Dynamics of Continuous, Discrete and Impulsive Systems
Series **A:** *Mathematical Analysis* **14 (2007)** 707-730
 http://www.watam.org

GENERATION AND CHARACTERIZATION OF NONLINEAR SEMIGROUPS ASSOCIATED TO SEMILINEAR EVOLUTION EQUATIONS INVOLVING "GENERALIZED" DISSIPATIVE OPERATORS

Paul Georgescu[1] and Naoki Shioji[2]

[1]Department of Mathematics
Technical University "Gh. Asachi" of Iasi
Bd. Copou 11, 700506, Iasi, Romania
[2]Department of Mathematics
Graduate School of Environment and Information Sciences
Yokohama National University, Tokiwadai, Hodogaya-ku
Yokohama 240-8501, Japan

Abstract. Given a linear operator A which satisfies a generalized dissipativity condition in terms of a "uniqueness function" w and its nonlinear continuous perturbation B in a real Banach space X, we discuss the construction of a nonlinear semigroup S providing mild solutions for the semilinear abstract Cauchy problem (SP) $u' = (A+B)u(t), t > 0$ and $u(0) = x$. It is shown that a subtangential condition and a semilinear stability condition are altogether necessary and sufficient for the generation of the semigroup S. A concrete example to which these generation results are applicable is also provided.
Keywords. "generalized" dissipative operators, nonlinear semigroups, generation theorems, semilinear evolution equations, semilinear stability condition, discrete schemes.
AMS (MOS) subject classification: 47H20, 47H06, 37L05.

1 Introduction

Since the fundamental paper [1] of Crandall and Liggett has been published, generation theory for nonlinear semigroups on arbitrary Banach spaces has evolved into a well-established subject, being used to treat a broad class of mathematical models due to its considerable unifying effect. A significant improvement of their theory has been made by Kobayashi in [6], who replaced the range condition used in [1] with a much less restrictive assumption, called the tangency range condition. Further, in their later paper [7], Kobayashi and Tanaka also succeeded in weakening Crandall and Liggett's classical dissipativity condition to a more general assumption of dissipativity with respect to a so-called uniqueness function, therefore opening the way for

[2]Partially supported by the Grant-in-Aid for Scientific Research (No. 02F00034) and the fund of Heisei 17 Joint reseach project in Graduate School of Environment and Information Sciences, Yokohama National University.

dealing with problems which fall outside the scope of Crandall and Liggett's theory.

However, in many important cases use is made of operators that can be decomposed as sums of linear dissipative operators and their nonlinear, continuous and possibly nondissipative perturbations. While these semilinear models have been found to describe accurately phenomena which exhibit nonlinear features, the particular structure of the operators involved in the mathematical model leads to significant qualitative properties of the solutions not appearing in the fully nonlinear case.

Of particular importance are generation theorems for nonlinear semigroups on arbitrary Banach spaces, in terms of necessary and sufficient conditions. As seen by Webb in [10], for a given nonlinear semigroup it is not always possible to associate in the classical sense a generator with a reasonably large domain, and this generator does not necessarily determine uniquely the semigroup even if it is densely defined. Hence one cannot expect a full equivalent of the celebrated Hille-Yosida theorem for the nonlinear case.

The aim of this paper is to study the generation of nonlinear semigroups associated to the semilinear problem

$$\text{(SP)} \qquad \begin{cases} u'(t) = (A+B)u(t), \quad t > 0; \\ u(0) = x \in D, \end{cases}$$

where A is the generator of a C_0-contraction semigroup on a Banach space $(X, |\cdot|)$ and $B : D \to X$ is a nonlinear, continuous perturbation defined on the closed set $D \subset X$. It is proved that the combination of a subtangential condition, in the form

$$\liminf_{h \downarrow 0} (1/h) d(T(h)x + hBx, D) = 0 \quad \text{for } x \in D,$$

and of a semilinear stability condition, given as

$$\liminf_{h \downarrow 0} (1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|) \le w(|x-y|) \quad \text{for } x, y \in D,$$

where w is an increasing uniqueness function, is a necessary and sufficient condition for the generation of a nonlinear semigroup S which provides mild solutions to (SP) and also satisfies the integral inequality

$$|S(t)x - S(t)y| \le |S(s)x - S(s)y| + \int_s^t w(|S(\tau)x - S(\tau)y|)d\tau$$

for $x, y \in D$ and $t \ge s \ge 0$.

The above result is our main theorem, which is fully stated in Section 2. Its proof makes use of a sequence $(u_n)_{n\ge1}$ of approximate solutions to (SP) depending upon small parameters $(\varepsilon_n)_{n\ge1}$, which is constructed using the subtangential condition. The uniform convergence of $(u_n)_{n\ge1}$ to a mild solution of (SP) is then shown as $\varepsilon_n \to 0$. Due to the semilinear nature of the equation under consideration, the proof does not follow Crandall and Liggett's classical argument, but rather a comparison argument, together with various estimates in terms of solutions of initial value problems for specific ordinary differential equations which involve the uniqueness function w.

Since the semilinear stability condition is not standard, even though it can be seen that it is well-suited to the semilinear structure of the problem under consideration, some significant situations in which this condition is satisfied are indicated. It is seen that if B is dissipative with respect to the uniqueness function w, then the semilinear stability condition follows from the subtangential condition and also that if the semilinear stability condition is satisfied, then the semilinear operator $A + B$ is strongly dissipative with respect to the same uniqueness function w mentioned in the statement of the semilinear stability condition. Moreover, an important feature of this condition is that it guarantees the uniqueness of the mild solution to (SP) for fixed initial data.

We use the main ideas of the approach devised (for the particular case $w(r) = w_0 r$) in Iwamiya, Oharu and Takahashi [5], to which our paper is related. Actually, the main result in [5] can be obtained as a particular case of our Theorem 2.1. Furthermore, some results and methods from Kobayashi and Tanaka [7], Nakagawa and Tanaka [9] and Iwamiya [4] are employed in order to complete our argument.

Our abstract theory is employed to establish the existence of a nonlinear semigroup which provides positive mild solutions for a concrete semilinear problem.

2 Main result

Let X be a real Banach space with norm $|\cdot|$. We define the semiinner products $[\cdot,\cdot]_\pm$ on X by

$$[x, y]_- = \lim_{h\uparrow 0} \frac{|x + hy| - |x|}{h} = \sup_{h<0} \frac{|x + hy| - |x|}{h}$$

and

$$[x, y]_+ = \lim_{h\downarrow 0} \frac{|x + hy| - |x|}{h} = \inf_{h>0} \frac{|x + hy| - |x|}{h}$$

for all $x, y \in X$. We also define $B(x_0, r) = \{x \in X, |x - x_0| \le r\}$ and $d(x_0, D) = \inf\{|x_0 - y|; y \in D\}$ for $x_0 \in X, D \subset X$ and $r > 0$. Given a continuous function $g : \mathbb{R} \to \mathbb{R}$, we denote by D^+, D_+, D^-, D_- its Dini derivatives, defined respectively by

$$(D^+ g)(t) = \limsup_{h\downarrow 0} \frac{g(t + h) - g(t)}{h}; \quad (D_+ g)(t) = \liminf_{h\downarrow 0} \frac{g(t + h) - g(t)}{h};$$

$$(D^- g)(t) = \limsup_{h\uparrow 0} \frac{g(t+h)-g(t)}{h}; \quad (D_- g)(t) = \liminf_{h\uparrow 0} \frac{g(t+h)-g(t)}{h}.$$

By a uniqueness function we mean a function $w : [0,\infty) \to \mathbb{R}$ which is continuous and satisfies condition (U) below.

(U) $w(0) = 0$ and $r \equiv 0$ is the unique solution of the initial value problem

$$\begin{cases} r'(t) = w(r(t)), & t > 0; \\ r(0) = 0. \end{cases}$$

An one-parameter family $S = \{S(t); t \geq 0\}$ of possibly nonlinear operators from D into itself is called a nonlinear semigroup on D if it satisfies the two properties below.

(S1) For $s, t \geq 0$ and $x \in D$, $S(t+s)x = S(t)S(s)x$ and $S(0)x = x$.
(S2) For $x \in D$, $u(\cdot) = S(\cdot)x$ is continuous on $[0, +\infty)$.

Given an operator $A : D(A) \subset X \to X$, it is said that A is dissipative, respectively strongly dissipative with respect to a continuous function w if

$$[x-y, x'-y']_- \leq w(|x-y|) \quad \text{for all } [x,x'],\, [y,y'] \in A,$$

respectively

$$[x-y, x'-y']_+ \leq w(|x-y|) \quad \text{for all } [x,x'],\, [y,y'] \in A.$$

If $w \equiv 0$, one obtains the classical definition of a dissipative, respectively of a strongly dissipative operator, while if $w(r) = w_0 r$, then the operator A is said to be w_0-dissipative, respectively w_0-strongly dissipative.

We consider the initial value problem

$$\text{(SP)} \qquad \begin{cases} u'(t) = (A+B)u(t), & t > 0; \\ u(0) = x \in D, \end{cases}$$

where D is a closed subset of X and the operators $A : D(A) \subset X \to X$ and $B : D \to X$ satisfy the following hypotheses

(A) A generates a C_0-contraction semigroup $T = \{T(t); t \geq 0\}$ on X;
(B) $B : D \to X$ is continuous.

It is then said that a function $u \in C([0,\infty); X)$ is a mild solution to (SP) if $u(t) \in D$ for $t \geq 0$ and the integral equation

$$u(t) = T(t)x + \int_0^t T(t-s)Bu(s)ds$$

is satisfied for each $t \geq 0$.

Our main result can now be stated as follows.

Theorem 2.1. *Suppose that the operators A and B satisfy hypotheses* (A)*, respectively* (B) *and the function $w : [0,\infty) \to \mathbb{R}$ is increasing, continuous and satisfies* (U)*. The following statements are then equivalent.*

(I) *There is a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D such that*

(I.a) $S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds;$

(I.b) $|S(t)x - S(t)y| \leq |S(s)x - S(s)y| + \int_s^t w(|S(\tau)x - S(\tau)y|)d\tau$

for $x, y \in D$ and $t \geq s \geq 0$.

(II) *The operators A and B satisfy the subtangential condition and the semilinear stability condition stated below*

(II.a) $\liminf_{h\downarrow 0}(1/h)d(T(h)x + hBx, D) = 0$ *for $x \in D$;*

(II.b) $\liminf_{h\downarrow 0}(1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|) \leq w(|x-y|)$

for $x, y \in D$.

In classical semigroup generation theory, condition (I.b) is replaced by a condition affirming that each $S(t)$ is a Lipschitz or locally Lipschitz operator in a prescribed sense, for example with "S is a contraction semigroup" or with another condition having a similar meaning. With regard to this, by a comparison argument (see Proposition 4.2), condition (I.b) implies that

$$\text{(QL)} \qquad |S(t)x - S(t)y| \leq m(t; |x-y|) \quad \text{for } t \in [0, \tau(|x-y|)),$$

where $m(t; |x-y|)$ is the maximal solution of the initial value problem

$$\begin{cases} r'(t) = w(r(t)), & t > 0; \\ r(0) = |x-y| \end{cases}$$

and $\tau(|x-y|)$ is its largest interval of existence, that is, a sort of Lipschitz estimate. Here, we cannot usually affirm that $\tau(|x-y|) = \infty$, so the estimate (QL) is not global.

However, we have used in this paper condition (I.b) instead of a Lipschitz estimate of type (QL) since if $w(r) = w_0 r$, then (I.b) becomes a well-known integral estimate, widely used in the theory of nonlinear semigroups generated by dissipative operators. Notice the difference between condition (I.b) given here and its similar counterpart in Iwamiya, Oharu and Takahashi [5], where the global existence of $m(t; |x-y|)$ is a priori assured.

3 Basic comparison results

Let $w \in C([0,\infty))$ with $w(0) \geq 0$ and let $\delta, \alpha \geq 0$. We shall denote by $m_\delta(t; \alpha)$ the maximal solution of the initial value problem

$$\begin{cases} r'(t) = w(r(t)) + \delta, & t > 0; \\ r(0) = \alpha \end{cases}$$

and by $[0, \tau_\delta(\alpha))$ its largest interval of existence. When $\delta = 0$, we shall sometimes omit the subscript δ, since there is no danger of confusion.

The following basic comparison result (Theorem 1.6.1 in Lakhsmikantham and Leela [8]), together with some of its corollaries, will be used to prove estimations which ensure the convergence of a sequence $(u_n)_{n\geq 1}$ of approximate solutions to (SP) depending on small error parameters to the unique exact mild solution of (SP).

Theorem 3.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is open and $g \in C(\Omega)$. Let $(t_0, u_0) \in \Omega$ and let $[t_0, t_0 + \tau)$ be the largest interval of existence on which the maximal solution $m(t; t_0, u_0)$ of the initial value problem*

$$\begin{cases} u'(t) = g(t, u(t)), & t > 0; \\ u(t_0) = u_0 \end{cases}$$

exists. Let $x \in C([t_0, t_0 + \tau))$ such that $(t, x(t)) \in \Omega$ for $t \in [t_0, t_0 + \tau)$, $x(t_0) \leq u_0$ and $Dx(t) \leq g(t, x(t))$ for some fixed Dini derivative D and for $t \in [t_0, t_0 + \tau)\backslash N$, N being an at most countable set. Then $x(t) \leq m(t; t_0, u_0)$ for $t \in [t_0, t_0 + \tau)$.

Note that in order to prove that a continuous function $u : [t_0, t_0 + \tau) \to \mathbb{R}$ is decreasing, it suffices to show that $Du \leq 0$ for $t \in [t_0, t_0 + \tau)\backslash N$, D being any Dini derivative and N being an at most countable set.

As a consequence, the following fundamental properties of the nonextendable maximal solution are obtained in Kobayashi and Tanaka [7], Lemma 5.1.

Lemma 3.1. *Let $\delta_0, \alpha_0 \geq 0$. Then the following properties* (i) *through* (iii) *hold*

(i) *If $\alpha \geq \alpha_0$ and $\delta \geq \delta_0$, then $\tau_\delta(\alpha) \leq \tau_{\delta_0}(\alpha_0)$ and $m_\delta(t; \alpha) \geq m_{\delta_0}(t; \alpha_0)$ for $t \in [0, \tau_\delta(\alpha))$.*

(ii) *If $\alpha \downarrow \alpha_0$ and $\delta \downarrow \delta_0$, then $\tau_\delta(\alpha) \uparrow \tau_{\delta_0}(\alpha_0)$ and $m_\delta(t; \alpha) \downarrow m_{\delta_0}(t; \alpha_0)$ uniformly on every compact subinterval of $[0, \tau_\delta(\alpha))$.*

(iii) *If $0 \leq s < \tau_{\delta_0}(\alpha_0)$, then $\tau_{\delta_0}(\alpha_0) - s \leq \tau_{\delta_0}(m_{\delta_0}(s; \alpha_0))$ and $m_{\delta_0}(t; m_{\delta_0}(s; \alpha_0)) = m_{\delta_0}(t + s; \alpha_0)$ for $t \in [0, \tau_{\delta_0}(\alpha_0) - s)$.*

Among other purposes, the following Lemma will be used to establish the uniqueness of the mild solution for the semilinear problem (SP).

Lemma 3.2. *Let $u, v \in C([0, \tau); X)$ satisfy the differential inequality*

$$D(|u(t) - v(t)|) \leq w(|u(t) - v(t)|) \quad \text{for } t \in [0, \tau)\backslash N,$$

where D is any Dini derivative and N is an at most countable set. Then

(i) *$|u(t) - v(t)| \leq m(t; |u(0) - v(0)|)$ for $t \in [0, \tau) \cap [0, \tau(|u(0) - v(0)|))$;*

(ii) *$|u(t) - v(t)| \leq |u(s) - v(s)| + \int_s^t w(|u(\xi) - v(\xi)|)d\xi$ for $0 \leq s \leq t < \tau$.*

Conversely, if (ii) *holds, then $D^+(|u(t) - v(t)|) \leq w(|u(t) - v(t)|)$ for $t \in [0, \tau)$.*

Proof. The first claim follows from Theorem 3.1. For the proof of the second claim, let us define $h : [0, \tau) \to \mathbb{R}$ by

$$h(t) = |u(t) - v(t)| - \int_0^t w(|u(\xi) - v(\xi)|)d\xi.$$

It is seen that $h \in C([0, \tau); X)$ and

$$Dh(t) = D(|u(t) - v(t)|) - w(|u(t) - v(t)|) \le 0 \quad \text{for } t \in [0, \tau)\backslash N.$$

It follows that h is decreasing, fact which implies the second estimate. The proof of the converse implication is obvious. □

Note that in the above results the function w is neither required to be an uniqueness function, nor assumed to be increasing.

Suppose now that w is increasing, in addition to its continuity. One obtains the following result.

Lemma 3.3. *Suppose that w is increasing. Then $m(t; u_0) + \alpha \le m(t; u_0 + \alpha)$ for all $t \in [0, \tau(u_0 + \alpha))$ and $u_0, \alpha \in \mathbb{R}_+$.*

Proof. Let us denote $u_1(t) = m(t; u_0) + \alpha$ and $u_2(t) = m(t; u_0 + \alpha)$. Then

$$u_1'(t) = m'(t; u_0) = w(m(t; u_0)) = w(u_1(t) - \alpha) \le w(u_1(t))$$
$$u_2'(t) = m'(t; u_0 + \alpha) = w(m(t; u_0 + \alpha)) = w(u_2(t)).$$

Since $u_1(0) = u_2(0) = u_0 + \alpha$, the conclusion follows from Theorem 3.1. □

Let us now particularize w to be an uniqueness function, not necessarily increasing. Given $K > 0$, we define

$$w^K(t) = \begin{cases} w(t) & \text{if } t \in [0, K]; \\ w(K) & \text{if } t > K. \end{cases}$$

We shall also denote by $m_\delta^K(t; \alpha)$ the maximal solution of the initial value problem

$$\begin{cases} r'(t) = w^K(r(t)) + \delta, & t > 0; \\ r(0) = \alpha. \end{cases}$$

Note that, since $w^K(\cdot)$ is bounded, $\tau_\delta^K(\alpha) = \infty$ for any $\delta, \alpha \in \mathbb{R}_+$. With these notations, it is seen that the following Lemma (Lemma 5.1 in Kobayashi and Tanaka [7]) holds.

Lemma 3.4. *Suppose that w is an uniqueness function and let $K > 0$. The following properties* (i) *and* (ii) *hold.*

(i) *If $\alpha \downarrow \alpha_0$ and $\delta \downarrow \delta_0$, then $m_\delta^K(t; \alpha) \downarrow m_{\delta_0}^K(t; \alpha_0)$ uniformly on any compact interval* $[0, T]$.

(ii) $m_0^K(t; 0) = 0$ *for* $t \ge 0$.

For related comparison results see also Iwamiya [4], Section 3.

4 Proof of the main theorem

We first prove the implication from (I) to (II). Let $x \in D$. Since $u(\cdot) = S(\cdot)x$ is a mild solution to (SP), it is seen from the definition of a mild solution that $\lim_{h\downarrow 0}(1/h)(S(h)x - T(h)x) = Bx$. Hence for each $\varepsilon > 0$ there is $\delta \in (0, \varepsilon]$ such that $(1/h)|S(h)x - T(h)x - hBx| \le \varepsilon$ for all $h \in (0, \delta]$, and therefore

$$(1/h)d(T(h)x + hBx, D) \le \varepsilon \quad \text{for all } h \in (0, \delta],$$

from which we obtain (II.a). Also, we note that

$$\begin{aligned}
&(1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|)\\
&\quad \le (1/h)(|T(h)x + hBx - S(h)x| + |T(h)y + hBy - S(h)y|)\\
&\qquad + (1/h)(|S(h)x - S(h)y| - |x-y|).
\end{aligned}$$

Passing to inferior limit as $h \downarrow 0$, and using the definition of a mild solution and the integral inequality (I.b), we obtain that

$$\begin{aligned}
&\liminf_{h\downarrow 0}(1/h)\big(|T(h)(x-y) + h(Bx - By)| - |x-y|\big)\\
&\quad \le \liminf_{h\downarrow 0}(1/h)\big(|S(h)x - S(h)y| - |x-y|\big)\\
&\quad \le \liminf_{h\downarrow 0}(1/h)\int_0^h w(|S(s)x - S(s)y|)ds.
\end{aligned}$$

Since w is continuous, we obtain (II.b). Therefore, the proof of the implication from (I) to (II) is completed.

We now prove the implication from (II) to (I), to which we devote the most part of this section. First, it is seen that the subtangential condition (II.a) holds uniformly in a local sense, the following result being obtained as a particular case of Theorem 3.1 from Georgescu and Oharu [2], for $\varphi = 0$.

Lemma 4.1. *Suppose that* (II.a) *holds. Let $x \in D, \varepsilon \in (0,1)$ and let $r = r(x,\varepsilon)$ be chosen such that $|Bx - By| \le \varepsilon/4$ and $\sup_{s\in[0,r]}|T(s)Bx - Bx| \le \varepsilon/4$ for each $y \in D \cap B(x,r)$. Choose $M \ge 0$ satisfying $|By| \le M$ for each $y \in D \cap B(x,r)$, and define $h(x,\varepsilon) = \sup\{h > 0; h(M+1) + \sup_{s\in[0,h]}|T(s)x - x| \le r\}$. Let $h \in [0, h(x,\varepsilon))$ and $y \in D$ satisfy $|y - T(h)x| \le h(M+1)$. Then for each $\eta > 0$ with $h + \eta \le h(x,\varepsilon)$ there is $z \in D \cap B(x,r)$ satisfying $(1/\eta)|z - T(\eta)y - \eta By| \le \varepsilon$.*

Remark 4.1. If in particular we let $h = 0$ and $y = x$ in Lemma 4.1, then it is seen that for every $\eta > 0$ with $\eta \le h(x,\varepsilon)$ there is $z \in D \cap B(x,r)$ such that $(1/\eta)|z - T(\eta)x - \eta Bx| \le \varepsilon$. This implies that the subtangential condition (II.a) is equivalent to its stronger form

(II.a)$'$ $\limsup_{h\downarrow 0}(1/h)d(T(h)x + hBx, D) = 0$ for $x \in D$.

Since the semilinear stability condition (II.b) is not necessarily standard, even though it seems to be the most suitable "dissipativity-like" condition for our semilinear problem, we first indicate a significant case in which this condition is satisfied.

Proposition 4.1. *In addition to* (A) *and* (B)*, assume that* (II.a) *holds and* B *is dissipative with respect to some continuous function* w*. Then the semilinear stability condition* (II.b) *holds for the same choice of* w*.*

Proof. Let $\varepsilon > 0$ and $x, y \in D$. From (II.a)$'$ we obtain that there are $h \in (0, \varepsilon]$ and $x_h, y_h \in D$ such that

$$|x_h - T(h)x - hBx| \le h\varepsilon, \quad |Bx_h - Bx| \le \varepsilon;$$
$$|y_h - T(h)y - hBy| \le h\varepsilon, \quad |By_h - By| \le \varepsilon.$$

Since B is dissipative with respect to the continuous function w, it is seen that

$$\begin{aligned}(1/h)(|x_h - y_h| - |(x_h - y_h) - h(Bx_h - By_h)|)\\ \le [x_h - y_h, Bx_h - By_h]_- \le w(|x_h - y_h|).\end{aligned}$$

Using the inequalities above, we obtain that

$$\begin{aligned}
&(1/h)(|T(h)(x-y) + h(Bx - By)| - |x - y|)\\
&\quad \le (1/h)(|x_h - T(h)x - hBx| + |y_h - T(h)y - hBy| + |x_h - y_h| - |x - y|)\\
&\quad \le (1/h)(|x_h - y_h| - |x - y|) + 2\varepsilon\\
&\quad \le (1/h)(|x_h - y_h| - |(x_h - y_h) - h(Bx_h - By_h)|)\\
&\qquad + (1/h)(|(x_h - y_h) - h(Bx_h - By_h)| - |x - y|) + 2\varepsilon\\
&\quad \le w(|x_h - y_h|) + (1/h)|x_h - T(h)x - hBx| + (1/h)|y_h - T(h)y - hBy|\\
&\qquad + |Bx_h - Bx| + |By_h - By| + (1/h)(|T(h)x - T(h)y| - |x - y|) + 2\varepsilon\\
&\quad \le w(|x_h - y_h|) + (1/h)(|T(h)x - T(h)y| - |x - y|) + 6\varepsilon.
\end{aligned}$$

Since T is a contraction semigroup and w is continuous, taking the inferior limit as $h \downarrow 0$ we obtain

$$\liminf_{h\downarrow 0}(1/h)(|T(h)(x-y) + h(Bx - By)| - |x - y|) \le w(|x - y|) + 6\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required conclusion. □

As pointed out in [5, Proposition 3.2], one can easily see that

$$\begin{aligned}\lim_{h\downarrow 0}(1/h)(|T(h)(x-y) + h(Bx - By)| - |x - y|)\\ = [x - y, (A+B)x - (A+B)y]_+\end{aligned}$$

for each $x, y \in D(A) \cap D$. Therefore the semilinear stability condition (II.b) implies the strong dissipativity of the semilinear differential operator $A + B$

with respect to the continuous function w. This is a natural result, since the subtangential condition (II.a) is weaker than the classical range condition, which couples the dissipativity condition, so in our setting we need an assumption stronger than the latter in order to prove our generation result.

An important feature of the classical dissipativity condition, to which our semilinear stability condition is related, is that it guarantees stability results which imply the uniqueness of the mild solution for given initial data. We shall see now that our semilinear stability condition (II.b) yields similar properties.

Proposition 4.2. *Suppose that $w : [0,\infty) \to \mathbb{R}$ is a continuous function, condition* (II.b) *is satisfied and u and v are mild solutions to* (SP) *with initial data $u(0) = x$, respectively $v(0) = y$. Then*

$$|u(t) - v(t)| \le m(t; |x - y|) \quad \text{for } t \in [0, \tau(|x - y|)). \tag{4.1}$$

In particular, if w is an uniqueness function and $x = y$, then $u \equiv v$ on $[0, \tau(|x - y|))$.

Proof. Let $t \in [0, \tau(|x-y|))$ and $h > 0$ such that $t+h \in [0, \tau(|x-y|))$. From the definition of a mild solution to (SP) we obtain

$$\begin{aligned}
&(1/h)(|u(t+h) - v(t+h)| - |u(t) - v(t)|)\\
&\quad =(1/h)\left|T(h)(u(t) - v(t)) + \int_t^{t+h} T(t+h-s)(Bu(s) - Bv(s))ds\right|\\
&\qquad - (1/h)|u(t) - v(t)|\\
&\quad \le(1/h)(|T(h)(u(t) - v(t)) + h(Bu(t) - Bv(t))| - |u(t) - v(t)|)\\
&\qquad + (1/h)\int_t^{t+h} |T(t+h-s)Bu(s) - Bu(t)|ds\\
&\qquad + (1/h)\int_t^{t+h} |T(t+h-s)Bv(s) - Bv(t)|ds.
\end{aligned}$$

From (II.b) and the continuity of B and T, we obtain that

$$D_+(|u(t) - v(t)|) \le w(|u(t) - v(t)|) \quad \text{for all } t \in [0, \tau(|x - y|)).$$

The desired inequality (4.1) follows now from Lemma 3.2. Also, if w is an uniqueness function, then $m(t; 0) = 0$ for $t > 0$, and so (4.1) implies our local uniqueness result. □

Once we have proved the previous stability result, we are ready to show that the global existence of the mild solutions to (SP) may be deduced from the corresponding local existence result via a classical extendability argument.

Proposition 4.3. *Suppose that $w : [0,\infty) \to \mathbb{R}$ is a continuous function, $w(0) \ge 0$,* (II.b) *holds and that for each $x \in D$ there is $T_x > 0$ such that a* *mild solution* $u(\cdot)$ *of* (SP) *with initial data* $u(0) = x$ *exists on* $[0, T_x]$. *Then for each* $x \in D$ *there exists a global mild solution of* (SP) *with initial data* $u(0) = x$.

Proof. Given $x \in D$, denote

$$T_x^{\max} = \sup\{T > 0;\ \text{there exists a mild solution } u(\cdot) \text{ of (SP)} \\ \text{with initial data } u(0) = x \text{ on } [0, T]\}.$$

We have to show that $T_x^{\max} = \infty$. Suppose to the contrary that $T_x^{\max} < \infty$. We now prove that $\lim_{t \uparrow T_x^{\max}} u(t)$ exists and lies in D. Then we may define $u(T_x^{\max}) = \lim_{t \uparrow T_x^{\max}} u(t)$ and use the local existence result to extend u beyond $T_x^{\max}$, hence contradicting $T_x^{\max}$'s definition. Let $h > 0$. Using Proposition 4.2, one obtains

$$|u(t+h) - u(t)| \leq m(t; |x - u(h)|) \text{ for } t \in [0, T_x^{\max} - h) \cap [0, \tau(|x - u(h)|)).$$

Since $\lim_{h \downarrow 0} |u(h) - x| = 0$, using Lemma 3.1 (ii) it is seen that $\tau(|x - u(h)|) > T_x^{\max}$ for small h, and so

$$\sup\{|u(t+h) - u(t)|; t \in [0, T_x^{\max} - h)\} \\ \leq \sup\{m(t; |x - u(h)|); t \in [0, T_x^{\max}]\} \quad \text{for small } h.$$

Using again Lemma 3.1, it is seen that

$$\limsup_{h \downarrow 0} \sup\{|u(t+h) - u(t)|; t \in [0, T_x^{\max} - h)\} \\ \leq \limsup_{h \downarrow 0} \sup\{m(t; |x - u(h)|); t \in [0, T_x^{\max}]\} = 0.$$

Hence the limit $y = \lim_{t \uparrow T_x^{\max}} u(t)$ exists and belongs to D. Now let $v : [0, T_y) \to X$ be a mild solution to (SP) with initial data $v(0) = y$. Then the function $z : [0, T_x^{\max} + T_y) \to X$ given by

$$z(t) = \begin{cases} u(t), & t \in [0, T_x^{\max}); \\ v(t - T_x^{\max}), & t \in [T_x^{\max}, T_x^{\max} + T_y) \end{cases}$$

is a mild solution to (SP) with initial data $z(0) = x$, its interval of existence being strictly larger than $[0, T_x^{\max})$, which is a contradiction. □

We now prove our local existence theorem.

A key result which will be used for the construction of the discrete local approximate solutions to (SP) is the following lemma, which establishes the existence of time-discretizing and respectively solution-discretizing sequences $(t_i)_{0 \leq i \leq N}$ and $(x_i)_{0 \leq i \leq N}$ enjoying a number of fundamental properties. Among these properties (i) through (vi) below, note the importance of (iii), which yields the a-priori boundedness of the sequence $(x_i)_{0 \leq i \leq N}$.

Lemma 4.2. *Suppose that condition* (II.a) *is satisfied. Let $x \in D$. Assume that $R > 0$ and $M > 0$ are such that $|By| \le M$ for $y \in D \cap B(x,R)$. Let $\tau > 0$ be small enough to satisfy*

$$\tau(M+1) + \sup_{t\in[0,\tau]} |T(t)x - x| \le R.$$

Then for each $\varepsilon \in (0,1)$ there exist sequences $(t_i)_{0\le i\le N}$ and $(x_i)_{0\le i\le N}$ such that

(i) $t_0 = 0$, $x_0 = x$, $t_N = \tau$;
(ii) $0 < t_{i+1} - t_i \le \varepsilon$ *for* $0 \le i \le N-1$;
(iii) $x_i \in D \cap B(x,R)$ *for* $0 \le i \le N$;
(iv) $|x_{i+1} - T(t_{i+1}-t_i)x_i - (t_{i+1}-t_i)Bx_i| \le (t_{i+1}-t_i)\varepsilon$ *for* $0 \le i \le N-1$;
(v) $|x_i - T(t_i)x| \le t_i(M+1)$ *for* $0 \le i \le N$;
(vi) *For* $0 \le i \le N-1$ *there is* $r_i \in (0,\varepsilon]$ *such that* $|By - Bx_i| \le \varepsilon/4$ *for* $y \in B(x_i,r_i) \cap D$, $\sup_{t\in[0,r_i]} |T(t)Bx_i - Bx_i| \le \varepsilon/4$ *and* $(t_{i+1}-t_i)(M+1) + \sup_{t\in[0,t_{i+1}-t_i]} |T(t)x_i - x_i| \le r_i$.

For the proof, see Iwamiya, Oharu and Takahashi [5], Lemma 4.1, or Georgescu and Oharu [2], Theorem 5.1 (for $\varphi = 0$).

Now, given a small parameter $\varepsilon \in (0,1)$, an approximate solution $u_\varepsilon : [0,\tau] \to X$ to (SP) may be constructed using the finite sequences $(t_i)_{0\le i\le N}$ in $[0,\tau]$ and $(x_i)_{0\le i\le N}$ in $B(x,R)$ obtained in the above lemma, as follows

$$u_\varepsilon(t) = \begin{cases} T(t-t_i)x_i + (t-t_i)Bx_i & \text{for } t \in [t_i,t_{i+1}),\ 0 \le i \le N-1; \\ T(\tau - t_{N-1})x_{N-1} + (\tau - t_{N-1})Bx_{N-1} & \text{for } t = \tau. \end{cases} \tag{4.2}$$

Then for $t \in [t_i, t_{i+1})$ and $0 \le i \le N-1$ we have

$$\begin{aligned} |x_{i+1} - u_\varepsilon(t)| &\le |x_{i+1} - T(t_{i+1}-t_i)x_i - (t_{i+1}-t_i)Bx_i| \\ &\quad + |T(t_{i+1}-t_i)x_i - T(t-t_i)x_i| + (t_{i+1}-t)|Bx_i| \\ &\le (t_{i+1}-t_i)\varepsilon + |T(t_{i+1}-t)x_i - x_i| + (t_{i+1}-t)|Bx_i| \\ &\le (t_{i+1}-t_i)(M+1) + |T(t_{i+1}-t)x_i - x_i| \le \varepsilon \end{aligned}$$

and in a similar way we may show that $|x_N - u_\varepsilon(\tau)| \le \varepsilon$. Hence

$$d(u_\varepsilon(t), D) \le \varepsilon \quad \text{for } t \in [0,\tau] \tag{4.3}$$

(note, however, that $u_\varepsilon(\cdot)$ does not necessarily take its values in D).

Now, for any small parameter $\varepsilon > 0$, (4.2) gives a method to construct an approximate solution $u_\varepsilon : [0,\tau] \to X$ to (SP) which satisfies (4.3). Our purpose is to show that, for a given null sequence $(\varepsilon_n)_{n\ge1}$, the corresponding sequence of approximate solutions $(u_{\varepsilon_n})_{n\ge1}$ is uniformly convergent to a continuous function $u : [0,\tau] \longrightarrow X$, which is a mild solution to (SP). The uniqueness of the mild solution will then be obtained from Proposition 4.2.

To this goal, we need to estimate the difference between two approximate solutions u_ε and $u_{\hat\varepsilon}$ corresponding to different small parameters ε and $\hat\varepsilon$. An important step in this direction is provided by the following lemma.

Lemma 4.3. *Assume that conditions* (II.a) *and* (II.b) *are satisfied. Let* $x, \hat{x} \in D$ *and* $\varepsilon, \hat{\varepsilon} \in (0, 1/3)$. *Suppose that* $r = r(x, \varepsilon)$ *and* $\hat{r} = \hat{r}(\hat{x}, \hat{\varepsilon})$ *are real numbers such that* $0 < r \leq \varepsilon$, $|Bx - By| \leq \varepsilon/4$, $|By| \leq M(x, \varepsilon)$ *for* $y \in D \cap B(x, r)$, $\sup_{s\in[0,r]} |T(s)Bx - Bx| \leq \varepsilon/4$ *and also* $0 < \hat{r} \leq \hat{\varepsilon}$, $|B\hat{x} - By| \leq \varepsilon/4$, $|By| \leq \hat{M}(\hat{x}, \hat{\varepsilon})$ *for* $y \in D \cap B(\hat{x}, \hat{r})$, $\sup_{s\in[0,\hat{r}]} |T(s)B\hat{x} - B\hat{x}| \leq \hat{\varepsilon}/4$ *for* $y \in D \cap B(\hat{x}, \hat{r})$ *and for some real numbers* $M(x, \varepsilon)$ *and* $\hat{M}(\hat{x}, \hat{\varepsilon})$. *Choose* $M > \max\{M(x, \varepsilon), \hat{M}(\hat{x}, \hat{\varepsilon})\}$. *Denote*

$$h(x, \varepsilon) = \sup\Big\{h > 0; h(M + 1) + \sup_{s\in[0,h]} |T(s)x - x| \leq r\Big\},$$
$$\hat{h}(\hat{x}, \hat{\varepsilon}) = \sup\Big\{h > 0; h(M + 1) + \sup_{s\in[0,h]} |T(s)\hat{x} - \hat{x}| \leq \hat{r}\Big\}$$

and let $y, \hat{y} \in D$, $h \in [0, h(x, \varepsilon))$ *and* $\hat{h} \in [0, \hat{h}(\hat{x}, \hat{\varepsilon}))$ *such that* $|y - T(h)x| \leq h(M + 1)$; $|\hat{y} - T(\hat{h})\hat{x}| \leq \hat{h}(M + 1)$. *Then for each* $\delta > 0$ *and* $\eta > 0$ *such that* $h + \eta \leq h(x, \varepsilon)$ *and* $\hat{h} + \eta \leq \hat{h}(\hat{x}, \hat{\varepsilon})$ *there exist* $z \in D \cap B(x, r)$ *and* $\hat{z} \in D \cap B(\hat{x}, \hat{r})$ *such that*

$$|z - T(\eta)y - \eta By| < 2\eta\varepsilon; \tag{4.4}$$
$$|\hat{z} - T(\eta)\hat{y} - \eta B\hat{y}| < 2\eta\hat{\varepsilon}; \tag{4.5}$$
$$|z - \hat{z}| \leq m_{\delta+\varepsilon+\hat{\varepsilon}}^{|x-\hat{x}|+\varepsilon+\hat{\varepsilon}}(\eta; |y - \hat{y}|). \tag{4.6}$$

Proof. Note first that $y \in D \cap B(x, r)$, since

$$|y - x| \leq |y - T(h)x| + |T(h)x - x| \leq h(M + 1) + |T(h)x - x| \leq r,$$

and also $\hat{y} \in D \cap B(\hat{x}, \hat{r})$. We construct sequences $(s_n)_{n\geq 0}$, $(x_n)_{n\geq 0}$, $(\hat{x}_n)_{n\geq 0}$ satisfying

(i) $s_0 = 0, x_0 = y$, $\hat{x}_0 = \hat{y}$;
(ii) $0 < s_n < s_{n+1}$ and $\lim_{n\to\infty} s_n = \eta$;
(iii) $|x_n - T(s_n - s_{n-1})x_{n-1} - (s_n - s_{n-1})Bx_{n-1}| \leq (s_n - s_{n-1})\varepsilon$;
(iv) $|\hat{x}_n - T(s_n - s_{n-1})\hat{x}_{n-1} - (s_n - s_{n-1})B\hat{x}_{n-1}| \leq (s_n - s_{n-1})\hat{\varepsilon}$;
(v) $|T(s_n - s_{n-1})(x_{n-1} - \hat{x}_{n-1}) + (s_n - s_{n-1})(Bx_{n-1} - B\hat{x}_{n-1})|$
$\leq |x_{n-1} - \hat{x}_{n-1}| + (s_n - s_{n-1})w(|x_{n-1} - \hat{x}_{n-1}|) + (s_n - s_{n-1})\delta$;
(vi) $|x_n - T(s_n)x_0| \leq s_n(M + 1)$;
(vii) $|\hat{x}_n - T(s_n)\hat{x}_0| \leq s_n(M + 1)$;
(viii) $x_n \in B(x, r) \cap D$;
(ix) $\hat{x}_n \in B(\hat{x}, \hat{r}) \cap D$

for each $n \geq 0$, properties (iii), (iv) and (v) being not formulated for $n = 0$. Set $s_0 = 0$, $x_0 = y$ and $\hat{x}_0 = \hat{y}$, so that (i), (vi), (vii), (viii) and (ix) are satisfied for $n = 0$. Suppose now that s_k, x_k and $\hat{x}_k$, $k = 0, 1, 2, \ldots, N$ have been defined in such a way that (i), (iii) through (ix) and the first half of (ii) are satisfied. Define

$$\begin{aligned}\overline{h}_N = \sup\{\xi > 0; s_N + \xi \le \eta; &|T(\xi)(x_N - \hat{x}_N) + \xi(Bx_N - B\hat{x}_N)| \\ &\le |x_N - \hat{x}_N| + \xi w(|x_N - \hat{x}_N|) + \xi\delta\}.\end{aligned}$$

From (II.b) one may see that $\overline{h}_N > 0$. Take $h_N \in (\overline{h}_N/2, \overline{h}_N)$ and define $s_{N+1} = s_N + h_N$ so that (v) is satisfied for $n = N+1$. From (II.a)$'$, one may find $x_{N+1}, \hat{x}_{N+1} \in D$ such that

$$\begin{aligned}&|x_{N+1} - T(s_{N+1} - s_N)x_N - (s_{N+1} - s_N)Bx_N| \le (s_{N+1} - s_N)\varepsilon; \\ &|\hat{x}_{N+1} - T(s_{N+1} - s_N)\hat{x}_N - (s_{N+1} - s_N)B\hat{x}_N| \le (s_{N+1} - s_N)\hat{\varepsilon},\end{aligned}$$

that is, (iii) and (iv) are satisfied for $n = N+1$. Then

$$\begin{aligned}|x_{N+1} - T(s_{N+1})x_0| &\le (s_{N+1} - s_N)M + |x_N - T(s_N)x_0| + (s_{N+1} - s_N)\varepsilon \\ &\le s_{N+1}(M+1)\end{aligned}$$

and similarly $|\hat{x}_{N+1} - T(s_{N+1})\hat{x}_0| \le s_{N+1}(M+1)$, so (vi) and (vii) are proved for $n = N+1$. Also,

$$\begin{aligned}&|x_{N+1} - T(s_{N+1} + h)x| \\ &\quad \le |x_{N+1} - T(s_{N+1})x_0| + |T(s_{N+1})x_0 - T(s_{N+1} + h)x| \\ &\quad \le (s_{N+1} + h)(M+1)\end{aligned}$$

and similarly $|\hat{x}_{N+1} - T(s_{N+1} + \hat{h})\hat{x}| \le (s_{N+1} + \hat{h})(M+1)$. Therefore,

$$|x_{N+1} - x| < (s_{N+1} + h)(M+1) + |T(s_{N+1} + h)x - x| \le r(x, \varepsilon)$$

and also $|\hat{x}_{N+1} - \hat{x}| < \hat{r}(\hat{x}, \hat{\varepsilon})$, so (viii) and (ix) are satisfied for $n = N+1$. It now remains to show that $\lim_{n\to\infty} s_n = \eta$. From (iii), (iv) and Lemma 5.2 in Iwamiya [4], we see that $(x_n)_{n\ge 0}$ and $(\hat{x}_n)_{n\ge 0}$ are convergent. We shall denote their limits by z, respectively by $\hat{z}$. Suppose that $\lim_{n\to\infty} s_n = s < \eta$. Then (II.b) implies that there exists $\xi \in (0, \eta)$ such that

$$|T(\xi)(z - \hat{z}) + \xi(Bz - B\hat{z})| \le |z - \hat{z}| + \xi w(|z - \hat{z}|) + (1/2)\xi\delta. \tag{4.7}$$

Choose $N \ge 1$ so that $s - s_n \le \xi/2$ for each $n \ge N$ and define $\xi_n = s - s_n + \xi$. It is seen that $s_n + \xi_n = s + \xi < \eta$ and also $\xi_n = s - s_n + \xi > \xi \ge 2(s - s_n) > 2h_n \ge \overline{h}_n$ for each $n \ge N$. This yields that

$$|T(\xi_n)(x_n - \hat{x}_n) + \xi_n(Bx_n - B\hat{x}_n)| > |x_n - \hat{x}_n| + \xi_n w(|x_n - \hat{x}_n|) + \xi_n\delta$$

for each $n \ge N$. Passing to limit as $n \to \infty$, we obtain that

$$|T(\xi)(z - \hat{z}) + \xi(Bz - B\hat{z})| \ge |z - \hat{z}| + \xi w(|z - \hat{z}|) + \xi\delta,$$

which contradicts (4.7). We now prove the required estimates (4.4), (4.5) and (4.6). It is easy to see that

$$\begin{aligned} x_n - T(s_n)y - s_nBy \\ &= \sum_{k=0}^{n-1} T(s_n - s_{k+1})[x_{k+1} - T(s_{k+1} - s_k)x_k - (s_{k+1} - s_k)Bx_k] \\ &\quad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)T(s_n - s_{k+1})Bx_k - s_nBy, \end{aligned}$$

and we can therefore obtain

$$\begin{aligned} |x_n - T(s_n)y - s_nBy| \\ &\le \sum_{k=0}^{n-1} |x_{k+1} - T(s_{k+1} - s_k)x_k - (s_{k+1} - s_k)Bx_k| \\ &\quad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|Bx_k - Bx| \\ &\quad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|T(s_n - s_{k+1})Bx - Bx| + s_n|Bx - By| \\ &\le \sum_{k=0}^{n-1} (s_{k+1} - s_k)\varepsilon + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|Bx_k - Bx| \\ &\quad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|T(s_n - s_{k+1})Bx - Bx| + s_n|Bx - By| \\ &\le s_n\varepsilon + s_n\varepsilon/4 + s_n\varepsilon/4 + s_n\varepsilon/4. \end{aligned}$$

Then

$$|x_n - T(s_n)y - s_nBy| \le 7s_n\varepsilon/4,$$

and it may be proved in a similar fashion that

$$|\hat{x}_n - T(s_n)\hat{y} - B\hat{y}| \le 7s_n\hat{\varepsilon}/4.$$

Passing to limit in the above estimates we obtain

$$|z - T(\eta)y - \eta By| < 2\eta\varepsilon; \quad |\hat{z} - T(\eta)\hat{y} - \eta B\hat{y}| < 2\eta\hat{\varepsilon},$$

that is, we obtain (4.4) and (4.5). Also, we see that

$$\begin{aligned} |x_{n+1} - \hat{x}_{n+1}| &\le |x_{n+1} - T(s_{n+1} - s_n)x_n - (s_{n+1} - s_n)Bx_n| \\ &\quad + |\hat{x}_{n+1} - T(s_{n+1} - s_n)\hat{x}_n - (s_{n+1} - s_n)B\hat{x}_n| \\ &\quad + |T(s_{n+1} - s_n)(x_n - \hat{x}_n) + (s_{n+1} - s_n)(Bx_n - B\hat{x}_n)| \\ &\le |x_n - \hat{x}_n| + (s_{n+1} - s_n)w(|x_n - \hat{x}_n|) + (s_{n+1} - s_n)(\delta + \varepsilon + \hat{\varepsilon}). \end{aligned} \tag{4.8}$$

Since $x_n \in B(x,r) \cap D$ and $\hat{x}_n \in B(\hat{x},\hat{r}) \cap D$, one has $|x_n - \hat{x}_n| \le |x-\hat{x}|+r+\hat{r}$ for $n \ge 0$. From (4.8), we see that

$$\begin{aligned}|x_{n+1} - \hat{x}_{n+1}| \le |x_n - \hat{x}_n| &+ (s_{n+1} - s_n) w^{|x-\hat{x}|+r+\hat{r}}(|x_n - \hat{x}_n|) \\ &+ (s_{n+1} - s_n)(\delta + \varepsilon + \hat{\varepsilon}).\end{aligned} \tag{4.9}$$

Denote

$$\begin{aligned}u_1(t) &= |x_n - \hat{x}_n| + t w^{|x-\hat{x}|+r+\hat{r}}(|x_n - \hat{x}_n|) + t(\delta + \varepsilon + \hat{\varepsilon}), \\ u_2(t) &= m^{|x-\hat{x}|+r+\hat{r}}_{\delta+\varepsilon+\hat{\varepsilon}}(t; |x_n - \hat{x}_n|).\end{aligned}$$

Then

$$\begin{aligned}u_1'(t) &= w^{|x-\hat{x}|+r+\hat{r}}(|x_n - \hat{x}_n|) + (\delta + \varepsilon + \hat{\varepsilon}) \\ &\le w^{|x-\hat{x}|+r+\hat{r}}(u_1(t)) + (\delta + \varepsilon + \hat{\varepsilon}), \\ u_2'(t) &= w^{|x-\hat{x}|+r+\hat{r}}(u_2(t)) + (\delta + \varepsilon + \hat{\varepsilon})\end{aligned}$$

and also

$$u_1(0) = |x_n - \hat{x}_n| = u_2(0).$$

From Theorem 3.1, one obtains that $u_1(t) \le u_2(t)$ for $t \ge 0$, so setting $t = s_{n+1} - s_n$ and using (4.9) we obtain that

$$|x_{n+1} - \hat{x}_{n+1}| \le m^{|x-\hat{x}|+r+\hat{r}}_{\delta+\varepsilon+\hat{\varepsilon}}(s_{n+1} - s_n; |x_n - \hat{x}_n|).$$

Using Lemma 3.1 (iii) and noting that $r \le \varepsilon$, $\hat{r} \le \hat{\varepsilon}$, we obtain by an easy induction argument that

$$|x_{n+1} - \hat{x}_{n+1}| \le m^{|x-\hat{x}|+\varepsilon+\hat{\varepsilon}}_{\delta+\varepsilon+\hat{\varepsilon}}(s_{n+1} - s_0; |x_0 - \hat{x}_0|).$$

Passing to limit as $n \to \infty$ we obtain the required estimate (4.6). $\square$

Remark 4.2. Suppose that (II.a) is satisfied. From Lemma 4.3, one may see using a limiting argument that condition (II.b) is equivalent to its stronger form

$$\begin{aligned}\limsup_{h \downarrow 0} (1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|) \\ \le w(|x-y|) \quad \text{for } x, y \in D.\end{aligned}$$

We are now ready to prove our local existence theorem.

Theorem 4.1. *Suppose that* (II.a) *and* (II.b) *are satisfied. Let $x \in D$ and let $R > 0$, $M > 0$ and $\tau > 0$ be such that $|By| \le M$ for $y \in D \cap B(x,R)$ and $\tau(M+1) + \sup_{t\in[0,\tau]} |T(t)x - x| \le R$. Then there exists a unique mild solution $u(\cdot)$ to* (SP) *on $[0,\tau]$ satisfying the initial condition $u(0) = x$.*

Proof. Let $\varepsilon_0 \in (0, 1/3)$ and let $(\varepsilon_n)_{n\geq 1}$ be any null sequence in $(0, \varepsilon_0)$. Referring to Lemma 4.2, for any $n \geq 1$ one may construct an auxiliary sequence $(r_i^n)_{i=0}^{N_n}$, a time-discretizing sequence $(t_i^n)_{i=0}^{N_n}$ in $[0, \tau]$ and an approximating sequence $(x_i^n)_{i=0}^{N_n}$ in $D \cap B(x, R)$ having properties (i) through (vi) listed in the statement of Lemma 4.2 for $\varepsilon = \varepsilon_n$ and $N = N_n$. Actually, as seen in Iwamiya, Oharu and Takahashi [5], Proposition 5.1, the above sequences may be constructed such that, for $m > n$, the partition $P_m = \{(t_k^m)_{k=0}^{N_m}\}$ is finer than $P_n = \{(t_i^n)_{i=0}^{N_n}\}$. We also note that $|x_k^n - x_l^m| \leq 2R$ for each $1 \leq k \leq N_n$ and $1 \leq l \leq N_m$.

We define a sequence of discrete approximate solutions $u_n(\cdot) : [0, \tau] \to X$ as mentioned before Lemma 4.3, by setting

$$u_n(t) = \begin{cases} T(t - t_i^n)x_i^n + (t - t_i^n)Bx_i^n & \text{for } t \in [t_i^n, t_{i+1}^n) \text{ and } 0 \leq i \leq N_n - 1, \\ T(\tau - t_{N_n-1}^n)x_{N_n-1}^n + (\tau - t_{N_n-1})Bx_{N_n-1}^n & \text{for } t = \tau. \end{cases}$$

It has already been seen that $d(u_n(t), D) \leq \varepsilon_n$ for $t \in [0, \tau]$. We now prove that $(u_n)_{n\geq 1}$ is uniformly convergent on $[0, \tau]$. To this end, let $t \in (0, \tau]$ and $1 \leq n < m$. Choose $0 \leq i < N_n - 1$ and $0 \leq j < N_m - 1$ such that $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$, or let $t = \tau$. We define a subdivision $(s_l)_{l=0}^{j+1}$ of $[0, t]$ by $s_l = t_l^m$ for $0 \leq l \leq j$ and $s_{j+1} = t$. We plan to estimate $|u_m(t) - u_n(t)|$ by using appropriate points $(z_l)_{l=0}^{j+1}$ and $(\hat{z}_l)_{l=0}^{j+1}$ with $z_0 = \hat{z}_0 = x$; these points will be found by Lemma 4.3. To this goal, we note that any $s_l, 0 \leq l \leq j$ is a point of P_m, but not necessarily of P_n, so we have to analyze two distinct situations.

The first situation is the situation in which s_l is a common point of P_m and P_n, that is, $s_l = t_k^n$ for some k. In this case, we may apply Lemma 4.3 for $x = x_k^n$, $\hat{x} = x_l^m$, $y = x$, $\hat{y} = \hat{x}$, $h = \hat{h} = 0$, $\eta = s_{l+1} - s_l$, $\delta = \varepsilon_m$ (note that the required hypotheses are automatically satisfied), and find z_{l+1} and $\hat{z}_{l+1}$ satisfying

$$|z_{l+1} - T(s_{l+1} - s_l)x_k^n - (s_{l+1} - s_l)Bx_k^n| < 2(s_{l+1} - s_l)\varepsilon_n, \tag{4.10}$$

$$|\hat{z}_{l+1} - T(s_{l+1} - s_l)x_l^m - (s_{l+1} - s_l)Bx_l^m| < 2(s_{l+1} - s_l)\varepsilon_m, \tag{4.11}$$

$$|z_{l+1} - \hat{z}_{l+1}| \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}(s_{l+1} - s_l; |x_k^n - x_l^m|). \tag{4.12}$$

The second situation is the situation in which s_l is not a common point for P_n and P_m, that is, $s_l \in (t_k^n, t_{k+1}^n)$ for some k. In this case, we shall employ a different choice of y and use Lemma 4.3 in a somehow "asymmetric" way with respect to the choice of "time-shifting" parameters h and $\hat{h}$. Namely, we let $x = x_k^n$, $\hat{x} = x_l^m$, $y = z_l$, $\hat{y} = \hat{x}$, $h = s_l - t_k^n$, $\hat{h} = 0$, $\eta = s_{l+1} - s_l$ and $\delta = \varepsilon_m$. Also, we note that the interval (t_k^n, t_{k+1}^n) may contain more than one uncommon point s_l. For all the uncommon points s_l, the choice for the intermediate comparison value y in Lemma 4.3 will be the corresponding z_l constructed in the previous step.

We first discuss the applicability of Lemma 4.3 in the second situation. Suppose that $t_k^n = s_{l_0}$. If $l = l_0 + 1$, that is, s_l is the first uncommon point in (t_k^n, t_{k+1}^n), then in order to apply Lemma 4.3 we need to verify first that $|z_l - T(s_l - t_k^n)x_k^n| \le (s_l - t_k^n)(M+1)$. Since the previous time-discretizing point s_{l_0} was a common point for P_n and P_m, (4.10) implies that

$$|z_{l_0+1} - T(s_{l_0+1} - t_k^n)x_k^n - (s_{l_0+1} - t_k^n)Bx_k^n| \le 2(s_{l_0+1} - t_k^n)\varepsilon_n$$

and therefore

$$|z_{l_0+1} - T(s_{l_0+1} - t_k^n)x_k^n| \le (s_{l_0+1} - t_k^n)(M + 2\varepsilon_n) \le (s_{l_0+1} - t_k^n)(M+1).$$

Applying Lemma 4.3 for the above-mentioned choice of parameters, we find z_{l_0+2} and $\hat{z}_{l_0+2}$ such that

$$\begin{aligned} &|z_{l_0+2} - T(s_{l_0+2} - s_{l_0+1})z_{l_0+1} - (s_{l_0+2} - s_{l_0+1})Bz_{l_0+1}| \\ &\qquad\qquad\qquad < 2(s_{l_0+2} - s_{l_0+1})\varepsilon_n \\ &|\hat{z}_{l_0+2} - T(s_{l_0+2} - s_{l_0+1})x_l^m - (s_{l_0+2} - s_{l_0+1})Bx_l^m| < 2(s_{l_0+2} - s_{l_0+1})\varepsilon_m, \\ &|z_{l_0+2} - \hat{z}_{l_0+2}| \le m_{2\varepsilon_m+\varepsilon_n}^{2R+1}(s_{l_0+2} - s_{l_0+1}; |z_{l_0+1} - x_l^m|). \end{aligned}$$

If s_{l_0+2} is a second uncommon point in (t_k^n, t_{k+1}^n), we see that

$$\begin{aligned} &|z_{l_0+2} - T(s_{l_0+2} - t_k^n)x_k^n| \\ &\quad \le |z_{l_0+2} - T(s_{l_0+2} - s_{l_0+1})z_{l_0+1} - (s_{l_0+2} - s_{l_0+1})Bz_{l_0+1}| \\ &\qquad + |T(s_{l_0+2} - s_{l_0+1})z_{l_0+1} - T(s_{l_0+2} - t_k^n)x_k^n| \\ &\qquad + (s_{l_0+2} - s_{l_0+1})(|Bz_{l_0+1} - Bx_k^n| + |Bx_k^n|) \\ &\quad \le 2(s_{l_0+2} - s_{l_0+1})\varepsilon_n + |z_{l_0+1} - T(s_{l_0+1} - t_k^n)x_k^n| \\ &\qquad + (s_{l_0+2} - s_{l_0+1})(M + \varepsilon_n/4) \\ &\quad < (s_{l_0+1} - t_k^n)(M+1) + (s_{l_0+2} - s_{l_0+1})(M + 9/4\varepsilon_n) \end{aligned}$$

and since $0 < \varepsilon_n \le \varepsilon_0 < 1/3$, it is seen that

$$|z_{l_0+2} - T(s_{l_0+2} - t_k^n)x_k^n| < (s_{l_0+2} - s_{l_0})(M+1),$$

so we are again in position to apply Lemma 4.3. The case of a next uncommon point $s_{l_0+3} \in (t_k^n, t_{k+1}^n)$ may be treated in the same way. We have therefore shown that we are also in position to apply Lemma 4.3 and construct the elements z_{l+1} and $\hat{z}_{l+1}$ even if the corresponding time-discretizing point s_l is not a common point for P_n and P_m. Hence if s_l is an uncommon point in (t_k^n, t_{k+1}^n) we may apply Lemma 4.3 and find z_{l+1} and $\hat{z}_{l+1} \in D$ satisfying

$$|z_{l+1} - T(s_{l+1} - s_l)z_l - (s_{l+1} - s_l)Bz_l| < 2(s_{l+1} - s_l)\varepsilon_n \tag{4.13}$$

$$|\hat{z}_{l+1} - T(s_{l+1} - s_l)x_l^m - (s_{l+1} - s_l)Bx_l^m| < 2(s_{l+1} - s_l)\varepsilon_m, \tag{4.14}$$

$$|z_{l+1} - \hat{z}_{l+1}| \le m_{2\varepsilon_m+\varepsilon_n}^{2R+1}(s_{l+1} - s_l; |z_l - x_l^m|). \tag{4.15}$$

As a conclusion, we are able to construct $(z_l)_{l=1}^{j+1}$ and $(\hat{z}_l)_{l=1}^{j+1}$ satisfying either (4.10) to (4.12), if s_l is a common point for P_m and P_n, or (4.13) to

(4.15), if s_l is not a common point for P_m and P_n. With those sequences $(z_l)_{l=1}^{j+1}$ and $(\hat{z}_l)_{l=1}^{j+1}$ at hand, we now estimate $|u_m(t) - u_n(t)|$. One sees that

$$|u_m(t) - u_n(t)| \le |u_m(t) - \hat{z}_{j+1}| + |\hat{z}_{j+1} - z_{j+1}| + |z_{j+1} - u_n(t)|. \tag{4.16}$$

Also, from (4.11) or (4.14), we see that

$$|u_m(t) - \hat{z}_{j+1}| = |\hat{z}_{j+1} - T(t - t_j^m)x_j^m - (t - t_j^m)Bx_j^m| \le 2(t - t_j^m)\varepsilon_m,$$

so we have obtained an estimation for the first term in the right-hand side of (4.16). If t_j^m is a common point for P_m and P_n, one also obtains as above, this time from (4.10), that $|u_n(t) - z_{j+1}| \le 2(t - t_j^n)\varepsilon_n$. Suppose now that t_j^m is not a common point for P_m and P_n, that is, $s_j \in (t_i^n, t_{i+1}^n)$. Then $t_i^n = s_{j_0}$ for some $j_0 < j$. It is easily seen that

$$\begin{aligned}
&|z_{j+1} - T(s_{j+1} - t_i^n)x_i^n - (s_{j+1} - t_i^n)Bx_i^n| \\
&\quad \le \sum_{l=j_0}^{j} |z_{l+1} - T(s_{l+1} - s_l)z_l - (s_{l+1} - s_l)Bz_l| \\
&\qquad + \sum_{l=j_0}^{j} (s_{l+1} - s_l)|Bz_l - Bx_i^n| \\
&\qquad + \sum_{l=j_0}^{j} (s_{l+1} - s_l)|T(s_{j+1} - s_{l+1})Bx_i^n - Bx_i^n| \\
&\quad \le 2\sum_{l=j_0}^{j} (s_{l+1} - s_l)\varepsilon_n + 2\sum_{l=j_0}^{j} (s_{l+1} - s_l)(\varepsilon_n/4) \\
&\quad \le 3\varepsilon_n(s_{j+1} - t_i^n).
\end{aligned} \tag{4.17}$$

Hence we may use the following estimation for the third term in the right-hand side of (4.16)

$$|z_{j+1} - u_n(t)| \le 3(s_{j+1} - t_i^n)\varepsilon_n$$

whether or not t_j^m is a common point of P_n and P_m. It now remains to estimate $|z_{j+1} - \hat{z}_{j+1}|$, and in order to do this we shall employ an recurrent argument.

We first indicate a general estimate which is to be used during our argument. It is easy to see that if $[t_k^n, t_{k+1}^n] = [s_{l_0}, s_{l_1}]$ for some k, l_0, l_1, then, in the same way as in the derivation of (4.17), one obtains

$$\begin{aligned}
|z_{l_1} - x_{k+1}^n| \le &|z_{l_1} - T(s_{l_1} - t_k^n)x_k^n - (s_{l_1} - t_k^n)Bx_k^n| \\
&+ |x_{k+1}^n - T(s_{l_1} - t_k^n)x_k^n - (s_{l_1} - t_k^n)Bx_k^n| \\
\le &4(t_{k+1}^n - t_k^n)\varepsilon_n.
\end{aligned} \tag{4.18}$$

Note that, in this estimate, the interval $[t_k^n, t_{k+1}^n]$, which corresponds to P_n, may or may not contain uncommon points for P_n and P_m, but the estimation does not change, and also that k, l_0, l_1 are not fixed here. Let $0 \le l \le j+1$. From (4.10) or (4.13), and from the construction of the discrete scheme, one sees that

$$\begin{aligned} |\hat{z}_l - x_l^m| &\le |\hat{z}_l - T(s_l - s_{l-1})x_{l-1}^m - (s_l - s_{l-1})Bx_{l-1}^m| \\ &\quad + |x_l^m - T(s_l - s_{l-1})x_{l-1}^m - (s_l - s_{l-1})Bx_{l-1}^m| \\ &\le 3(s_l - s_{l-1})\varepsilon_m, \end{aligned} \tag{4.19}$$

whether or not s_l is a common point for P_m and P_n. If s_l is a common point for P_m and P_n, we see from (4.12), (4.19), (4.18) and Lemma 3.3 that

$$\begin{aligned} |z_{l+1} - \hat{z}_{l+1}| &\le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_{l+1} - s_l, |x_k^n - z_l| + |z_l - \hat{z}_l| + |z_l - \hat{x}_l^m|) \\ &\le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_{l+1} - s_l; |z_l - \hat{z}_l| + 3(s_l - s_{l-1})\varepsilon_m + 4(t_k^n - t_{k-1}^n)\varepsilon_n). \end{aligned}$$

If s_l is not a common point for P_m and P_n, we see from (4.15), (4.19) and Lemma 3.3 that

$$\begin{aligned} |z_{l+1} - \hat{z}_{l+1}| &\le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_{l+1} - s_l; |z_l - \hat{z}_l| + |\hat{z}_l - x_l^m|) \\ &\le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_{l+1} - s_l; |z_l - \hat{z}_l| + 3(s_l - s_{l-1})\varepsilon_m). \end{aligned}$$

Since $|z_1 - \hat{z}_1| \le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_1; 0)$ and also, from Lemmas 3.2 and 3.3,

$$m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(t; m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s; \alpha + \alpha_1) + \alpha_2) \le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(t + s; \alpha + \alpha_1 + \alpha_2)$$

for any $t, s, \alpha, \alpha_1, \alpha_2 \ge 0$, we see by an easy induction argument that

$$|z_{j+1} - \hat{z}_{j+1}| \le m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(s_{j+1}; 4\varepsilon_n t_{i+1}^n + 3\varepsilon_m t_{j+1}^m).$$

Summarizing, if $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$, then

$$\begin{aligned} |u_m(t) - u_n(t)| &\le 2(t - t_j^m)\varepsilon_m + 3(t_{j+1}^m - t_i^n)\varepsilon_n \\ &\quad + m^{2R+1}_{2\varepsilon_m+\varepsilon_n}(t_{j+1}^m; 4\varepsilon_n t_{i+1}^n + 3\varepsilon_m t_{j+1}^m), \end{aligned}$$

and a similar estimate may be obtained for $t = \tau$. The use of Lemma 3.4 yields that $(u_n(t))_{n\ge1}$ is uniformly convergent on $[0, \tau]$ to a function u. Since $d(u_n(t), D) \le \varepsilon_n$ for $t \in [0, \tau]$, it follows that $u(t) \in D$ for $t \in [0, \tau]$. The fact that u is a mild solution to (SP), its continuity and the semigroupal property are easily obtained as in the proof of Theorem 6.1 in Georgescu and Oharu [2], and our local existence theorem is proved.

Using Proposition 4.3, one obtains that there exists a nonlinear semigroup $S = \{S(t); t \ge 0\}$ on D such that for each $x \in D, u(t) = S(t)x$ is a global mild solution to (SP).

It now remains to prove (I.b). Let now $x, y \in D$ and denote $u(t) = S(t)x$, $v(t) = S(t)y$. As done in the proof of Proposition 4.2, it is easily seen that

$$D_+(|u(t) - v(t)|) \le w(|u(t) - v(t)|),$$

and the desired inequality follows from Lemma 3.2 (ii). □

5 A concrete example

In this section we make an attempt to describe the application of our abstract generation results to the study of a concrete semilinear model. Our purpose is to treat the initial value problem formulated as follows:

$$\text{(IVP)} \quad \begin{cases} u_t(t,x) = \Delta_x u(t,x) + g(u(t,x)), & t > 0,\ x \in \mathbb{R}^N, \quad u(t,x) \geq 0; \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N, \quad u_0(x) \geq 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$ and

$$\text{(QLE)} \qquad |g(t) - g(s)| \leq w(|t-s|) \quad \text{for all } t, s \in \mathbb{R},$$

w being an increasing uniqueness function.

Let us denote

$$\begin{aligned} X = C_0(\mathbb{R}^N) &= \{u \in C(\mathbb{R}^N);\ u(x) \to 0 \text{ as } |x| \to \infty\}, \\ C_0^\infty(\mathbb{R}^N) &= \{u \in C^\infty(\mathbb{R}^N);\ u(x) \to 0 \text{ as } |x| \to \infty\}, \\ C_c^\infty(\mathbb{R}^N) &= \{u \in C^\infty(\mathbb{R}^N);\ \operatorname{supp} u \text{ is compact}\}. \end{aligned}$$

For $u \in C_0(\mathbb{R}^N)$, denote also

$$\|u\| = \sup_{x \in \mathbb{R}^N} |u(x)|, \quad m(u) = \inf_{x \in \mathbb{R}^N} u(x), \quad M(u) = \sup_{x \in \mathbb{R}^N} u(x).$$

We attempt to establish the existence of positive mild solutions for the initial value problem (IVP) in $(X, \|\cdot\|)$. To this goal, let us define the operator $A_1 : D(A_1) \subset C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ by

$$D(A_1) = \{u \in C_0^\infty(\mathbb{R}^N); \Delta u \in C_0^\infty(\mathbb{R}^N)\}, \quad A_1 u = \Delta u$$

and denote by A the closure of A_1 in X. Let us also denote

$$D = \{u \in X;\ u(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$$

and define the operator $B : D \to X$ by

$$(Bu)(x) = g(u(x)) \quad \text{for all } u \in X \text{ and } x \in \mathbb{R}^N.$$

In order to apply our abstract generation result, Theorem 2.1, we first show that A is m-dissipative and $D(A) = \{u \in C_0(\mathbb{R}^N);\ \Delta u \in C_0(\mathbb{R}^N)\}$, that is, A is the part of Δ in $C_0(\mathbb{R}^N)$. To this purpose, we prepare the following intermediary estimation.

Lemma 5.1. *For each $\lambda > 0$ and each $u \in R(I - \lambda A_1)$, one has*

$$m(u) \leq [(I - \lambda A_1)^{-1} u](x) \leq M(u) \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof. Fix $\lambda > 0$ and $u \in R(I - \lambda A_1)$. Let $v \in C_0^\infty(\mathbb{R}^N)$ such that $v - \lambda A_1 v = u$. Then $u \in C_0^\infty(\mathbb{R}^N)$ and hence $\lim_{|x|\to\infty} u(x) = 0$, which implies that $m(u) \leq 0$ and $M(u) \geq 0$. Let $\varepsilon > 0$ be arbitrary, but fixed, and set $M_\varepsilon = M(u) + \varepsilon$, $m_\varepsilon = m(u) - \varepsilon$. It is seen that $v - \lambda\Delta v < M_\varepsilon$ for all $x \in \mathbb{R}^N$ and hence $(v - M_\varepsilon) - \lambda\Delta(v - M_\varepsilon) < 0$ for all $x \in \mathbb{R}^N$. Multiplying the last inequality by $(v - M_\varepsilon)^+$, integrating over $\mathbb{R}^N$ and noting that $\operatorname{supp}(v - M_\varepsilon)^+$ is compact, it is seen that

$$\int_{\mathbb{R}^N} |(v - M_\varepsilon)^+|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla(v - M_\varepsilon)^+|^2 dx \leq 0,$$

from which we infer that $v(x) \leq M_\varepsilon$ for all $x \in \mathbb{R}^N$. Similarly, $m_\varepsilon \leq v(x)$ for all $x \in \mathbb{R}^N$. Since $\varepsilon > 0$ was arbitrary, we obtain the desired inequality. □

Lemma 5.2. *For each $\lambda > 0$ and each $u \in R(I - \lambda A)$, one has*

$$m(u) \leq [(I - \lambda A)^{-1}u](x) \leq M(u) \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof. Fix $\lambda > 0$ and $u \in R(I - \lambda A)$. Let $v \in D(A)$ such that $v - \lambda A v = u$. Then $[v, (u - v)/\lambda] \in A$ and, from the definition of the operator A, there exists $([v_n, w_n])_{n\geq 1} \subset A_1$ such that $v_n \to v$ and $w_n \to (u - v)/\lambda$ as $n \to \infty$. Let $u_n = v_n + \lambda w_n$. It is easy to see that $u_n \to u$ and also, from Lemma 5.1, $m(u_n) \leq v_n(x) \leq M(u_n)$ for all $x \in \mathbb{R}^N$. Passing to limit as $n \to \infty$, it is seen that $m(u) \leq v(x) \leq M(u)$ for all $x \in \mathbb{R}^N$, that is, the desired inequality. □

From the above lemma we infer that $\|(I - \lambda A)^{-1}u\| \leq \|u\|$ and hence A is dissipative. Let us define $A_2 : D(A_2) \subset C_0(\mathbb{R}^N) \to C_0(\mathbb{R}^N)$ by

$$D(A_2) = \{u \in C_0(\mathbb{R}^N); \Delta u \in C_0(\mathbb{R}^N)\}, \quad A_2 u = \Delta u,$$

that is, A_2 is the part of Δ in $C_0(\mathbb{R}^N)$. We now attempt to show that $A \equiv A_2$.

First, let us show that A_2 is closed. To this goal, let $([u_n, v_n])_{n\geq 1} \subset A_2$ such that $[u_n, v_n] \to [u, v]$ in $C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)$ as $n \to \infty$. Since $([u_n, v_n])_{n\geq 1} \subset A_2$, by a limiting argument it is seen that $[u, v] \in A$, and it can be inferred from our previous considerations that $[u, v] \in A_2$, which implies that A_2 is closed.

It is easy to see that $A_1 \subset A_2$ and passing to closures in $C_0(\mathbb{R}^N)$ we may infer that $A \subset A_2$. It now remains to show that $A_2 \subset A$. Let $[u, v] \in A_2$. By the definition of A_2, $u, v \in C_0(\mathbb{R}^N)$ and $-\Delta u = v$. Set

$$\rho(x) = \begin{cases} e^{\frac{1}{|x|^2 - 1}} & \text{if } |x| < 1; \\ 0 & \text{if } |x| > 1, \end{cases} \qquad \rho_n(x) = n^N \rho(nx) \Big/ \int_{\mathbb{R}^N} \rho(x)dx.$$

Define $u_n = \rho_n * u$ and $v_n = \rho_n * v$. Since $\rho_n \in C_c^\infty(\mathbb{R}^N)$ and $u, v \in C_0(\mathbb{R}^N)$, it is seen that the regularized functions u_n and v_n belong to $C^\infty(\mathbb{R}^N)$. Also, $u_n \to u$ and $v_n \to v$ uniformly on compact subsets of $\mathbb{R}^N$. Let $\varepsilon > 0$. Since $u \in C_0(\mathbb{R}^N)$, there is $R > 0$ such that $|u(x)| < \varepsilon$ for all $|x| > R$. Then

$|u_n(x)| < \varepsilon$ for each $n \geq 1$ and $x \in \mathbb{R}^N$ with $|x| > R+1$, so $u_n \in C_0(\mathbb{R}^N)$ and similarly $v_n \in C_0(\mathbb{R}^N)$. Since $\Delta u_n = \Delta u * \rho_n = v_n$, it is seen that $[u_n, v_n] \in A_1$ for each $n \geq 1$, which implies that $[u, v] \in A$. Consequently, we also have $A_2 \subset A$, which implies that $A_2 \equiv A$.

We next show that $R(I - A) = C_0(\mathbb{R}^N)$, i.e. A is m-dissipative. First, let us prove that $R(I - A) \supset C_c^\infty(\mathbb{R}^N)$.

Lemma 5.3. *For each $v \in C_c^\infty(\mathbb{R}^N)$ there is $u \in D(A)$ such that $u - Au = v$.*

Proof. Fix $v \in C_c^\infty(\mathbb{R}^N)$ and $p > N$. Then there is $u \in W^{1,p}(\mathbb{R}^N)$ such that $u - \Delta u = v$. Since $v \in C_c^\infty(\mathbb{R}^N)$, by the regularity theorem it is seen that $u \in W^{m,p}(\mathbb{R}^N)$ for all $m \geq 1$. From Morrey's embedding theorem, $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ and so $\lim_{|x|\to\infty} w(x) = 0$ for all $w \in W^{1,p}(\mathbb{R}^N)$. Combining the above, it is seen that $u \in C_0^\infty(\mathbb{R}^N)$, hence $u \in D(A)$ and $u - Au = v$. Moreover, from Lemma 5.2, one has that $-\|v\| \leq u(x) \leq \|v\|$ for all $x \in \mathbb{R}^N$. □

Fix now $v \in C_0(\mathbb{R}^N)$. Then there are $(u_n)_{n\geq 1} \subset C_c^\infty(\mathbb{R}^N)$ and $(v_n)_{n\geq 1} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n - \Delta u_n = v_n$ for all $n \geq 1$ and $v_n \to v$ as $n \to \infty$. By the previous argument, $\|u_n - u_m\| \leq \|v_n - v_m\|$ for all $m, n \geq 1$, which implies the existence of $u \in C_0(\mathbb{R}^N)$ such that $u_n \to u$ as $n \to \infty$. Since $[u_n, u_n - v_n] \in A_1$, passing to limit as $n \to \infty$ it is seen that $[u, u - v] \in A$. Therefore $R(I - A) = C_0(\mathbb{R}^N)$ and A is m-dissipative.

It is easy to see that B is continuous on D. Since w is increasing, using (QLE) one sees that

$$|(Bu)(x) - (Bv)(x)| = |g(u(x)) - g(v(x))| \leq w(\|u - v\|) \quad \text{for all } x \in \mathbb{R}^N$$

and so $\|Bu - Bv\| \leq w(\|u - v\|)$ for all $u, v \in D$. From Proposition 4.1, it is seen that the semilinear stability condition (II.b) is satisfied.

We now prove that the subtangential condition (II.a) holds. Since

$$m(u) \leq [(I - \lambda A)^{-1} u](x) \leq M(u) \quad \text{for all } \lambda > 0,\ u \in X \text{ and } x \in \mathbb{R}^N,$$

it is seen that $(I - \lambda A)^{-1}(D) \subset D$ for all $\lambda > 0$ and so, using the exponential formula, $T(t)D \subset D$ for all $t \geq 0$.

Fix now $u \in D$ and $\varepsilon > 0$. Since g is continuous and $g(0) = 0$, there is $\delta > 0$ such that $|t| \leq \delta$ implies $|g(t)| \leq \varepsilon$. Also, since $u \in C_0(\mathbb{R}^N)$, there is $\eta > 0$ such that

$$\inf_{x\in\mathbb{R}^N} g(u(x)) + \delta/(2h) \geq 0 \quad \text{for all } h \in (0, \eta).$$

Fix $x \in \mathbb{R}^N$. If $|u(x)| \leq \delta$, then

$$(1/h)[(T(h)u)(x) + hg(u(x))] \geq g(u(x)) \geq -\varepsilon.$$

If $|u(x)| > \delta$, then

$$\begin{aligned}(1/h)&[(T(h)u)(x) + hg(u(x))] \\ &= (1/h)[(T(h)u)(x) - u(x)] + (1/h)[u(x) + g(u(x))] \\ &\geq (\delta - \delta/2)/h + g(u(x)) \geq 0.\end{aligned}$$

Therefore

$$(1/h)d(T(h)u + hBu, D) \leq \varepsilon \quad \text{for all } h \in (0, \eta)$$

and so the subtangential condition (II.a) is satisfied. With these preliminaries, our initial value problem (IVP) can be reformulated as a semilinear initial value problem in X, of the form

$$\text{(SP)} \qquad \begin{cases} U'(t) = (A+B)U(t), & t > 0; \\ U(0) = x \in D, \end{cases}$$

where $U(t) = u(t, \cdot)$, and it is seen from our previous considerations that hypotheses (A), (B) and conditions (II.a), (II.b) are satisfied. We can now apply our abstract generation result, Theorem 2.1, and conclude that there exists a semigroup $T = \{T(t); t \geq 0\}$ which provides positive mild solutions for the initial value problem (IVP) for any positive initial data $u_0 \in C_0(\mathbb{R}^N)$.

Acknowledgments

The authors are grateful to Professor Naoki Tanaka for sending them a preprint of his recent work [9].

References

[1] M. Crandall and T. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, *Amer. J. Math.*, **93** (1971), 265-298.

[2] P. Georgescu and S. Oharu, Generation and characterization of locally Lipschitzian semigroups associated with semilinear evolution equations, *Hiroshima Math. J.*, **31** (2001), 133-169.

[3] J. A. Goldstein, S. Oharu and T. Takahashi, Semilinear Hille-Yosida theory: the approximation theorem and groups of operators, *Nonlinear. Anal.*, **13** (1989), 325-339.

[4] T. Iwamiya, Global existence of mild solutions to semilinear differential equations in Banach spaces, *Hiroshima Math. J.*, **16** (1986), 499-530.

[5] T. Iwamiya, S. Oharu and T. Takahashi, Characterization of nonlinearly perturbed semigroups, in Functional Analysis and Related Topics, 1991, Kyoto, 85-102, Lecture Notes in Math., Vol. 1540, Springer, Berlin-New York, 1993.

[6] Y. Kobayashi, Difference approximations of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups, *J. Math. Soc. Japan*, **27** (1975), 640-665.

[7] Y. Kobayashi and N. Tanaka, Nonlinear semigroups and evolution governed by "generalized" dissipative operators, *Adv. Math. Sci. Appl.*, **3** (1993/1994), special issue, 401-426.

[8] V. Laksmikantham and S. Leela, Nonlinear differential equations in abstract spaces, International Series in Nonlinear Mathematics: Theory, Methods and Applications 2, Pergamon Press, Oxford-New York, 1981.

[9] Y. Nakagawa and N.Tanaka, Approximation of nonlinear semigroups generated by operators satisfying dissipative conditions in terms of uniqueness functions, preprint.

[10] G. Webb, Continuous nonlinear perturbations of linear accretive operators in Banach spaces, *J. Funct. Anal.*, **10** (1972), 191-203.

Received September 2005; revised May 2006.