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[2] C.Costara, D.Popa, *Berkeley Preliminary Exams - culegere de probleme*, Ed. Ex-Ponto, Constanţa, 2000.

[3] Gh.Radu, *Algebra categoriilor şi functorilor*, Ed Junimea, Iaşi, 1988.

[4] J.Rotman, *Homological Algebras*, Van Nostrand Reinhold Comp., 1972.

[5] P.N.de Souza, J.N.Silva, *Berkeley Problems in Matematics*, Springer-Verlag, 2000.

[6] M.Ţena, *Algebră - structuri fundamentale pentru liceu*, Ed. Corint, Bucureşti, 1996.

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ON A FIRST ORDER NONLINEAR RECURRENCE DEFINED USING THE MODULUS FUNCTION

by **Paul Georgescu** and **Gabriel Popa**

On concern in this note is the limit of the first order nonlinear recurrence defined by $x_{n+1} = |x_n - a_n|$, $x_1 \in \mathbb{R}$, under some hypotheses on the sequence $(a_n)_{n\geq 1}$, which will be made precise afterwards.

Let $(x_n)_{n\geq 1}$ be the sequence of real numbers defined by:

(R) $$x_{n+1} = |x_n - a_n|,\ n \geq 1,\ x_1 \in \mathbb{R},$$

$(a_n)_{n\geq 1}$ being a sequence of real numbers which satisfies

(P) $$a_n > 0,\ \forall n \in \mathbb{N}^*.$$

We shall treat the following cases:

I. $a_n \to 0$ as $n \to \infty$; II. $(a_n)_{n\geq 1}$ strictly increasing and bounded;
III. $(a_n)_{n\geq 1}$, strictly increasing and unbounded.

In the first case, one obtains the following result.

Theorem 1. *Let $(x_n)_{n\geq 1}$ be the sequence defined by* (R), *where $(a_n)_{n\geq 1}$ is a null sequence which satisfies* (P). *Then $(x_n)_{n\geq 1}$ is convergent.*

Proof. Removing x_1 if necessary, we may suppose that $x_n \geq 0$, $\forall n \in \mathbb{N}^*$. Let us denote $I_1 = \{n \in \mathbb{N}^*;\ x_n \geq a_n\}$ and $I_2 = \{n \in \mathbb{N}^*;\ x_n < a_n\}$.

Case 1. If I_2 is finite, then $\exists n_1 \in \mathbb{N}^*$ such that $x_n \geq a_n$, $\forall n \geq n_1$, so $x_{n+1} = x_n - a_n$, $\forall n \geq 1$. Then $(x_n)_{n\geq 1}$ is an ultimately decreasing sequence of positive numbers and so it is convergent.

Case 2. If I_1 is finite, then $\exists\, n_2 \in \mathbb{N}^*$ such that $0 \le x_n \le a_n, \forall\, n \ge n_2$ and since $(a_n)_{n\ge1}$ is a null sequence, $(x_n)_{n\ge1}$ is also a null sequence.

Case 3. If I_1 and I_2 are both infinite, let us detail I_1 as

$$I_1 = \{x_{n_1}, \dots, x_{n'_1}, x_{n_2}, \dots, x_{n'_2}, \dots\}$$

with $n_k \le n'_k$, $n'_k < n_{k+1}$ and I_1 contains together with x_{n_k} and $x_{n'_k}$ all the intermediary terms x_i, $n_k < i < n'_k$. In order to insure the convergence of $(x_n)_{n\ge1}$, our goal is to conveniently majorize x_i for $i \in I_1$.

Denote $S_k = \{x_{n_k}, \dots, x_{n'_k}\}$. Since $x_{n_k+i+1} = x_{n_k+i} - a_{n_k+i}, \forall\, i \in \in \overline{0, n_{k+1} - n_k - 1}$, we see that $x_{n_k} \ge x_{n_k+1} \ge \dots \ge x_{n'_k}$ for each k. Also, since $x_{n_k} = a_{n_k-1} - x_{n_k}$, one has that $x_{n_k} \le a_{n_k-1}$ for each k, so $x_i \le a_{n_k-1}$, $\forall\, i \in S_k$. Together with the definition of I_2 this obviously yields that $(x_n)_{n\ge1}$ is a null sequence.

Remark 1. It is clear from the above proof that in Cases 2 and 3 one obtains that $\lim\limits_{n\to\infty} x_n = 0$, result which does not necessarily hold in the first case (for instance, if $(a_n)_{n\ge1} = \left(\dfrac{1}{n(n+1)}\right)_{n\ge1}$ and $x_1 > 1$, $(x_n)_{n\ge1}$ is not a null sequence). With respect to this, a necessary and sufficient condition for the finiteness of I_2 is:

(C) $\quad \exists n_0 \in \mathbb{N}$ such that $x_{n_0} \ge \sum\limits_{k=n_0}^{\infty} a_n$.

In view of this remark, we may state the following consequence of Theorem 1.

Corollary 1. *Let $(x_n)_{n\ge1}$ be the sequence defined by* (R), *where $(a_n)_{n\ge1}$ is a null sequence which satisfies* (P) *and also for which $\sum\limits_{n=1}^{\infty} a_n = +\infty$. Then $(x_n)_{n\ge1}$ is a null sequence.*

We now indicate some applications of this result.

Problem 1 (*P.Georgescu, G.Popa*). *Let $(x_n)_{n\ge1}$ be the sequence defined by $x_{n+1} = \left|x_n - b_n f\left(\dfrac{1}{n}\right)\right|$, where $(b_n)_{n\ge1}$ is a sequence such that $0 < m \le b_n \le M, \forall\, n \ge 1$, for some $m, M \in \mathbb{R}$ and $f : (0,\infty) \to (0,\infty)$ is a function such that $\lim\limits_{x\to0_+} \dfrac{f(x)}{x} \in \mathbb{R}^*_+$. Then $\lim\limits_{n\to\infty} x_n = 0$.*

Solution. Here $a_n = b_n f\left(\dfrac{1}{n}\right)$ is a null sequence which satisfies (P). Since the series $\sum\limits_{n=1}^{\infty} b_n \cdot \dfrac{1}{n}$ is divergent, $\sum\limits_{n=1}^{\infty} a_n$ diverges also and we may apply Corollary 1.

Proof. One has $x_{n_0+2k} = x_{n_0} + \sum_{i=0}^{k-1}(a_{n_0+2i+1} - a_{n_0+2i})$. Since $(x_{2k})_{k\geq 1}$ and $(x_{2k+1})_{k\geq 0}$ have finite limits l_1 and l_2 and $x_{n+1} = a_n - x_n, \forall\, n \geq n_0$, we see that $l_1 + l_2 = c$. Therefore, a necessary and sufficient condition for the convergence of $(x_n)_{n\geq 1}$ is $\lim\limits_{n\to\infty} x_{n_0+2k} = \frac{c}{2}$, which finishes the proof.

Remark 2. The convergence condition in Theorem 2 may be reformulated as $x_1 + \sum_{i=0}^{\infty}(a_{n_0+2i+1} - a_{n_o+2i}) = \frac{c}{2} + \sum_{i=1}^{n_0-1} a_i$.

Theorem 2 may be used in solving the following problems.

Problem 3 (*P.Georgescu, G.Popa*). *Prove that the sequence* $(x_n)_{n\geq 1}$ *defined by* $x_{n+1} = \left|x_n - \left(2 - \frac{1}{n}\right)\right|$, $n \geq 1$ *and* $x_1 = 3$, *is divergent.*

Solution. In this case, $n_0 = 3$, $x_{n_0} = \frac{1}{2}$, $c = 2$.

Then, $x_{3+2n} = x_3 + \sum_{i=0}^{n-1}\left(2 - \frac{1}{3+2i+1} - 2 + \frac{1}{3+2i}\right)$ and it is seen that $\lim\limits_{n\to\infty} x_{3+2n} = \ln 2$, so $(x_n)_{n\geq 1}$ is divergent, since for the convergence it would be required to have $\lim\limits_{n\to\infty} x_{3+2n} = \frac{1}{2}\lim\limits_{n\to\infty}\left(2 - \frac{1}{n}\right) = 1$.

Problem 4 (*P.Georgescu, G.Popa*). *Prove that the sequence* $(x_n)_{n\geq 1}$ *defined by* $x_{n+1} = \left|x_n - \left(1 + \frac{1}{n}\right)^n\right|$, $x_1 \in \mathbb{R}$, *diverges for infinitely many* x_1 *and converges for infinitely many* x_1.

Solution. Let $n_0 \geq 5$ be arbitrary, but fixed. In order to have $n_0 = \min\{i; x_i < a_i\}$, x_1 should satisfy:

$$\sum_{i=1}^{n_0-1}\left(1 + \frac{1}{i}\right)^i \leq x_1 < \sum_{i=1}^{n_0}\left(1 + \frac{1}{i}\right)^i. \tag{1}$$

As seen earlier, for the convergence of $(x_n)_{n\geq 1}$ we need also:

$$x_1 + \sum_{i=0}^{\infty}\left(\left(1 + \frac{1}{n_0+2i+1}\right)^{n_0+2i+1} - \left(1 + \frac{1}{n_0+2i}\right)^{n_0+2i}\right) = \sum_{i=0}^{n_0-1}\left(1 + \frac{1}{i}\right)^i + \frac{e}{2}. \tag{2}$$

Given $n_0 \geq 5$, it is seen that (1) and (2) are both satisfied for a single x_1 (which increases as n_0 increases). To see this, we may use the following estimates from [1]:

$$e - \frac{e}{2n+2} > \left(1 + \frac{1}{n}\right)^n > e - \frac{e}{2n+1},\ \forall\, n \geq 3,$$

to conclude that:

Proof. One has $x_{n_0+2k} = x_{n_0} + \sum_{i=0}^{k-1}(a_{n_0+2i+1} - a_{n_0+2i})$. Since $(x_{2k})_{k\geq 1}$ and $(x_{2k+1})_{k\geq 0}$ have finite limits l_1 and l_2 and $x_{n+1} = a_n - x_n, \forall\, n \geq n_0$, we see that $l_1 + l_2 = c$. Therefore, a necessary and sufficient condition for the convergence of $(x_n)_{n\geq 1}$ is $\lim_{n\to\infty} x_{n_0+2k} = \frac{c}{2}$, which finishes the proof.

Remark 2. The convergence condition in Theorem 2 may be reformulated as $x_1 + \sum_{i=0}^{\infty}(a_{n_0+2i+1} - a_{n_o+2i}) = \frac{c}{2} + \sum_{i=1}^{n_0-1} a_i$.

Theorem 2 may be used in solving the following problems.

Problem 3 (*P.Georgescu, G.Popa*). *Prove that the sequence $(x_n)_{n\geq 1}$ defined by $x_{n+1} = \left|x_n - \left(2 - \frac{1}{n}\right)\right|$, $n \geq 1$ and $x_1 = 3$, is divergent.*

Solution. In this case, $n_0 = 3$, $x_{n_0} = \frac{1}{2}$, $c = 2$.

Then, $x_{3+2n} = x_3 + \sum_{i=0}^{n-1}\left(2 - \frac{1}{3+2i+1} - 2 + \frac{1}{3+2i}\right)$ and it is seen that $\lim_{n\to\infty} x_{3+2n} = \ln 2$, so $(x_n)_{n\geq 1}$ is divergent, since for the convergence it would be required to have $\lim_{n\to\infty} x_{3+2n} = \frac{1}{2}\lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 1$.

Problem 4 (*P.Georgescu, G.Popa*). *Prove that the sequence $(x_n)_{n\geq 1}$ defined by $x_{n+1} = \left|x_n - \left(1 + \frac{1}{n}\right)^n\right|$, $x_1 \in \mathbb{R}$, diverges for infinitely many x_1 and converges for infinitely many x_1.*

Solution. Let $n_0 \geq 5$ be arbitrary, but fixed. In order to have $n_0 = \min\{i; x_i < a_i\}$, x_1 should satisfy:

$$\sum_{i=1}^{n_0-1}\left(1+\frac{1}{i}\right)^i \leq x_1 < \sum_{i=1}^{n_0}\left(1+\frac{1}{i}\right)^i. \tag{1}$$

As seen earlier, for the convergence of $(x_n)_{n\geq 1}$ we need also:

$$x_1 + \sum_{i=0}^{\infty}\left(\left(1+\frac{1}{n_0+2i+1}\right)^{n_0+2i+1} - \left(1+\frac{1}{n_0+2i}\right)^{n_0+2i}\right) = \sum_{i=0}^{n_0-1}\left(1+\frac{1}{i}\right)^i + \frac{\mathrm{e}}{2}. \tag{2}$$

Given $n_0 \geq 5$, it is seen that (1) and (2) are both satisfied for a single x_1 (which increases as n_0 increases). To see this, we may use the following estimates from [1]:

$$\mathrm{e} - \frac{\mathrm{e}}{2n+2} > \left(1+\frac{1}{n}\right)^n > \mathrm{e} - \frac{\mathrm{e}}{2n+1},\ \forall\, n \geq 3,$$

to conclude that:

$$\sum_{i=0}^{\infty}\left[\left(1+\frac{1}{n_0+2i+1}\right)^{n_0+2i+1}-\left(1+\frac{1}{n_0+2i}\right)^{n_0+2i}\right]<\frac{6e}{2(n_0+1)+1},$$

so the above sum is majorized by $\frac{e}{2}$ for $n_0 \geq 5$. Therefore one of the estimates in (1) is obtained, the other being obvious.

One, therefore, has infinitely many x_1 for which $(x_n)_{n\geq 1}$ is convergent (a countable subset of $\mathbb{R}$) and also infinitely many x_1 for which $(x_n)_{n\geq 1}$ is divergent (the complementary set of the above).

We now study the case in which $(a_n)_{n\geq 1}$ is strictly increasing and unbounded (that is, Case III).

As seen in Lemma 1, the subsequences $(x_{2k})_{k\geq 1}$ and $(x_{2k+1})_{k\geq 0}$ have limits, finite or not. However, since $\lim\limits_{n\to\infty} a_n = +\infty$, if $(x_n)_{n\geq 1}$ has a limit, this can be only $+\infty$.

Let us denote $T_n = \sum\limits_{k=1}^{n}(-1)^k a_k$. It is obvious that $T_{2k} > 0$ and $T_{2k+1} < 0$ and an easy computation yields that:

$$x_{n_0+k} = (-1)^{n_0+k-1}T_{n_0+k-1} - (-1)^{n_0+k-1}(T_{n_0} + (-1)^{n_0}x_{n_0}), \quad (3)$$

where $n_0 = \min\{i, x_i < a_i\}$.

We now obtain the following result.

Theorem 3. *Let $(x_n)_{n\geq 1}$ be the sequence defined by (R), with $(a_n)_{n\geq 1}$ satisfying* (P) *and being strictly increasing and unbounded. Then, $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if $\lim\limits_{n\to\infty}(-1)^n T_n = +\infty$ and, in this case, $\lim\limits_{n\to\infty}\frac{x_n}{(-1)^n T_n} = +\infty$.*

The proof is straight forward and it is based on (3).

We may use this result to solve the following problem:

Problem 5 ([1], Problem 295, p.105, modified). *Let $x_0 > 0$. Find the limit of the sequence $(x_n)_{n\geq 0}$ which verifies $x_{n+1} = |x_n - n|$, $\forall\, n \in \mathbb{N}$, and prove that $\lim\limits_{n\to\infty}\frac{x_n}{n} = \frac{1}{2}$.*

Solution. Here $(-1)^n T_n = \begin{cases} \frac{n}{2}, & \text{if } n = 2k \\ \frac{n+1}{2}, & \text{if } n = 2k+1 \end{cases}$, so $(-1)^n T_n \to \infty$ (and therefore $x_n \to \infty$ as well), and $\lim\limits_{n\to\infty}\frac{x_n}{(-1)^n T_n} = 1$, which immediately implies that $\lim\limits_{n\to\infty}\frac{x_n}{n} = \frac{1}{2}$.

Remark 3. One does not necessarily obtain that $\lim\limits_{n\to\infty}\frac{x_n}{a_n} = \frac{1}{2}$ for all $(a_n)_{n\geq 1}$; this quotient may not have a limit in some cases. This may be seen if $(a_n)_{n\geq 1} = (n^2)_{n\geq 1}$, for which $\lim\limits_{n\to\infty}\frac{x_{2n}}{a_{2n}} = \frac{1}{4}$ and $\lim\limits_{n\to\infty}\frac{x_{2n+1}}{a_{2n+1}} = \frac{3}{4}$.

The above considerations may be extended to systems of recurrences. We hereby indicate an example.

Problem 6 (*P.Georgescu, G.Popa*). *Let* $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *be the sequences defined by* $\begin{cases} x_{n+1} = \left|y_n - \dfrac{1}{n}\right| \\ y_{n+1} = \left|x_n - \sin\dfrac{1}{n^2}\right| \end{cases}$, $n \geq 1$, $x_1, y_1 \in \mathbb{R}$. *Prove that* $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$.

Solution. We see that $y_{n+2} = \left|\left|y_n - \dfrac{1}{n}\right| - \sin\dfrac{1}{(n+1)^2}\right| = |y_n - c_n|$, with $c_n = \begin{cases} \dfrac{1}{n} + \sin\dfrac{1}{(n+1)^2}, & \text{if } y_n > \dfrac{1}{n} \\ \dfrac{1}{n} - \sin\dfrac{1}{(n+1)^2}, & \text{if } y_n < \dfrac{1}{n} \end{cases}$.

It is seen that $\lim\limits_{n\to\infty} c_n = 0$ and $\sum\limits_{k=1}^{\infty} c_{2k} = \sum\limits_{k=0}^{\infty} c_{2k+1} = \infty$, so Corollary 1 implies that $(y_{2k+1})_{k\geq 0}$ and $(y_{2k})_{k\geq 1}$ are null sequences, so $(y_n)_{n\geq 1}$ is a null sequence. Since $x_{n+1} = \left|y_n - \dfrac{1}{n}\right|$ for $n \geq 1$, we deduce that $(x_n)_{n\geq 1}$ is also a null sequence.

References

[1] Andrei Vernescu, *Sequences of real numbers: a problem book* (Romanian), Bucharest University Publishing House, 2000.

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EXAMENE ŞI CONCURSURI

CONCURSUL DE MATEMATICĂ „FLORICA T.CÂMPAN" FAZA INTERJUDEŢEANĂ, 25 MAI 2002, IAŞI

prezentare de **Dan Brânzei**

O proiectare de competiţie reprezintă un vis, dar poate fi fapt pozitiv dacă gândeşte să măsoare profund calităţi şi deveniri şi să completeze demersuri didactice. Prima întrupare a competiţiei arată ce a fost bun în proiect şi ce se mai poate îmbunătăţi. A doua întrupare, dacă este reuşită, se instituie în săgeată spre viitor. A doua întrupare a fazei interjudeţene a concursului *Florica T. Câmpan* a fost o reuşită: subiecte corecte, uneori ghiduşe, cerând