stable provided that $\mathcal{R}_{01} < 1$ and unstable provided that $\mathcal{R}_{01} > 1$. This completes the proof. □

We now turn our attention toward investigating the stability of E_{02}. As seen above, this equilibrium exists if and only if $\mathcal{R}_{01} > 1$. We first see that the Jacobian matrix of the system (1) at E_{02} is given by:

$$J(E_{02}) = \begin{bmatrix} -\frac{\Lambda\beta_1}{\mu} & -\mu & -\frac{\beta_2\mu}{\beta_1} \\ \frac{\Lambda\beta_1 - \mu^2}{\mu} & 0 & \frac{\beta_2\mu}{\beta_1} - \frac{\beta_3(\Lambda\beta_1 - \mu^2)}{\mu\beta_1} \\ 0 & 0 & \frac{\beta_3(\Lambda\beta_1 - \mu^2)}{\mu\beta_1} - (\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu) \end{bmatrix}. \tag{5}$$

We then find that $J(E_{02})$ has the following eigenvalues:

$$\lambda_1 = -\mu, \quad \lambda_2 = \mu(1 - \mathcal{R}_{01}), \quad \lambda_3 = \left(\varepsilon f\left(\frac{\rho\psi}{e+\theta}\right) + \mu\right)\left(\frac{\beta_3\mu(\mathcal{R}_{01} - 1)}{\beta_1(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)} - 1\right).$$

While λ_1 is always negative, λ_2 is also negative whenever E_{02} exists, and consequently, the stability of E_{02} is determined by the sign of λ_3. Let us define:

$$\mathcal{R}_{02} = \frac{\beta_3\mu(\mathcal{R}_{01} - 1)}{\beta_1(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)}, \tag{6}$$

which is to be interpreted as being the average number of conformable spreaders turned into obstinate spreaders by a single obstinate spreader when introduced into a population which is at the semi-trivial equilibrium. From now on, $\mathcal{R}_{02}$ will be called the obstinate spreaders influence number. Using the above considerations, it is then seen that the following result holds true.

Theorem 2 *The ignorant and conformable spreader-only equilibrium* $E_{02} = (\frac{\mu}{\beta_1}, \frac{\mu}{\beta_1}(\mathcal{R}_{01} - 1), 0)$ *is locally asymptotically stable provided that* $0 < \mathcal{R}_{02} < 1$ *and unstable provided that* $\mathcal{R}_{02} > 1$.

Note again that in our model, neither the conformable spreaders naturally progress to the obstinate spreader compartment, nor the obstinate spreaders have the capability to turn ignorants directly into obstinate spreaders. Consequently, it is natural to have the stability (or lack thereof) of E_{02} linked to the influence of obstinate spreaders upon the conformable spreaders only. Also, while $\mathcal{R}_{01}$ is akin to a basic reproduction number, $\mathcal{R}_{02}$ is more like an invasion reproduction number, being related to the capability of a compartment to disrupt an equilibria involving the other compartment only.

To find the coordinates of the rumor-prevailing equilibrium $E^* = (I^*, L^*, S^*)$, one needs to solve the equilibrium system:

$$\begin{aligned} \Lambda - \beta_1 L^* I^* - \beta_2 S^* I^* - \mu I^* &= 0, \\ \beta_1 L^* I^* + \beta_2 S^* I^* - \beta_3 L^* S^* - \mu L^* &= 0, \\ \beta_3 L^* S^* - \varepsilon S^* f(\frac{\rho\psi}{e+\theta}) - \mu S^* &= 0. \end{aligned} \tag{7}$$

One easily sees that:

$$L^* = \frac{\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu}{\beta_3}, \quad I^* = \frac{\Lambda}{\beta_1 L^* + \beta_2 S^* + \mu},$$

while S^* is a root of the equation $AS^{*^2} - BS^* + C = 0$, with:

$$\begin{aligned} &A = \beta_2\beta_3 L^*, \quad B = \beta_1\beta_3 L^{*^2} + \mu(\beta_2 + \beta_3)L^* - \Lambda\beta_2, \\ &C = -\mu\beta_1(L^*)^2(\mathcal{R}_{02} - 1). \end{aligned} \tag{8}$$

It is obvious that $\mathcal{R}_{02} > 1$ implies that $C < 0$. Also:

$$L^* > \frac{-\mu(\beta_2 + \beta_3) + \sqrt{\left[\mu(\beta_2 + \beta_3)\right]^2 + 4\beta_1\beta_2\beta_3\Lambda}}{2\beta_1\beta_3},$$

that is:

$$\mathcal{R}_{02} < \mathcal{R}_{02}^*,$$

where:

$$\mathcal{R}_{02}^{*} = \frac{2\beta_3(\mathcal{R}_{01}-1)}{-(\beta_2+\beta_3)+\sqrt{(\beta_2+\beta_3)^2+\frac{4}{\mu^2}\beta_1\beta_2\beta_3\Lambda}},$$

implies that $B > 0$. On the basis of the above considerations, we then obtain the following existence result.

Theorem 3 *The following statements hold.*

(i) *If* $\mathcal{R}_{02} > 1$, *then* (2) *has a unique rumor-prevailing equilibrium* $E^* = (I^*, L^*, S^*)$, *in which* $S^* = \frac{-B+\sqrt{B^2-4AC}}{2A}$.

(ii) *If* $\mathcal{R}_{02} = 1 > \mathcal{R}_{02}^*$, *then* (2) *has a unique rumor-prevailing equilibrium* $E^* = (I^*, L^*, S^*)$, *in which* $S^* = \frac{-B}{A}$.

(iii) *If* $\mathcal{R}_{02}^* < \mathcal{R}_{02} < 1$ *and* $B^2 - 4AC = 0$, *then* (2) *has a unique rumor-prevailing equilibrium* $E^* = (I^*, L^*, S^*)$, *in which* $S^* = \frac{-B}{2A}$.

(iv) *If* $\mathcal{R}_{02}^* < \mathcal{R}_{02} < 1$ *and* $B^2 - 4AC > 0$, *then* (2) *has two rumor-prevailing equilibria* $E_1^* = (I_1^*, L_1^*, S_1^*)$ *and* $E_2^* = (I_2^*, L_2^*, S_2^*)$, *in which* $S_{1,2}^* = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$.

(v) *If* $\mathcal{R}_{02}^* < \mathcal{R}_{02} < 1$ *and* $B^2 - 4AC < 0$, *then* (2) *has no rumor-prevailing equilibrium.*

(vi) *If* $\mathcal{R}_{02} < \min\{\mathcal{R}_{02}^*, 1\}$, *then* (2) *has no rumor-prevailing equilibrium.*

3.4 Backward bifurcation

While the stability analysis for E_{01} and E_{02} is rather straightforward, as it relies on a sign analysis, E^* is not really amenable to this kind of approach. Instead, we shall now perform a bifurcation analysis using Theorem 4.1 of Castillo-Chavez and Song (2004), a theoretical result based on the Center Manifold Theory, which will yield consequences regarding the stability of E^* as a byproduct. To conform to the framework of Castillo-Chavez and Song (2004), we use the following notations:

$$I = x_1, \quad L = x_2, \quad S = x_3.$$

The system (2) can then be written in vector form as:

$$\frac{dX}{dt} = H(X),$$

where $X = (x_1, x_2, x_3)^T$ and $H = (h_1, h_2, h_3)^T$, with:

$$h_1 \doteq \Lambda - \beta_1 x_1 x_2 - \beta_2 x_1 x_3 - \mu x_1, \quad h_2 \doteq \beta_1 x_1 x_2 + \beta_2 x_1 x_3 - \beta_3 x_2 x_3 - \mu x_2,$$
$$h_3 \doteq \beta_3 x_2 x_3 - (\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)x_3.$$

Let us choose β_3 as being the bifurcation parameter. While it is certainly possible to choose other bifurcation parameters, we intended to emphasize the influence of obstinate spreaders via both this choice of a bifurcation parameter and stating Theorem 4 chiefly in terms of $\mathcal{R}_{02}$, the obstinate spreaders influence number. When $\mathcal{R}_{02} = 1$, β_3 takes the critical value:

$$\beta_3^* = \frac{\beta_1(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)}{\mu(\mathcal{R}_{01}-1)}.$$

The Jacobian matrix $J(E_{02})$ evaluated for $\beta_3 = \beta_3^*$ is given by:

$$J(E_{02})|_{\beta_3=\beta_3^*} = \begin{bmatrix} -\frac{\Lambda\beta_1}{\mu} & -\mu & -\frac{\beta_2\mu}{\beta_1} \\ \mu(\mathcal{R}_{01}-1) & 0 & \frac{\mu}{\beta_1}\big(\beta_2 - \beta_3^*(\mathcal{R}_{01}-1)\big) \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has a right eigenvector $w = (w_1, w_2, w_3)^T$ associated with the eigenvalue 0 at $\beta_3 = \beta_3^*$, with:

$$w_1 = \frac{\beta_3^*(\mathcal{R}_{01}-1) - \beta_2}{\beta_1(\mathcal{R}_{01}-1)}, \quad w_2 = \frac{\beta_1\Lambda\big(\beta_2 - \beta_3^*(\mathcal{R}_{01}-1)\big) - \beta_2\mu^2(\mathcal{R}_{01}-1)}{\beta_1\mu^2(\mathcal{R}_{01}-1)},$$
$$w_3 = 1$$

and a left eigenvector $v = (v_1, v_2, v_3)$ associated with the eigenvalue 0, with:

$$v_1 = v_2 = 0, \quad v_3 = 1.$$

To apply the bifurcation result presented in Castillo-Chavez and Song (2004), let us compute the quantities involving the second-order partial derivatives indicated therein, given by:

$$a = \Sigma_{k,i,j=1}^3 v_k w_i w_j \frac{\partial^2 h_k}{\partial x_i \partial x_j}(E_{02}, \beta_3^*), \qquad b = \Sigma_{k,i=1}^3 v_k w_i \frac{\partial^2 h_k}{\partial x_i \partial \beta_3^*}(E_{02}, \beta_3^*)$$

Consequently:

$$a = v_3 w_2 w_3 \frac{\partial^2 h_3}{\partial x_2 \partial x_3} + v_3 w_3 w_2 \frac{\partial^2 h_3}{\partial x_3 \partial x_2} = 2\beta_3^* w_2 w_3.$$

Also:

$$b = \Sigma_{i=1}^3 v_3 w_i \frac{\partial^2 h_3}{\partial x_i \partial \beta_3^*} = \frac{\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu}{\beta_3^*} > 0.$$

By applying Theorem 4.1 of Castillo-Chavez and Song (2004), we obtain the following result.

Theorem 4 *The rumor-prevailing equilibrium E^* is locally asymptotically stable for $\mathcal{R}_{02} > 1$ but close to 1 and the system (2) has a backward bifurcation at $\mathcal{R}_{02} = 1$ if one of the following equivalent conditions is satisfied:*

(i) $\beta_2 > \frac{\beta_1(\varepsilon f(\frac{\rho\psi}{e+\theta})+\mu)}{\mu}$ *and* $\Lambda > \frac{\beta_2\mu^2(\mathcal{R}_{01}-1)}{\beta_1(\beta_2 - \frac{\beta_1(\varepsilon f(\frac{\rho\psi}{e+\theta})+\mu)}{\mu})}$*;*

(ii) $\beta_2 > \frac{\beta_1\Lambda(\varepsilon f(\frac{\rho\psi}{e+\theta})+\mu)}{\mu\big(\beta_1\Lambda - \mu^2(\mathcal{R}_{01}-1)\big)}$ *and* $\Lambda > \frac{\mu^2(\mathcal{R}_{01}-1)}{\beta_1}$*;*

(iii) $\beta_2 > \frac{4\mu(\mathcal{R}_{01}-1)(\varepsilon f(\frac{\rho\psi}{e+\theta})+\mu)}{\Lambda}$ *and* $\beta_1 \in (\beta_1^*, \beta_2^*)$*, in which:*

$$\beta_1^* = \frac{\Lambda\beta_2 - \sqrt{(\Lambda\beta_2)^2 - 4\beta_2\Lambda\mu(\mathcal{R}_{01}-1)(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)}}{2\frac{\Lambda}{\mu}(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)},$$

$$\beta_2^* = \frac{\Lambda\beta_2 + \sqrt{(\Lambda\beta_2)^2 - 4\beta_2\Lambda\mu(\mathcal{R}_{01}-1)(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)}}{2\frac{\Lambda}{\mu}(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu)}.$$

4 The nonstandard finite difference (NSFD) scheme

In this section, we make use of the nonstandard finite difference (NSFD) scheme approach devised by Mickens (1994), and further elaborated by Anguelov and Lubuma (2001). This approach is proved to be more appropriate than the classical Euler and Runge–Kutta approximation schemes, because, as far as the existence and positivity of the equilibria are concerned, it is dynamically consistent with the original continuous-time model (2), regardless of the step size taken in the numerical simulations. Most importantly, the dynamical consistency guarantees the positivity of solutions, contrary to other numerical schemes that may produce negative or spurious approximate solutions. The main features of a NSFD scheme are as follows (see also Anguelov and Lubuma (2003), Mickens (2007), Garba et al. (2001) for further details).

- The standard denominator value $\Delta t = h$ of the discrete derivatives is replaced by a more general function $\phi(h)$, which satisfies the condition $\phi(h) = h + o(h^2)$.
- Nonlinear as well as linear terms are approximated in a nonlocal way using more than one mesh point.

4.1 The discretized model

By employing a NSFD scheme, we hereby construct the following discretized version of the model (2):

$$\begin{aligned}
\frac{I_{n+1}-I_n}{\phi_1(h,\mu)} &= \Lambda - \beta_1 I_{n+1}L_n - \beta_2 I_{n+1}S_n - \mu I_{n+1},\\
\frac{L_{n+1}-L_n}{\phi_2(h,\mu)} &= \beta_1 L_n I_{n+1} + \beta_2 I_{n+1}S_n - \beta_3 L_{n+1}S_n - \mu L_{n+1},\\
\frac{S_{n+1}-S_n}{\phi_3\big(h,\mu,\varepsilon f(\frac{\rho\psi}{e+\theta})\big)} &= \beta_3 S_n L_{n+1} - \varepsilon f(\frac{\rho\psi}{e+\theta})S_{n+1} - \mu S_{n+1},
\end{aligned} \tag{9}$$

in which:

$$\phi_1(h,\mu) = \phi_2(h,\mu) = \frac{e^{\mu h}-1}{\mu}, \quad \phi_3\big(h,\mu,\varepsilon f(\frac{\rho\psi}{e+\theta})\big) = \frac{e^{\mu+\varepsilon f(\frac{\rho\psi}{e+\theta})}-1}{\mu+\varepsilon f(\frac{\rho\psi}{e+\theta})}.$$

Hereinafter, $I_n = I(t_n)$, $L_n = L(t_n)$, and $S_n = S(t_n)$ are the sizes of the of ignorant, conformable spreader, and obstinate spreader populations, respectively, at time $t_n = n\Delta t$. All other parameters have the same meaning and values given in Table 1.

4.2 Positivity of solutions

In this section, we establish the positivity of the approximate solutions constructed via the NSFD scheme, as seen in the following result.

Theorem 5 *The solutions I_n, L_n, S_n of the discretized model (9) are positive at all times t_n, for $n = 1, 2, \cdots$, provided that they start with positive initial conditions I_0, L_0, and S_0.*

Table 1 State variables and parameters of the model

Symbol	Quantity	Default value	Source
I	Ignorant population	Variable	
L	Conformable spreader (latent) population	Variable	
S	Obstinate spreader population	Variable	
R	Stifler population	Variable	
P	Inhibiting mechanism	Variable	
Λ	Recruitment rate of ignorants	$4 - 400$	Chen (2019)
β_1	Probability that an ignorant contacts a latent and then turns into a new conformable spreader	$10^{-4} - 10^{-3}$	Chen (2019)
β_2	Probability that an ignorant contacts an obstinate spreader and then turns into a conformable spreader	$10^{-4} - 5 \cdot 10^{-1}$	Chen (2019)
β_3	Probability that a conformable spreader contacts an obstinate spreader and then turns into an obstinate spreader	$5 \cdot 10^{-4} - 5 \cdot 10^{-1}$	estimated
μ	Emigration rate	$4 \cdot 10^{-3} - 4 \cdot 10^{-1}$	Chen (2019)
ε	Effective obstinate spreader-to-stifler coefficient	$4 \cdot 10^{-4} - 4 \cdot 10^{-2}$	Chen (2019)
ψ	Release rate of inhibiting mechanisms	$1 - 20$	Estimated
e	Natural clearance rate of inhibiting mechanisms	0.5	Estimated
θ	Decay rate of inhibiting mechanisms	$0 - 1$	Estimated
ρ	Effectiveness of implementation	0.5	Estimated
δ	Rescaling constant	$0.01 - 0.99$	Estimated

Proof By solving (9) at step $n+1$, we obtain that:

$$\begin{aligned} I_{n+1} &= \frac{\phi_1 \Lambda + I_n}{1+\phi_1(\beta_1 L_n + \beta_2 S_n + \mu)} \doteq j_1(I_n, L_n, S_n), \\ L_{n+1} &= \frac{L_n(1+\phi_2\beta_1 I_{n+1}) + \phi_2\beta_2 I_{n+1} S_n}{1+\phi_2(\beta_3 S_n + \mu)} \doteq j_2(L_n, S_n, I_{n+1}), \\ S_{n+1} &= \frac{S_n(1+\phi_3\beta_3 L_{n+1})}{1+\phi_3\big(\varepsilon f(\frac{\rho\psi}{e+\theta}) + \mu\big)} \doteq j_3(S_n, L_{n+1}). \end{aligned} \tag{10}$$

The desired positivity property follows now from an easy induction argument. Consequently, the solutions of (9) are always positive whenever they start with positive initial data. ◻

4.3 The fixed points of the discretized model

The fixed points of the discretized model, corresponding to the equilibria of the continuous model, can be obtained by solving the following algebraic equations:

$$I^* = j_1(I^*, L^*, S^*), \quad L^* = j_2(L^*, S^*, I^*), \quad S^* = j_3(S^*, L^*). \tag{11}$$

It is easy to see that the system (11) has the semi-trivial solutions $(\frac{\Lambda}{\mu}, 0, 0)$ and $(\frac{\mu}{\beta_1}, \frac{\Lambda\beta_1-\mu^2}{\mu\beta_1}, 0)$, which are also the semi-trivial equilibria of (2). As for the positive solution, its components are determined as shown below.

From the third equation in (11), we see that $L^* = \frac{\varepsilon f(\frac{\rho\psi}{e+\theta})+\mu}{\beta_3}$. Also, solving the first equation of (11) for I^*, we obtain $I^* = \frac{\Lambda}{\beta_1 L^*+\beta_2 S^*+\mu}$. Substituting the values of L^* and I^* into the second equation of (11), one notes that S^* is the positive solution of the following quadratic equation:

$$A{S^*}^2 + BS^* + C = 0,$$

in which the coefficients are same as those given in (8), which implies that the fixed points of the NSFD scheme coincide with the equilibria of (2).

4.4 Necessary conditions for the fixed horizon optimal control regime

So far, we have investigated the dynamics of continuous and discrete models of rumor spreading which are subject to two control mechanisms: attitude adjustment via educational or punitive measures and enhancement of strength and effectiveness of the rumor control mechanisms via more generous budgeting.

Our optimal control problem can be posed as the following question: What is the best-combined control regime of attitude adjustment and budgeting for the discrete model described above, so that the population size of the spreader population is reduced to a minimal level?

In what follows, we focus on the optimal control problem with augmented states, which can be solved by means of the discrete minimum principle introduced by Goodwin *et al.* in Goodwin et al. (2005). Even though optimal control theory for systems of difference equations seems to be an adequate tool for the social sciences, it is perhaps underused in population dynamics. In our case, the control variables are ψ, the budgeting function, and i, the attitude adjustment function. In what follows, we shall derive the necessary optimality conditions for the control mechanisms that minimize the size of the obstinate spreader population. To this end, let us define:

$$\{X_n\} \doteq \left\{\left[X_n^1, X_n^2, X_n^3\right]^T\right\} \doteq \left\{\left[I_n, L_n, S_n\right]^T\right\},$$
$$\{V_n\} \doteq \left\{\left[V_n^1, V_n^2\right]^T\right\} \doteq \left\{\left[\psi_n, i_n\right]^T\right\},$$

in which $n = 0, 1, \cdots, N-1$. Here, N is the (discrete) optimization horizon. Our control problem can then be stated in the following form:

$$\begin{aligned} \text{Minimize } \nu_N(X_n, V_n) &= X_N^3 + \Sigma_{n=0}^{N-1}(X_n^3 + c_\psi (V_n^1)^2 + c_i (V_n^2)^2) \\ &= S_N + \Sigma_{n=0}^{N-1}(S_n + c_\psi \psi_n^2 + c_i i_n^2) \end{aligned} \tag{12}$$

subject to:

$$\begin{aligned} X_{n+1}^1 = I_{n+1} &= \frac{\phi_1 \Lambda + I_n}{1 + \phi_1(\beta_1 L_n + \beta_2 S_n + \mu)} \doteq j_1(X_n, V_n), \\ X_{n+1}^2 = L_{n+1} &= \frac{L_n\left(1 + \frac{(\phi_1 \Lambda + I_n)\phi_2 \beta_1}{1+\phi_1(\beta_1 L_n + \beta_2 S_n + \mu)}\right) + \frac{(\phi_1 \Lambda + I_n)\phi_2 \beta_2 S_n}{1+\phi_1(\beta_1 L_n + \beta_2 S_n + \mu)}}{1 + \phi_2(\beta_3 S_n + \mu)} \\ &\doteq j_2(X_n, V_n), \\ X_{n+1}^3 = S_{n+1} &= \frac{S_n\left(1 + \phi_3^n \beta_3 \frac{L_n\left(1+\frac{(\phi_1 \Lambda + I_n)\phi_2 \beta_1}{1+\phi_1(\beta_1 L_n + \beta_2 S_n + \mu)}\right) + \frac{(\phi_1 \Lambda + I_n)\phi_2 \beta_2 S_n}{1+\phi_1(\beta_1 L_n + \beta_2 S_n + \mu)}}{1+\phi_2(\beta_3 S_n + \mu)}\right)}{1 + \phi_3^n\left(\varepsilon f\left(\frac{\rho \psi_n}{e+\theta_n}\right) + \mu\right)} \\ &\doteq j_3(X_n, V_n), \\ X_0 &= \bar{X}, \end{aligned} \tag{13}$$

in which c_ψ and c_i are suitable weights. Also, $\bar{X}$ is the initial state and:

$$\phi_3^n = \frac{e^{\mu + \varepsilon f\left(\frac{\rho \psi_n}{e+\theta_n}\right)} - 1}{\mu + \varepsilon f\left(\frac{\rho \psi_n}{e+\theta_n}\right)}, \quad \theta_n = (1 - \lambda_n)\bar{\theta}, \quad \lambda_n = \exp(-\gamma i_n + \delta).$$

To apply the Karush–Kuhn–Tucker (KKT) optimality conditions, we need to verify the constraint qualification that the gradients of the equality constraints be linearly independent when evaluated at the minimizers. Subsequently, we define a new variable:

$$X \doteq \left[X_0^T, X_1^T, \cdots, X_N^T, V_0^T, V_1^T, \cdots, V_{N-1}^T\right]^T \in \mathbb{R}^{5N+3},$$

which comprises all the variables with respect to which the optimization is performed. Now, let:

$$j(X_n, V_n) = [j_1(X_n, V_n), j_2(X_n, V_n), j_3(X_n, V_n)]^T.$$

We can then write the state equations (13) as $3(N+1)$ equality constraints on X as follows:

$$h(X) \doteq \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_N \end{bmatrix} = \begin{bmatrix} \bar{X} - X_0 \\ j(X_0, V_0) - X_1 \\ \vdots \\ j(X_{N-1}, V_{N-1}) - X_N \end{bmatrix} = 0. \tag{14}$$

Let us also define:

$$\frac{\partial j}{\partial X_k} \doteq \begin{bmatrix} \frac{\partial j_1}{\partial X_k^1} & \frac{\partial j_1}{\partial X_k^2} & \frac{\partial j_1}{\partial X_k^3} \\ \frac{\partial j_2}{\partial X_k^1} & \frac{\partial j_2}{\partial X_k^2} & \frac{\partial j_2}{\partial X_k^3} \\ \frac{\partial j_3}{\partial X_k^1} & \frac{\partial j_3}{\partial X_k^2} & \frac{\partial j_3}{\partial X_k^3} \end{bmatrix}, \quad \frac{\partial j}{\partial V_k} \doteq \begin{bmatrix} \frac{\partial j_1}{\partial V_k^1} & \frac{\partial j_1}{\partial V_k^2} \\ \frac{\partial j_2}{\partial V_k^1} & \frac{\partial j_2}{\partial V_k^2} \\ \frac{\partial j_3}{\partial V_k^1} & \frac{\partial j_3}{\partial V_k^2} \end{bmatrix}, \tag{15}$$

for $k = 0, 1, \cdots, N-1$. We then compute the $(3N+3) \times (5N+3)$ Jacobian matrix of the vector-valued function $h(X)$ in (14) as:

$$\frac{\partial h}{\partial X} = \begin{bmatrix} -I_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{\partial j}{\partial X_0} & -I_3 & 0 & \cdots & 0 & 0 & \frac{\partial j}{\partial V_0} & 0 & \cdots & 0 \\ 0 & \frac{\partial j}{\partial X_1} & -I_3 & \cdots & 0 & 0 & 0 & \frac{\partial j}{\partial V_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\partial j}{\partial X_{N-1}} & -I_3 & 0 & 0 & \cdots & \frac{\partial j}{\partial V_{N-1}} \end{bmatrix}, \tag{16}$$

where 0 denotes zero matrices of appropriate dimensions, and I_3 denotes the 3×3 identity matrix. Since the matrix $\frac{\partial h}{\partial X}$ in (16) has full row rank for all $X \in \mathbb{R}^{5N+3}$, the gradients of the equality constraints (14) are linearly independent in $\mathbb{R}^{5N+3}$, which implies that the constraint qualification required by the optimality conditions of Theorem 2.5.5 (Karush–Kuhn–Tucker Necessary Conditions) in Goodwin et al. (2005) is satisfied for all $X \in \mathbb{R}^{5N+3}$.

Next, we introduce the Lagrange multipliers $\lambda_{-1} \in \mathbb{R}^3$ for the initial state equation in (13), and the other Lagrange multipliers:

$$\{\lambda_0, \lambda_1, \cdots, \lambda_{N-1}\}, \ \lambda_k \in \mathbb{R}^3, \ k = 0, \ \cdots, N-1,$$

for the other state equations in (13), and denote:

$$\lambda = [\lambda_{-1}^T, \lambda_0^T, \cdots, \lambda_{N-1}^T]^T.$$

We then construct the following Hamiltonian function $\mathcal{L} : \mathbb{R}^{3N+3} \times \mathbb{R}^{2N} \times \mathbb{R}^{3N+3} \to \mathbb{R}$:

$$\mathcal{H}(X, \lambda) = X_N^3 + \Sigma_{n=0}^{N-1} (X_n^3 + c_\psi (V_n^1)^2 + c_i (V_n^2)^2) + \Sigma_{k=-1}^{N-1} \lambda_k^T h_{k+1}. \tag{17}$$

According to the KKT conditions, we obtain the following necessary condition for the existence of the optimal rumor spreading control.

Theorem 6 *A necessary condition for the sequences* $\{X_0^*, \cdots, X_N^*\}$, $\{V_0^*, \cdots, V_{N-1}^*\}$ *to be the minimisers of the optimal control problem* (12) *and* (13) *is that there exists a sequence of Lagrange multipliers* $\lambda^* \doteq \{\lambda_{-1}^*, \lambda_0^*, \cdots, \lambda_{N-1}^*\}$, *such that the following conditions hold:*

(i) $X^* \doteq \left[(X_0^*)^T, \cdots, (X_N^*)^T, (V_0^*)^T, \cdots, (V_{N-1}^*)^T\right]^T$ *satisfies the state equations:* (13).
(ii) $(\lambda_{k-1}^*)^T = \frac{\partial \mathcal{H}(X^*, \lambda^*)}{\partial X_k}$ *for* $k = 0, 1, \cdots, N-1$, *and* $(\lambda_{N-1}^*)^T = (0, 0, 1)^T$.
(iii) $\frac{\partial \mathcal{H}(X^*, \lambda^*)}{\partial V_k} = 0$ *for* $k = 0, 1, \cdots, N-1$.

5 Numerical simulations

In this section, we perform several numerical simulations to illustrate the modeling considerations and theoretical findings presented in the previous sections. We first choose $\Lambda = 40$,

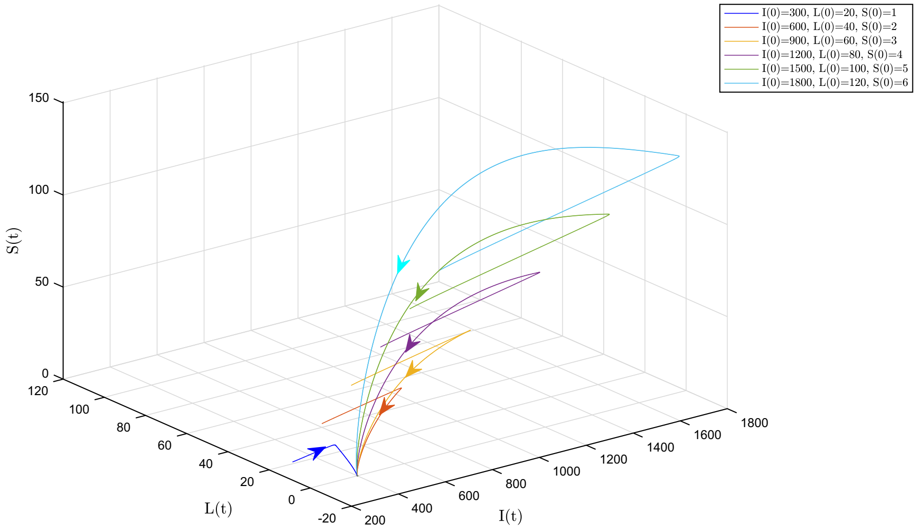

Fig. 2 The dynamical behaviors of the ignorant, conformable spreader and obstinate spreader populations, respectively, for $\Lambda = 40$, $\beta_1 = 0.0002$, $\beta_2 = 0.0001$, $\beta_3 = 0.5$, $\mu = 0.1$, $\varepsilon = 0.0022$, $\psi = 10$, $e = 0.5$, $\theta = 0.2$, $\rho = 0.5$ and for several distinct initial conditions

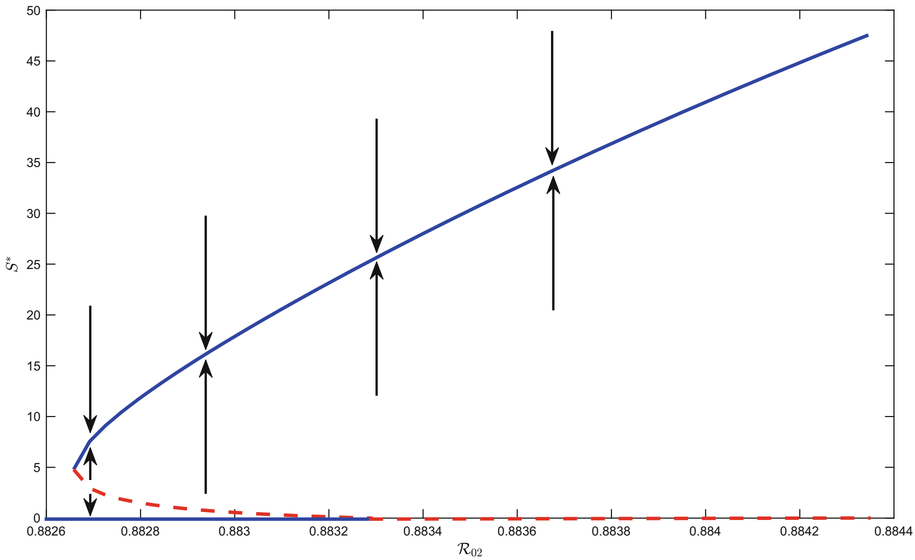

Fig. 3 A backward bifurcation. The solid blue curve and solid blue line segment depict the stable equilibria, while the dotted red curve and the dotted line segment depict the unstable equilibria. Here, $\Lambda = 400$, $\beta_1 = 0.0002$, $\beta_2 = 0.5$, $\beta_3 \in (0.00049, 0.00053)$, $\mu = 0.1$, $\varepsilon = 0.0022$, the other parameters having the values given in the caption of Fig. 2

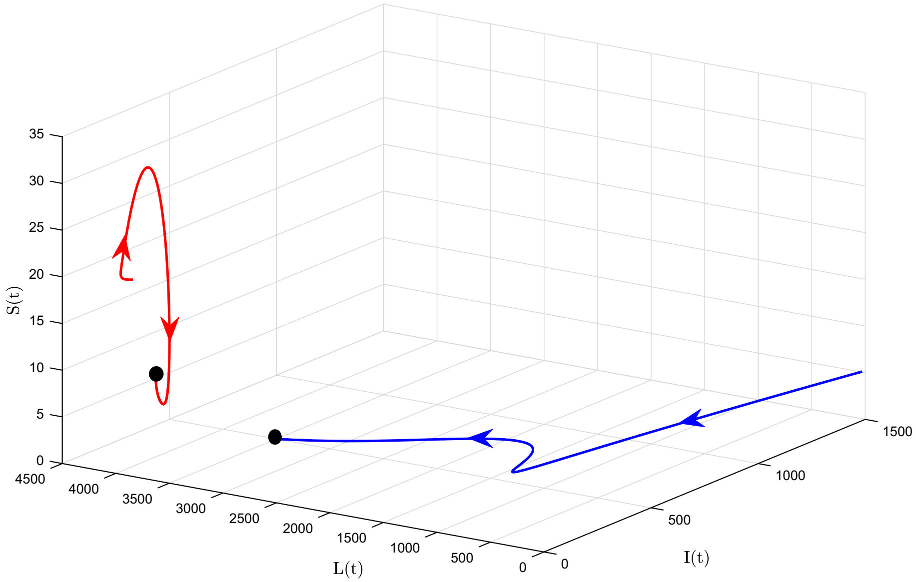

Fig. 4 The dynamical behaviors of ignorant, conformable, and obstinate spreader populations, respectively. The blue curve and the red curve denote solutions tending to the stable equilibria (one is the rumor-prevailing equilibrium and the other is the rumor-free equilibrium), respectively. Here, $\Lambda = 400$, $\beta_1 = 0.0002$, $\beta_2 = 0.5$, $\beta_3 = 0.00049654$, $\mu = 0.1$, and $\varepsilon = 0.0022$, the other parameters having the values given in the caption of Fig. 2

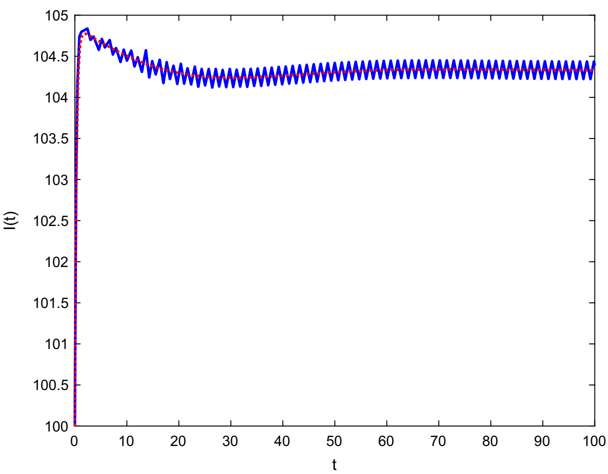

Fig. 5 Superimposed continuous (blue) and discrete (red) dynamical behaviors of the ignorant population for $\Lambda = 400$, $\beta_1 = 0.001$, $\beta_2 = 0.002$, $\beta_3 = 0.0005$, $\mu = 0.1$, $\varepsilon = 0.0022$, $\psi = 10$, $e = 0.5$, $\theta = 0.2$, $\rho = 0.5$, $h = 0.1$, and the initial conditions (100, 3700, 10) (color figure online)

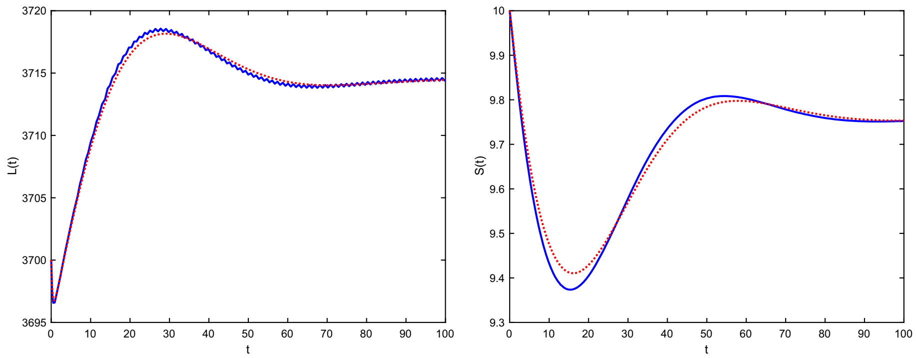

Fig. 6 The superimposed continuous (blue) and discrete (red) dynamical behaviors of the conformable spreader population and of the obstinate spreader population, respectively. The parameter values and initial conditions have the values given in the caption of Fig. 5 (color figure online)

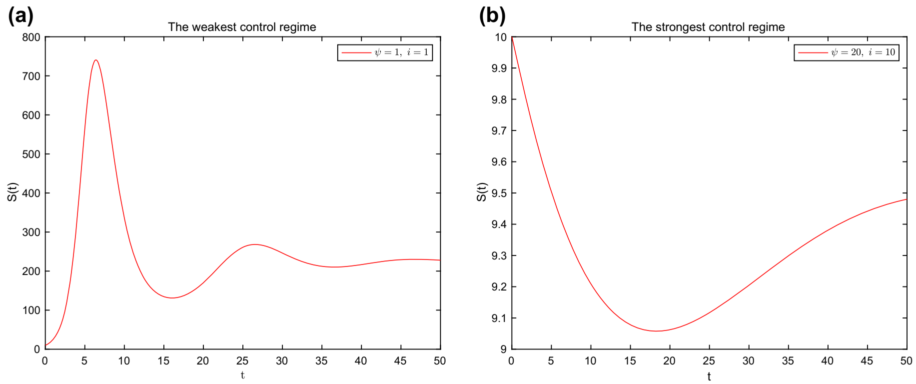

Fig. 7 The discrete dynamical behavior of the obstinate spreader population for $\Lambda = 400$, $\beta_1 = 0.001$, $\beta_2 = 0.002$, $\beta_3 = 0.0005$, $\mu = 0.1$, $\varepsilon = 0.0022$, $e = 0.5$, $\theta = 0.8$, $\gamma = 0.09$, $\delta = 0.98$, $\rho = 0.5$, $h = 0.01$, and the initial conditions (100, 3700, 10). (a) $\psi_n \equiv 1$ and $i_n \equiv 1$; (b) $\psi_n \equiv 20$ and $i_n \equiv 10$

$\beta_1 = 0.0002$, $\beta_2 = 0.0001$, $\beta_3 = 0.5$, $\mu = 0.1$, $\varepsilon = 0.0022$, $\psi = 10$, $e = 0.5$, $\theta = 0.2$, $\rho = 0.5$, and $f(P) = \frac{6}{1+2e^{-P}}$. It is then seen that $\mathcal{R}_{01} = 0.8 < 1$, which implies that the ignorant-only equilibrium $E_{01} = (400, 0, 0)$ is stable and the rumor dies out, as shown in Fig. 2.

To depict a bifurcation curve, we choose $\Lambda = 400$, $\beta_1 = 0.0002$, $\beta_2 = 0.5$, $\beta_3 \in (0.00049, 0.00053)$, $\mu = 0.1$, $\varepsilon = 0.0022$, $\psi = 10$, $e = 0.5$, $\theta = 0.2$, $\rho = 0.5$ and $f(P) = \frac{800}{1+2e^{-P}}$, and see that $\mathcal{R}^*_{02} \approx 0.8827$. As shown in Fig. 3, a backward bifurcation occurs.

Furthermore, for $\beta_3 = 0.00049654$, which implies that $\mathcal{R}^*_{02} \approx 0.8828$, there exist a stable rumor-prevailing equilibrium and a stable rumor-free equilibrium in system (2), as shown in Fig. 4.

To illustrate the claim that our discrete model (9) is dynamically consistent with the original continuous-time model (2), we fix $\Lambda = 400$, $\beta_1 = 0.001$, $\beta_2 = 0.0022$, $\beta_3 = 0.0005$, $\mu = 0.1$, $\varepsilon = 0.002$, $\psi = 10$, $e = 0.5$, $\theta = 0.2$, $\rho = 0.5$, $h = 0.01$, and the initial condition (100, 3700, 10), the (superimposed) results being shown in Figs. 5–6.

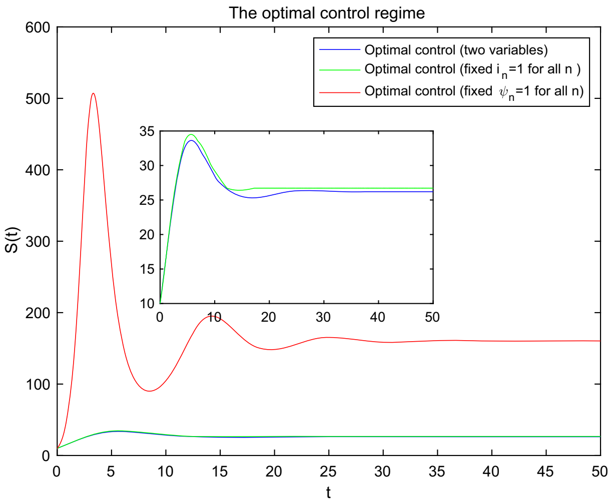

Fig. 8 Behavior of the spreader population in three scenarios: the optimal combined control regime (with $\psi_n \in [1, 20]$ and $i_n \in [1, 10]$ as control sets), the optimal single control regime ($\psi_n \in [1, 20]$, $i_n \equiv 1$), another optimal single control regime ($\psi_n \equiv 1$, $i_n \in [1, 10]$). Here, $\Lambda = 400$, $\beta_1 = 0.001$, $\beta_2 = 0.002$, $\beta_3 = 0.0005$, $\mu = 0.1$, $\varepsilon = 0.0022$, $e = 0.5$, $\bar{\theta} = 0.8$, $\gamma = 0.09$, $\delta = 0.98$, $\rho = 0.5$, $h = 0.01$, and the initial conditions are (100, 3700, 10)

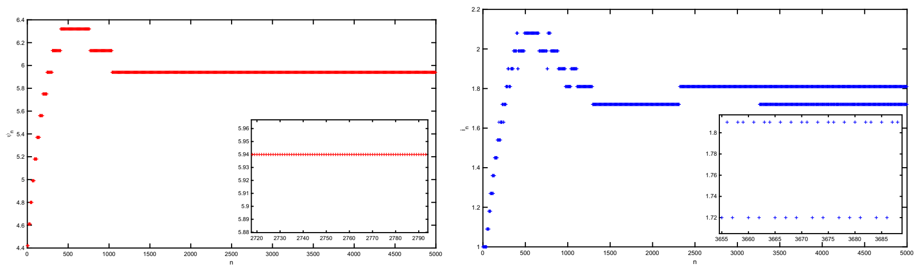

Fig. 9 Impact of varying the release rate of inhibiting mechanisms ($\psi_n \in [1,\ 20]$) and the self-adjusting mechanisms ($i_n \in [1,\ 10]$), respectively, upon the optimal combined control regime. The parameter values and initial conditions have the values given in the caption of Fig. 8. $e = 0.5$,

To describe results attained via distinct fixed horizon control regimes of the system (9), we choose $\Lambda = 400$, $\beta_1 = 0.001$, $\beta_2 = 0.002$, $\beta_3 = 0.0005$, $\mu = 0.1$, $\varepsilon = 0.0022$, $\psi_n \in [1, 20]$, $e = 0.5$, $\bar{\theta} = 0.5$, $\gamma = 0.1$, $\delta = 0.01$, $i_n \in [1, 10]$, $\rho = 0.5$, $h = 0.01$, $c_\psi = 0.0015$, and $c_i = 0.001$, the numerical particulars under different combinations of ψ_n and i_n, $n = 0, 1, \cdots, N-1$, being given in Figs. 7–9.

Figure 7 illustrates the weakest and strongest control regimes, respectively. Two control variables (ψ and i) are employed to optimize the objective function ν_N, as shown in Figs. 8–9.

In addition, the numerical results shown in Fig. 9 show that the low i and "medium" ψ determine the optimal outcomes.

6 Concluding remarks

The present paper attempts to formulate and analyze a rumor spreading model which is subject to two control mechanisms, namely to budgeting input and to an attitude adjustment function. The existence of the semi-trivial equilibria and of the positive (rumor-prevailing) equilibrium is completely characterized in terms of two threshold parameters, which describe the influence of conformable and obstinate spreaders, respectively, and are closely related to the basic reproduction number and to the invasion reproduction number of mathematical epidemiology, respectively. However, although our model draws from mathematical epidemiology, it is not functionally equivalent to a disease propagation model, as it involves a variable which is unrelated to rumor spreading status (or to disease status, in an epidemiological context).

A backward bifurcation analysis is performed, which leads to characterizing the stability of the rumor-prevailing equilibrium as a byproduct. To preserve the positivity of the approximating solutions and to keep consistency with the continuous model, a nonstandard finite difference scheme is employed. The existence of optimal controls is investigated, necessary conditions being obtained. It is observed that a combination of both controls has the most significant impact upon the size of the spreader population and that a combination of "medium" budgeting and low self-adjustment leads to optimal results.

Author Contributions C.W. and H.Z conceived the study, participated in model formulation and model analysis, and helped drafting the intermediary versions of the manuscript. P.G. participated in model formulation and model analysis, helped drafting the intermediary versions of the manuscript, and drafted the final version of the manuscript. T.L. carried out the stability analysis of the model steady states and helped drafting the intermediary versions of the manuscript. B. Z. helped drafting the intermediary versions of the manuscript. All authors read and approved the final manuscript.

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Availability of data and code. The choices of parameter values have been stated in the body of the paper. The graphs were produced using MATLAB version 2018a, available from https://www.mathworks.com/products/matlab.html.

Declarations

Conflict of interest The authors declare no competing interests.

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