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Generation and characterization of locally Lipschitzian semigroups associated with semilinear evolution equations

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ABSTRACT. Nonlinear continuous perturbations of (C_0)-semigroups are treated from the point of view of the theory of semigroups of nonlinear operators. Given a (C_0)-semigroup $T(t)$ with generator A in a Banach space X, a general class of nonlinear perturbations is introduced by means of a l.s.c. functional φ. Generation and characterization of locally Lipschitzian semigroups are discussed in terms of semilinear stability condition and subtangential condition. The local Lipschitz continuity and growth condition for the semigroups are restricted by a lower semicontinuous functional φ on a Banach space X under consideration. In the case in which both φ and the domain D of the perturbing operator B are convex, it is shown that the semilinear stability condition is replaced by the standard quasidissipativity condition, and that a Hille-Yosida type theorem is obtained. Moreover, generation and characterization of locally Lipschitzian groups are investigated.

1. Introduction

As widely recognized, the theory of semilinear evolution equations plays an important role in the studies of semilinear problems arising in various fields. Of special interest are generation theorems for the corresponding nonlinear semigroups in terms of necessary and sufficient conditions. A start for the semilinear Hille-Yosida theory in a general Banach space framework was made by Oharu and Takahashi in [13], and their results were significantly generalized in [12], where fundamental properties of the semilinear generators are also investigated.

One of the features of our semilinear Hille-Yosida theory is that generation theorems are not necessarily covered by well-known results concerning nonlinear contraction semigroups. An important example in this sense is a generalized Kortweg-deVries equation, which is treated in [2] and the

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global well-posedness is obtained using a suitable semilinear Hille-Yosida type theorem.

The paper is concerned with semilinear equations of the form

$$u'(t) = (A + B)u(t); \qquad t > 0. \tag{SE}$$

Here A is the generator of a (C_0)-semigroup on a Banach space X and B is a possibly nonlinear operator in X which is defined on a subset D of X. It is assumed that B is continuous in a local sense with respect to a lower semicontinuous functional φ on X such that $D \subset D(\varphi) = \{v \in X; \varphi(v) < \infty\}$. The functional φ also restricts the growth of mild solutions $u(t)$ to (SE) in the sense that $\varphi(u(t))$ satisfies an exponential growth condition. The nonlinear operator B is not necessarily assumed to be quasidissipative by itself. In fact, there are many cases where the nonlinear part B is not quasidissipative but the whole semilinear operator $A + B$ is quasidissipative. Furthermore, the level sets $D_\alpha = \{v \in D; \varphi(v) \le \alpha\}$, $\alpha \ge 0$ are assumed to be closed. In this setting we discuss generation and characterization of a nonlinear semigroup on D which provides solutions to (SE) in a generalized sense.

The generation theorem is treated under a combination of a subtangential condition, a growth condition and a semilinear stability condition. One of the main points of our argument is to deal with the case in which D and φ are not necessarily convex. In the convex case we show that the above-mentioned conditions may be replaced by the well-known range condition, exact exponential growth condition and quasidissipativity condition, and so a semilinear Hille-Yosida theorem is obtained.

In general, the resulting semigroups are not differentiable with respect to t and it is not expected to find their infinitesimal generators in a standard way. In this paper a notion of semilinear infinitesimal generator is introduced, although convexity of the functionals φ and domains D are essential to make it possible to investigate such generators, as discussed in [12]. See also [3], [11] and [13].

This paper is organized as follows: In Section 2 our main results are stated along with remarks and comments. Section 3 deals with the so-called local uniformity of the subtangential condition. In Section 4, this local uniformity is applied to discuss the relationship between the semilinear stability condition and quasidissipativity condition. Moreover, a uniqueness theorem for the mild solution to (SE) is also given. Section 5 is devoted to the construction of approximate solutions to (SP) through a precise discrete scheme consistent with (SP). Our main result is established in Section 6 in the case in which D is not necessarily convex. The corresponding results under convexity assumptions are given in Section 7. Here a generation theorem for nonlinear operator groups is also established.

2. Main results

Let $(X, |\cdot|)$ be a Banach space, D a subset of X and $\varphi : X \to [0, \infty]$ a l.s.c. functional on X such that $D \subset D(\varphi) = \{x \in X; \varphi(x) < \infty\}$. We denote by X^* the dual space of X and, given $x \in X$ and $f \in X^*$, $\langle x, f \rangle$ stands for the value of f at x. The duality mapping of X is the function $F : X \to 2^{X^*}$ defined by $Fx = \{x^* \in X^*; \langle x, x^* \rangle = |x|^2 = |x^*|^2\}$. Given a pair x and y in X, we define the lower and upper semiinner products $\langle y, x \rangle_i$, $\langle y, x \rangle_s$ by the infimum and supremum of the set $\{\langle y, f \rangle; f \in Fx\}$, respectively.

A nonlinear operator $B : D \subset X \to X$ is said to be locally quasidissipative (respectively, strongly locally quasidissipative) on $D(B)$ with respect to φ if for each $\alpha \geq 0$ there exist $\omega_\alpha \in \mathbf{R}$ such that

$$\langle Bx - By, x - y \rangle_i \leq \omega_\alpha |x - y|^2 \qquad \text{for } x, y \in D_\alpha,$$

(respectively,

$$\langle Bx - By, x - y \rangle_s \leq \omega_\alpha |x - y|^2 \qquad \text{for } x, y \in D_\alpha).$$

For further properties of the duality mapping and those of quasidissipative operators, see [10] or [14].

By a locally Lipschitzian semigroup on D with respect to φ is meant a one-parameter family $S = \{S(t); t \geq 0\}$ of (possibly nonlinear) operators from D into itself satisfying the following three conditions below:

(S1) For $x \in D$ and $s, t \geq 0$, $S(s)S(t)x = S(t+s)x$, $S(0)x = x$.

(S2) For $x \in D$, $S(\cdot)x \in C([0, \infty), X)$.

(S3) For each $\alpha > 0$ and each $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbf{R}$ such that

$$|S(t)x - S(t)y| \leq e^{\omega t}|x - y| \qquad \text{for } x, y \in D_\alpha \text{ and } t \in [0, \tau].$$

We consider the semilinear evolution problem

$$\text{(SP)} \qquad u'(t) = (A + B)u(t), \quad t > 0; \qquad u(0) = x \in D.$$

The semilinear problem (SP) may, sometimes, not have strong solutions and the variation of constants formula is employed to obtain solutions in a generalized sense. It is then said that a function $u(\cdot) \in C([0, \infty); X)$ is a mild solution to (SP) if $u(t) \in D$ for $t \geq 0$, $Bu(\cdot) \in C([0, \infty); X)$ and the integral equation

$$u(t) = T(t)x + \int_0^t T(t - s)Bu(s)ds$$

is satisfied for each $t \geq 0$.

In this paper we are concerned with the case in which (SP) is well-posed in the sense of semigroups. We say that a semigroup S is associated with (SP), if it provides mild solutions to (SP) in the sense that for each $x \in D$ the function $u(\cdot) = S(\cdot)x$ is a mild solution to (SP).

In what follows, the operators A and B are assumed to satisfy the following conditions:

(A) $A : D(A) \subset X \to X$ generates a (C_0)-semigroup $T = \{T(t); t \geq 0\}$ on X such that $|T(t)x| \leq e^{\omega t}|x|$ for $x \in X$, $t \geq 0$ and some $\omega \in \mathbf{R}$.

(B) The level set D_α is closed for each $\alpha \geq 0$ and $B : D \subset X \to X$ is continuous on each D_α.

We now state our main result.

THEOREM 1. *Let $a, b \geq 0$ and suppose that* (A) *and* (B) *hold. Then the following statements are equivalent:*

(I) *There is a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D satisfying the following properties:*

(I.1) $S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds$ *for* $t \geq 0$ *and* $x \in D$.

(I.2) *For $\alpha > 0$ and $\tau > 0$ there is $\omega_1 = \omega_1(\alpha, \tau) \in \mathbf{R}$ such that*

$$|S(t)x - S(t)y| \leq e^{\omega_1(\alpha,\tau)t}|x - y|$$

for $x, y \in D_\alpha$ and $t \in [0, \tau]$.

(I.3) $\varphi(S(t)x) \leq e^{at}(\varphi(x) + bt)$ *for* $x \in D$ *and* $t \geq 0$.

(II) *The semilinear operator $A + B$ satisfies the explicit subtangential condition and semilinear stability condition stated below:*

(II.1) *For $x \in D$ and $\varepsilon > 0$ there is $(h, x_h) \in (0, \varepsilon] \times D$ such that*

$$(1/h)|T(h)x + hBx - x_h| \leq \varepsilon \quad \text{and} \quad \varphi(x_h) \leq e^{ah}(\varphi(x) + (b + \varepsilon)h).$$

(II.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\varliminf_{h \downarrow 0}(1/h)[|T(h)(x - y) + h(Bx - By)| - |x - y|] \leq \omega_\alpha|x - y|$$

for $x, y \in D_\alpha$.

Moreover, if the subset D and the functional φ are assumed to be convex, then (I), (II) *and the following statements are equivalent:*

(III) *The semilinear operator $A + B$ satisfies the following density condition, quasidissipativity condition and range condition:*

(III.1) *The domain $D(A + B) = D(A) \cap D$ is dense in D.*

(III.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\langle (A + B)x - (A + B)y, x - y \rangle_i \leq \omega_\alpha|x - y|^2$$

for each $x, y \in D(A) \cap D_\alpha$.

(III.3) *For $\alpha > 0$ there is $\lambda_0 = \lambda_0(\alpha) \in (0, 1/a)$ such that for each $x \in D_\alpha$ and $\lambda \in (0, \lambda_0)$ there is $x_\lambda \in D(A) \cap D$ satisfying*

$$x_\lambda - \lambda(A+B)x_\lambda = x \quad \textit{and} \quad \varphi(x_\lambda) \le (1-a\lambda)^{-1}(\varphi(x)+b\lambda).$$

(IV) *The semilinear operator $A+B$ satisfies the density condition, quasidissipativity condition and implicit subtangential condition which permits errors as stated below*

(IV.1) *$D(A)\cap D$ is dense in D.*

(IV.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\langle (A+B)x - (A+B)y, x-y\rangle_i \le \omega_\alpha |x-y|^2$$

for $x, y \in D(A)\cap D_\alpha$.

(IV.3) *For $\alpha > 0$ and $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\alpha,\varepsilon)$ such that for $\lambda \in (0,\lambda_0)$ and $x \in D_\alpha$ there exist $x_\lambda \in D(A)\cap D$ and $z_\lambda \in X$ satisfying $|z_\lambda| < \varepsilon$,*

$$x_\lambda - \lambda(A+B)x_\lambda = x + \lambda z_\lambda,$$

$$\varphi(x_\lambda) \le (1-\lambda a)^{-1}(\varphi(x) + (b+\varepsilon)\lambda).$$

(V) *The semilinear operator $A+B$ satisfies the quasidissipativity condition and sequential implicit subtangential condition stated below:*

(V.1) *For each $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\langle (A+B)x - (A+B)y, x-y\rangle_i \le \omega_\alpha |x-y|^2$$

for $x, y \in D(A)\cap D_\alpha$.

(V.2) *For each $x \in D$ there exists a null sequence $\{h_n\}$ of positive numbers and a sequence $\{x_n\}$ in $D(A)\cap D$ such that*

(V.2a) $\lim_{n\to\infty} (1/h_n)|x_n - h_n(A+B)x_n - x| = 0,$

(V.2b) $\overline{\lim_{n\to\infty}} (1/h_n)[\varphi(x_n) - \varphi(x)] \le a\varphi(x) + b,$

(V.2c) $\lim_{n\to\infty} |x_n - x| = 0.$

The equivalence between (I) and (III) may be regarded as a semilinear version of Hille-Yosida theorem, while the implication from (II) to (I) gives a generation theorem in the nonconvex case. The application of our result to semilinear approximation theory will be discussed in the forthcoming paper [1]. It is important to establish Theorem 1 in general Banach spaces. In fact, in many significant evolution equations such as quasilinear conservation laws and convection reaction-diffusion systems are formulated in the nonreflexive Lebesgue space $L^1(\Omega)$. Also, a Kisyński space $c(X)$ employed in a convergence theorem is always nonreflexive even if X is a Hilbert space. Since the implicit subtangential condition stated in (IV) permits errors, the implication from (IV) to (I) is particularly useful for establishing the well-posedness of concrete evolution problems. A similar result is obtained in [5] under much stronger assumptions, although our paper is affected by the work done in [5]. Also, some of the results and methods devised in [4], [9], [11] and [16] are applied here. It should be also noted that the representation $A+B$ of a semilinear operator must be restricted in such a way that $BS(\cdot)x \in C([0,\infty);X)$ for x in D. In particular, A may be written $(A+wI)-wI$. Hence, in what follows we assume that A generates a (C_0)-contraction semigroup for the sake of simplicity.

3. Local uniformity in the subtangential condition

In the proof of our main theorem it is straightforward to check the implication from (I) to (II). In fact, by Lemma 3.1 in [12], $\lim_{h\downarrow 0}(1/h)|S(h)x - T(h)x - hBx| = 0$ for each $x \in D$. This shows that for each $\varepsilon > 0$ there is $h_\varepsilon \in (0,\varepsilon]$ such that $(1/h)|S(h)x - T(h)x - hBx| \le \varepsilon$ for $h \in (0,h_\varepsilon]$. Moreover, $\varphi(S(h)x) \le e^{ah}(\varphi(x)+bh)$, and it follows that (II.1) holds for $x_h = S(h)x$. In order to derive (II.2), we observe that

$$\begin{aligned}
&(1/h)(|T(h)(x-y)+h(Bx-By)| - |x-y|)\\
&\quad\le (1/h)(|T(h)x+hBx-S(h)x| + |T(h)y+hBy-S(h)y|)\\
&\qquad+(1/h)(|S(h)x-S(h)y| - |x-y|).
\end{aligned}$$

Passing to the inferior limit as $h\downarrow 0$ we get

$$\begin{aligned}
&\varliminf_{h\downarrow 0}(1/h)(|T(h)(x-y)+h(Bx-By)| - |x-y|)\\
&\quad\le \varliminf_{h\downarrow 0}(1/h)(|S(h)x-S(h)y| - |x-y|)\\
&\quad\le \left(\lim_{h\downarrow 0}\ \omega_1(\alpha,h)\right)|x-y|
\end{aligned}$$

for $x, y \in D_\alpha$, which shows that (II.2) is satisfied.

Therefore, the most of the proof must be devoted to show the implication from (II) to (I). To this end, we necessitate making full use of the subtangential condition (II.1) together with the continuity of B on a level set D_α. In this section we discuss the so-called local uniformity of the subtangential condition (II.1).

THEOREM 3.1. *Suppose that* (II.1) *hold. Let* $x \in D$, $\varepsilon \in (0,1)$, $\beta > \varphi(x)$, *and let* $r = r(x,\beta,\varepsilon)$ *be chosen such that*

$$|Bx - By| \le \varepsilon/4 \quad \textit{and} \quad \sup_{s\in[0,r]} |T(s)Bx - Bx| \le \varepsilon/4 \tag{3.1}$$

for each $y \in D_\beta \cap B(x,r)$,

where $B(x,r) = \{y; |y-x| \le r\}$. *We then choose* $M \ge 0$ *satisfying*

$$(3.2) \qquad |By| \le M \qquad \text{for each } y \in D_\beta \cap B(x,r),$$

and define

$$(3.3) \qquad h(x,\beta,\varepsilon) = \sup\left\{h > 0; h(M+1) + \sup_{s \in [0,h]} |T(s)x - x| \le r \text{ and } e^{ah}(\varphi(x) + (b+\varepsilon)h) \le \beta\right\}.$$

Let $h \in [0, h(x,\beta,\varepsilon))$ *and* $y \in D$ *satisfy*

$$(3.4) \qquad |y - T(h)x| \le h(M+1) \qquad \text{and} \qquad \varphi(y) \le e^{ah}(\varphi(x) + (b+\varepsilon)h).$$

Then for each $\eta > 0$ *with* $h + \eta \le h(x,\beta,\varepsilon)$ *there is* $z \in D_\beta \cap B(x,r)$ *satisfying*

$$(3.5) \qquad (1/\eta)|z - T(\eta)y - \eta B(y)| \le \varepsilon \qquad \text{and} \qquad \varphi(z) \le e^{a\eta}(\varphi(y) + (b+\varepsilon)\eta).$$

Proof. Let $h \in [0, h(x,\beta,\varepsilon))$ and $y \in D$ satisfy (3.4). The definition (3.3) means that $h(x,\beta,\varepsilon)$ specifies a maximal forward time interval which does not exceed the modulus of continuity, $r(x,\beta,\varepsilon)$, of B. We often use the fact that $h \le r$ which follows from (3.3).

We then find the desired elements z by constructing a sequence $\{s_n\}_{n\ge 0}$ in $[0,\eta]$ and the associated sequence $\{x_n\}_{n\ge 0}$ in D such that

(i) $s_0 = 0,\ x_0 = y,\ 0 < s_{n-1} < s_n < \eta;$

(ii) $\lim_{n\to\infty} s_n = \eta,\ \lim_{n\to\infty} x_n = z;$

(iii) $|x_n - T(s_n - s_{n-1})x_{n-1} - (s_n - s_{n-1})Bx_{n-1}| \le (\varepsilon/4)(s_n - s_{n-1});$

(iv) $|x_n - T(s_n + h)x| \le (s_n + h)(M+1)$

(v) $\varphi(x_n) \le e^{a(s_n - s_{n-1})}(\varphi(x_{n-1}) + (b+\varepsilon/4)(s_n - s_{n-1}));$

(vi) $\varphi(x_n) \le e^{as_n}(\varphi(x_0) + (b+\varepsilon/4)s_n);$

(vii) $x_n \in D_\beta \cap B(x,r)$

for each $n \ge 0$. Here, (i) is considered for $n \ge 1$; estimates (iii) and (v) are not formulated for $n = 0$.

The proof is given through an induction argument. First, x_0 satisfies (iv) by the first inequality in (3.4). Estimate (vi) is trivial for $n = 0$. From (3.3) and (3.4) one obtains

$$|y - x| \le h(M+1) + |T(h)x - x| \le r \qquad \text{and} \qquad \varphi(y) \le \beta.$$

Hence (iv), (vi) and (vii) are satisfied for $n = 0$. Estimates (iii) and (v) are not considered for $n = 0$. The first step of the induction argument is completed in this sense.

Suppose now that $\{s_n\}_{n=0}^{N}$ and $\{x_n\}_{n=0}^{N}$ have been constructed in such a way that (i) and (iii) through (vii) hold for $0 \le n \le N$. We first note that, by (vii), $|Bx_N| \le M$. Then by (II.1) one finds $\xi \in (0, \eta)$ and $x_{N,\xi} \in D$ such that $s_N + \xi < \eta$ and

$$(1/\xi)|x_{N,\xi} - T(\xi)x_N - \xi Bx_N| \le \varepsilon/4, \tag{3.6}$$

$$\varphi(x_{N,\xi}) \le e^{a\xi}(\varphi(x_N) + (b + \varepsilon/4)\xi).$$

Let $\bar{h}_N$ be the supremum of such numbers ξ; hence $\bar{h}_N > 0$. We then choose an appropriate number $h_N \in (\bar{h}_N/2, \bar{h}_N)$ and set $s_{N+1} = s_N + h_N$. Also, we define x_{N+1} to be an element $x_{N,\xi}$ which is obtained for $\xi = h_N$ by (3.6). It should be noted that s_{N+1} and x_{N+1} are constructed without properties (iii) and (iv) with $n = N$ (which do not make sense for $n = 0$). It is seen from the construction that (iii) and (v) hold for $n = N + 1$. Also, applying (iii), we have

$$\begin{aligned} |x_{N+1} - T(s_{N+1} - s_N)x_N| &\le |x_{N+1} - T(s_{N+1} - s_N)x_N - (s_{N+1} - s_N)Bx_N| \\ &\quad + (s_{N+1} - s_N)|Bx_N| \\ &\le (s_{N+1} - s_N)(M + 1). \end{aligned}$$

Condition (iv) with $n = N$ then implies

$$\begin{aligned} |x_{N+1} - T(s_{N+1} + h)x| &\le |x_{N+1} - T(s_{N+1} - s_N)x_N| \\ &\quad + |T(s_{N+1} - s_N)x_N - T(s_{N+1} + h)x| \\ &\le (s_{N+1} - s_N)(M + 1) + |x_N - T(s_N + h)x| \\ &\le (s_{N+1} + h)(M + 1), \end{aligned}$$

which shows that (iv) is valid for $n = N + 1$. Moreover, the above estimate implies

$$\begin{aligned} |x_{N+1} - x| &\le |x_{N+1} - T(s_{N+1} + h)x| + |T(s_{N+1} + h)x - x| \\ &\le (s_{N+1} + h)(M + 1) + |T(s_{N+1} + h)x - x| \end{aligned}$$

and so

$$|x_{N+1} - x| \le (\eta + h)(M + 1) + \sup_{s \in [0, \eta + h]} |T(s)x - x| \le r. \tag{3.7}$$

Since (v) holds for $n = N + 1$, we have

$$e^{-as_{n+1}}\varphi(x_{n+1}) \le e^{-as_n}\varphi(x_n) + e^{-as_n}(b + \varepsilon/4)(s_{n+1} - s_n) \qquad \text{for } 0 \le n \le N. \tag{3.8}$$

Summing up the inequalities in (3.8) with respect to $n = 0, \ldots, N$ gives

$$\varphi(x_{N+1}) \le e^{as_{N+1}}\varphi(x_0) + \sum_{k=0}^{N} e^{a(s_{N+1}-s_k)}(b+\varepsilon/4)(s_{k+1}-s_k) \tag{3.9}$$

$$\le e^{as_{N+1}}(\varphi(x_0) + (b+\varepsilon/4)(s_{N+1}-s_0)),$$

which implies the desired estimate (vi) for $n = N+1$. Finally, combining (3.4) and (vi) for $n = N+1$ we see that

$$\varphi(x_{N+1}) \le e^{a(s_{N+1}+h)}(\varphi(x) + (b+\varepsilon)(s_{n+1}+h)).$$

This, together with (3.3) implies that $\varphi(x_{N+1}) \le \beta$ and, since $|x_{N+1} - x| \le r$, x_{N+1} satisfies (vii) for $n = N+1$. Thus we may extend the sequences $\{s_n\}_{n=0}^N$ and $\{x_n\}_{n=0}^N$ up to $N+1$. By induction, it is concluded that we can construct a sequence $\{s_n\}_{n\ge 0}$ in $[0,\eta]$ and $\{x_n\}_{n\ge 0}$ in D with the properties (i) and (iii) through (vii).

It now remains to prove that (ii) holds for the sequences $\{s_n\}_{n\ge 0}$ and $\{x_n\}_{n\ge 0}$ constructed above. To this goal, we need the following lemmas given in [5].

Lemma 3.1. *Let $\{s_n\}_{n\ge 0}$ be a nondecreasing sequence and $\{x_n\}_{n\ge 0}$ a sequence in D. Then the following identity holds:*

$$x_n - T(s_n - s_0)x_0 - \sum_{k=0}^{n-1}(s_{k+1}-s_k)T(s_n - s_{k+1})Bx_k \tag{3.10}$$

$$= \sum_{k=0}^{n-1} T(s_n - s_{k+1})[x_{k+1} - T(s_{k+1}-s_k)x_k - (s_{k+1}-s_k)Bx_k].$$

Lemma 3.2. *Let $\{s_n\}_{n\ge 0}$ be a nondecreasing sequence and $\{x_n\}_{n\ge 0}$ a sequence in D satisfying $|Bx_n| \le M$ and*

$$|x_{n+1} - T(s_{n+1}-s_n)x_n - (s_{n+1}-s_n)Bx_n| \le (s_{n+1}-s_n)\varepsilon$$

for $n \ge 0$. If $s_n \uparrow s$ as $n \to \infty$, then the sequence $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in X.

We now verify (ii) by contradiction. In view of the construction of the sequence $\{s_n\}_{n\ge 0}$, we see that s_n converges to some $s \le \eta$. Hence, by Lemma 3.2 the sequence $\{x_n\}_{n\ge 0}$ in D_β is convergent in X to some z and $z \in D_\beta$ by the closedness of D_β.

Suppose then that $s < \eta$. Let $\delta \in (0, \eta - s)$. Then we may apply (II.1) to find an element $z_\delta \in D$ such that

$$(1/\delta)|T(\delta)z + \delta Bz - z_\delta| \le \varepsilon/5, \tag{3.11}$$

and

$$\varphi(z_\delta) \le e^{a\delta}(\varphi(z) + (b + \varepsilon/5)\delta). \tag{3.12}$$

Let N be an integer such that $s - s_n \le \delta/2$ for $n \ge N$. Let $n \ge N$ and let $\bar{h}_n$ be the supremum of $h > 0$ such that $s_n + h < \eta$ and (3.6) holds for N replaced by n, as it was considered earlier. In the construction of the sequences $\{s_n\}_{n \ge 0}$ and $\{x_n\}_{n \ge 0}$ we have chosen $h_n \in (\bar{h}_n/2, \bar{h}_n)$ and set $s_{n+1} = s_n + h_n$ and $x_{n+1} = x_{N,h}$ with $n = N$ and $h = h_n$. Hence $0 < \bar{h}_n < 2h_n < 2(s - s_n) \le \delta < \eta - s$, and so $s_n + \delta < s_n + \eta - s < \eta$. By the maximality of $\bar{h}_n$, this means that we must have either $(1/\delta)|z_\delta - T(\delta)x_n - \delta B x_n| > (\varepsilon/4)$, or $\varphi(z_\delta) > e^{a\delta}(\varphi(x_n) + (b + \varepsilon/4)\delta)$ for infinitely many $n \ge N$. Passing here to limit as $n \to \infty$ we get either $(1/\delta)|z_\delta - T(\delta)z - \delta B z| \ge \varepsilon/4$ or $\varphi(z_\delta) \ge e^{a\delta}(\varphi(z) + (b + \varepsilon/4)\delta)$, which contradicts either (3.12) or (3.11).

Thus it is concluded that $\lim_{n\to\infty} s_n = \eta$. Finally, we demonstrate that the element z satisfies (3.5). Using Lemma 3.1, we obtain

$$\begin{aligned}
&x_n - T(s_n)y - s_n B y \\
&\quad = \sum_{k=0}^{n-1} T(s_n - s_{k+1})[x_{k+1} - T(s_{k+1} - s_k)x_k - (s_{k+1} - s_k)Bx_k] \\
&\qquad + \sum_{k=0}^{n-1} (s_{k+1} - s_k) T(s_n - s_{k+1})[Bx_k - Bx] \\
&\qquad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)(T(s_n - s_{k+1})Bx - Bx) + s_n(Bx - By)
\end{aligned}$$

and so, by (iii)

$$\begin{aligned}
&|x_n - T(s_n)y - s_n B y| \\
&\quad \le \sum_{k=0}^{n-1} (s_{k+1} - s_k)\varepsilon/4 + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|Bx_k - Bx| \\
&\qquad + \sum_{k=0}^{n-1} (s_{k+1} - s_k)|T(s_n - s_{k+1})Bx - Bx| + s_n|Bx - By|.
\end{aligned} \tag{3.13}$$

In view of (3.3) and the fact that $s_n < \eta$ for all n, we see that $s_n - s_{k+1} < h(x, \beta, \varepsilon) \le r(x, \beta, \varepsilon)$ for $0 \le k \le n-1$ and $n \ge 1$. Hence combining (3.1) and (3.13) implies

$$|x_n - T(s_n)y - s_n B y| \le s_n\varepsilon/4 + s_n\varepsilon/4 + s_n\varepsilon/4 + s_n\varepsilon/4$$

Letting $n \to \infty$, we obtain

$$(1/\eta)|z - T(\eta)y - \eta By| \le \varepsilon$$

Also, it is seen from (vi) that

$$\varphi(z) \le e^{a\eta}(\varphi(y) + (b + \varepsilon/4)\eta).$$

Thus it is concluded that z is the desired element. The proof of Theorem 3.1 is now complete. □

REMARK 3.1. If in particular $h = 0$ and $y = x$ in Theorem 3.1, then the above assertion states that for every $\eta > 0$ with $\eta \le h(x, \beta, \varepsilon)$ there is $z \in D_\beta \cap B(x, r)$ such that

$$(1/\eta)|z - T(\eta)x - \eta Bx| \le \varepsilon, \qquad \varphi(z) \le e^{a\eta}(\varphi(x) + (b + \varepsilon)\eta).$$

This means that we can make full use of the subtangential condition (II.1) coupled with the continuity of B on level sets D_α, as we will see in the following sections.

4. The semilinear stability condition

The explicit subtangential condition (II.1) is essentially weaker than the range condition (III.3). Therefore it is natural to impose a stronger condition than the standard quasidissipativity condition (IV.2) in order to compensate the gap. Following the previous work [5], we here employ condition (II.2), which we call the semilinear stability condition. In order to compare these conditions, we first show that if B itself is assumed to be locally quasidissipative with respect to the l.s.c. functional φ, then the semilinear stability condition (II.2) follows from the subtangential condition (II.1).

THEOREM 4.1. *Suppose that A is ω_A-dissipative and for each $\alpha > 0$ B is $\omega_{B,\alpha}$-quasidissipative on D_α. Then* (II.1) *implies* (II.2) *with* $\omega_\alpha = \omega_A + \varliminf_{\beta \downarrow \alpha} \omega_{B,\beta}$.

PROOF. Let $\alpha > 0$, $x, y \in D_\alpha$, $\beta > \alpha$ and $\varepsilon > 0$. Then, as mentioned in Remark 3.1, one finds $h \in (0, \varepsilon]$ and x_h, $y_h \in D_\beta$ such that

$$(1/h)|T(h)x + hBx - x_h| \le \varepsilon, \qquad (1/h)|T(h)y + hBy - y_h| \le \varepsilon \tag{4.1}$$

$$|Bx - Bx_h| \le \varepsilon, \qquad |By - By_h| \le \varepsilon \tag{4.2}$$

$$\varphi(x_h) \le e^{ah}(\varphi(x) + (b + \varepsilon)h), \qquad \varphi(y_h) \le e^{ah}(\varphi(y) + (b + \varepsilon)h). \tag{4.3}$$

Hence we obtain

$$(4.4)\qquad \begin{aligned}&(1/h)(|T(h)(x-y)+h(Bx-By)|-|x-y|)\\ &\quad\le(1/h)(|T(h)x+hBx-x_h|+|T(h)y+hBy-y_h|)\\ &\qquad+(1/h)(|x_h-y_h|-|x-y|)\\ &\quad\le 2\varepsilon+(1/h)(|x_h-y_h|-|x-y|).\end{aligned}$$

Since B is quasidissipative on D_β, we have

$$|x_h-y_h|\le(1-h\omega_{B,\beta})^{-1}|(x_h-y_h)-h(Bx_h-By_h)|.$$

Writing $x_h-y_h-h(Bx_h-By_h)$ as $(x_h-T(h)x-hBx)-(y_h-T(h)y-hBy)-h(Bx_h-Bx)+h(By_h-By)+(T(h)x-T(h)y)$ and applying (4.1), (4.2), we obtain

$$(4.5)\qquad |x_h-y_h|\le(1-h\omega_{B,\beta})^{-1}[e^{\omega_A h}|x-y|+4h\varepsilon].$$

Combining (4.4) and (4.5) gives

$$(4.6)\qquad \begin{aligned}&(1/h)(|T(h)(x-y)+h(Bx-By)|-|x-y|)\\ &\quad\le 2\varepsilon+(e^{\omega_A h}|x-y|+4h\varepsilon-(1-h\omega_{B,\beta})|x-y|)/(h(1-h\omega_{B,\beta}))\\ &\quad\le 2\varepsilon+(e^{\omega_A h}-1)|x-y|/(h(1-h\omega_{B,\beta}))\\ &\qquad+4\varepsilon/(1-h\omega_{B,\beta})+\omega_{B,\beta}|x-y|/(1-h\omega_{B,\beta}).\end{aligned}$$

Taking the inferior limit on both sides of (4.6) as $h\downarrow 0$ we have

$$\varliminf_{h\downarrow 0}(1/h)(|T(h)(x-y)+h(Bx-By)|-|x-y|)\le 6\varepsilon+(\omega_A+\omega_{B,\beta})|x-y|.$$

Since ε and β are arbitrary so far as $\varepsilon>0$ and $\beta>\alpha$, we obtain the desired result. □

The next proposition states that the semilinear stability condition (II.2) implies the so-called strong quasidissipativity condition.

THEOREM 4.2. *If $x,y\in D\cap D(A)$, we have*

$$\begin{aligned}&\lim_{h\downarrow 0}(1/h)[|T(h)(x-y)+h(Bx-By)|-|x-y|]|x-y|\\ &\qquad=\langle(A+B)x-(A+B)y,x-y\rangle_s.\end{aligned}$$

PROOF. Let $x,y\in D(A)\cap D$ and fix any $x^*\in F(x-y)$. Then

$$\begin{aligned}&(1/h)(|T(h)(x-y)+h(Bx-By)|-|x-y|)|x-y|\\ &\quad\ge(1/h)(\langle T(h)(x-y)+h(Bx-By),x^*\rangle-\langle x-y,x^*\rangle)\\ &\quad=\langle(1/h)(T(h)x-x)+Bx-(1/h)(T(h)y-y)-By,x^*\rangle\end{aligned}$$

for $h > 0$. Hence

$$\underline{\lim}_{h \downarrow 0}[(1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|)]|x-y|$$
$$\geq \langle (A+B)x - (A+B)y, x^* \rangle.$$

Since x^* was arbitrary, we get

$$\underline{\lim}_{h \downarrow 0}(1/h)[|T(h)(x-y) + h(Bx - By)| - |x-y|]|x-y| \tag{4.7}$$
$$\geq \langle (A+B)x - (A+B)y, x-y \rangle_s$$

In order to derive the desired identity, we choose any element $x_h^* \in F(T(h)(x-y) + h(Bx - By))$ for $h \in (0,1]$. Then there is an upper bound $M > 0$ such that

$$|x_h^*| = |T(h)(x-y) + h(Bx - By)| \leq M$$

for $h \in (0,1]$. Hence, by Alaoglu's theorem, the generalized sequence $\{x_h^*\}_{h \geq 0}$ has a weak* cluster point x^*. Since $T(h)(x-y) + h(Bx - By) \to x - y$ in X as $h \downarrow 0$, it follows that for any $\varepsilon > 0$ there is $h_\varepsilon \in (0, \varepsilon)$ such that

$$|\langle x-y, x^* \rangle - \langle T(h_\varepsilon)(x-y) + h_\varepsilon(Bx - By), x_{h_\varepsilon}^* \rangle| < \varepsilon.$$

This implies that

$$\langle x-y, x^* \rangle = \lim_{h \downarrow 0} |T(h)(x-y) + h(Bx - By)|^2 = |x-y|^2. \tag{4.8}$$

On the other hand, for any $v \in X$ and $\varepsilon > 0$, there is $h(\varepsilon) \in (0, \varepsilon)$ such that $|\langle v, x^* - x_{h(\varepsilon)}^* \rangle| < \varepsilon$. Hence

$$|\langle v, x^* \rangle| \leq |\langle v, x_{h(\varepsilon)}^* \rangle| + \varepsilon \leq |v|\,|x_{h(\varepsilon)}^*| + \varepsilon$$
$$\leq |v|\,|T(h(\varepsilon))(x-y) + h(\varepsilon)(Bx - By)| + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have

$$|\langle v, x^* \rangle| \leq |v|\,|x-y|.$$

This shows that

$$|x^*| \leq |x-y|,$$

and so that $x^* \in F(x-y)$ by (4.8). Since

$$(1/h)(|T(h)(x-y) + h(Bx - By)| - |x-y|)|T(h)(x-y) + h(Bx - By)|$$
$$\leq \langle (1/h)(T(h)x - x) + Bx - (1/h)(T(h)y - y) - By, x_h^* \rangle,$$

we see in the same way as in the derivation of (4.8) that

$$(4.9)\qquad \overline{\lim_{h\downarrow 0}}(1/h)[|T(h)(x-y)+h(Bx-By)|-|x-y|]|x-y| \le \langle (A+B)x-(A+B)y, x^*\rangle \le \langle (A+B)x-(A+B)y, x-y\rangle_s.$$

Combining (4.7) and (4.9) we obtain the desired relation. □

We now demonstrate that the semilinear stability condition (II.2) guarantees the uniqueness of mild solutions to the semilinear problem (SP).

Theorem 4.3. *Suppose that the semilinear stability condition* (II.2) *holds. Let* $u(\cdot)$ *and* $v(\cdot)$ *be mild solutions with initial data* $u(0), v(0)$ *in* D_α*, respectively. Then*

$$|u(t)-v(t)| \le e^{\omega_\beta t}|u(0)-v(0)|$$

for $\tau > 0$, $\beta > e^{a\tau}(\alpha + b\tau)$ *and* $t \in [0,\tau]$.

Proof. Let $t \in [0,\tau]$. In view of the definition of mild solutions to (SP) we have

$$\begin{aligned}
&(1/h)(|u(t+h)-v(t+h)|-|u(t)-v(t)|)\\
&\quad = (1/h)\left|T(h)(u(t)-v(t)) + \int_t^{t+h} T(t+h-s)(Bu(s)-Bv(s))ds\right|\\
&\qquad - (1/h)|u(t)-v(t)|\\
&\quad \le (1/h)(|T(h)(u(t)-v(t)) + h(Bu(t)-Bv(t))| - |u(t)-v(t)|)\\
&\qquad + (1/h)\int_t^{t+h} |T(t+h-s)Bu(s)-Bu(t)|ds\\
&\qquad + (1/h)\int_t^{t+h} |T(t+h-s)Bv(s)-Bv(t)|ds
\end{aligned}$$

and so

$$\begin{aligned}
(4.10)\quad &\underline{\lim}_{h\downarrow 0}(1/h)(|u(t+h)-v(t+h)|-|u(t)-v(t)|)\\
&\quad \le \underline{\lim}_{h\downarrow 0}(1/h)(|T(h)(u(t)-v(t)) + h(Bu(t)-Bv(t))| - |u(t)-v(t)|)\\
&\qquad + \overline{\lim_{h\downarrow 0}}(1/h)\int_t^{t+h} |T(t+h-s)Bu(s)-Bu(t)|ds\\
&\qquad + \overline{\lim_{h\downarrow 0}}(1/h)\int_t^{t+h} |T(t+h-s)Bv(s)-Bv(t)|ds.
\end{aligned}$$

Since $\varphi(u(s)) \le e^{as}(\varphi(u(0)) + bs) \le \beta$ for $s \in [0, \tau + h]$ and $h > 0$ sufficiently small, the continuity of B on D_β implies

$$\overline{\lim_{h\downarrow 0}}(1/h)\int_t^{t+h} |T(t+h-s)Bu(s) - Bu(t)|ds = 0 \tag{4.11}$$

and

$$\overline{\lim_{h\downarrow 0}}\int_t^{t+h} |T(t+h-s)Bv(s) - Bv(t)|ds = 0. \tag{4.12}$$

The relations (4.10), (4.11) and (4.12) together imply

$$\underline{D}_+|u(t) - v(t)| \le \omega_\beta |u(t) - v(t)|,$$

or

$$\underline{D}_+(e^{-\omega_\beta t}|u(t) - v(t)|) \le 0. \tag{4.13}$$

From this we obtain the desired result

$$|u(t) - v(t)| \le e^{\omega_\beta t}|u(0) - v(0)|. \qquad \square$$

5. Construction of the approximate solutions

In this section we discuss the construction of approximate solutions to the problem (SP) in terms of method of discretization in time. First, we prepare a result for constructing local approximate solutions.

THEOREM 5.1. *Suppose that condition* (II.1) *is satisfied. Let* $x \in D$, $R > 0$, $\beta > \varphi(x)$ *and let* $M > 0$ *be such that* $|By| \le M$ *for* $y \in D_\beta \cap B(x, R)$. *Let* $\tau > 0$ *and* $\varepsilon_0 \in (0, 1)$ *be chosen so that*

$$\tau(M+1) + \sup_{t\in[0,\tau]} |T(t)x - x| \le R \qquad \text{and} \qquad e^{a\tau}(\varphi(x) + (b + \varepsilon_0)\tau) < \beta.$$

Then for each $\varepsilon \in (0, \varepsilon_0]$ *there exist a sequence* $\{t_i\}_{i=0}^N$ *and a sequence* $\{x_i\}_{i=0}^N$ *in* $D_\beta \cap B(x, R)$ *such that*

(i) $t_0 = 0$, $x_0 = x$, $t_N = \tau$;

(ii) $0 < t_{i+1} - t_i \le \varepsilon$ *for* $0 \le i \le N - 1$;

(iii) $|x_i - T(t_i)x| \le t_i(M+1)$ *and* $\varphi(x_i) \le e^{at_i}(\varphi(x) + (b+\varepsilon)t_i)$ *for* $0 \le i \le N$;

(iv) $|x_{i+1} - T(t_{i+1} - t_i)x_i - (t_{i+1} - t_i)Bx_i| \le (t_{i+1} - t_i)\varepsilon$ *and* $\varphi(x_{i+1}) \le e^a(t_{i+1} - t_i)(\varphi(x_i) + (b+\varepsilon)(t_{i+1} - t_i))$ *for* $0 \le i \le N - 1$;

(v) $x_i \in D_\beta \cap B(x, R)$ *for* $0 \le i \le N$;

(vi) *For* $0 \le i \le N - 1$ *there is* $r_i \in (0, \varepsilon]$ *such that* $|By - Bx_i| \le \varepsilon/4$ *for* $y \in B(x_i, r_i) \cap D_\beta$, $\sup_{t\in[0,r_i]} |T(t)Bx_i - Bx_i| \le \varepsilon/4$ *and* $(t_{i+1} - t_i)(M+1) +$

$$\sup_{t \in [0, t_{i+1} - t_i]} |T(t)x_i - x_i| \le r_i.$$

PROOF. First, it is interesting to compare the above assertion with the proof of Theorem 3.1. In Theorem 3.1 η and z in (3.5) were attained by infinite sequences $\{s_n\}_{n \ge 1}$ and $\{x_n\}_{n \ge 1}$, since uniformity in the subtangential condition with respect to h and y is required. Here τ and x can be attained in a finite number of steps $\{(s_i, x_i)\}_{i=0}^N$. Obviously, τ is understood to be small and so we obtain only local existence.

Set $t_0 = 0$ and $x_0 = x$. Suppose that $\{t_i\}_{i=0}^n$ and $\{x_i\}_{i=0}^n$ have been constructed in such a way that conditions (i) through (vi) are fulfilled. We then define

$$r_n = \sup\left\{ r \in (0, \varepsilon]; |By - Bx_n| \le \varepsilon/4 \text{ for } y \in D_\beta \cap B(x_n, r), \right. \\ \left. \text{and } \sup_{s \in [0, r]} |T(s)Bx_n - Bx_n| \le \varepsilon/4 \right\}. \tag{5.1}$$

and

$$\eta_n = \sup\left\{ t > 0; t(M+1) + \sup_{s \in [0, t]} |T(s)x_n - x_n| \le r_n \text{ and} \right. \\ \left. e^{at}(\varphi(x_n) + (b + \varepsilon)t) \le \beta \right\}. \tag{5.2}$$

Referring to the proof of Theorem 3.1, we define $h_n = \min(\tau - t_n, \eta_n)$ and $t_{n+1} = t_n + h_n$. Applying Theorem 3.1 with $h = 0$, $\eta = h_n$, $y = x = x_n$ and $r = r_n$, one finds $x_{n+1} \in D_\beta \cap B(x_n, r_n)$ satisfying

$$|x_{n+1} - T(h_n)x_n - h_nBx_n| \le \varepsilon h_n \qquad \text{and} \qquad \varphi(x_{n+1})e^{ah_n}(\varphi(x_n) + (b + \varepsilon)h_n).$$

By the induction hypothesis (i) through (vi), it is easily seen that

$$\begin{aligned} |x_{n+1} - T(t_{n+1})x| &\le h_n + t_n(M+1) + h_nM \\ &= t_{n+1}(M+1) \end{aligned}$$

and thereby

$$|x_{n+1} - x| \le t_{n+1}(M+1) + |T(t_{n+1})x - x| \le \tau(M+1) + \sup_{s \in [0, \tau]} |T(s)x - x| \le R$$

and

$$\begin{aligned} \varphi(x_{n+1}) &\le e^{a(t_{n+1} - t_n)}(\varphi(x_n) + (b + \varepsilon)(t_{n+1} - t_n)) \\ &\le e^{at_{n+1}}(\varphi(x) + (b + \varepsilon)t_{n+1}). \end{aligned}$$

Thus we have constructed sequences $\{t_i\}_{i=0}^{n+1}$ and $\{x_i\}_{i=0}^{n+1}$ satisfying (i) through (vi). It now remains to show that τ can be attained in some finite, say N, steps. Suppose to the contrary that this is not the case. Then we would obtain two infinite sequences $\{t_i\}_{i\geq 0}$ in $[0,\tau)$ and $\{x_i\}_{i\geq 0}$ in $D_\beta \cap B(x,R)$. Hence x_i would converge to some $z \in D_\beta \cap B(x,R)$. It should be noted here that $\varphi(z) < \beta$ by (iii).

Let $\bar{r} \in (0, \varepsilon/2)$ and $\bar{k}_\varepsilon \geq 1$ be such that $|T(s)Bz - Bz| \leq \varepsilon/12$ for $s \in [0,\bar{r}]$,

$$|By - Bz| \leq \varepsilon/6 \qquad \text{for each } y \in D_\beta \cap B(z,\bar{r}) \tag{5.3}$$

and

$$|Bx_k - Bz| \leq \varepsilon/12 \qquad \text{and} \qquad |x_k - z| \leq \min(\varepsilon/12, \bar{r}/16) \tag{5.4}$$

for each $k \geq \bar{k}_\varepsilon$. Then $|T(s)Bx_k - Bx_k| \leq |T(s)Bz - Bz| + 2|Bx_k - Bz| \leq \varepsilon/4$ for each $k \geq \bar{k}_\varepsilon$ and $s \in [0,\bar{r}]$. Also, (5.4) implies that $B(x_k, \bar{r}/2) \cap D_\beta \subseteq B(z,\bar{r}) \cap D_\beta$ for $k \geq \bar{k}_\varepsilon$. Hence, for $k \geq \bar{k}_\varepsilon$ and $y \in D_\beta \cap B(x_k, \bar{r}/2)$, we have

$$\begin{aligned} |By - Bx_k| &\leq |By - Bz| + |Bx_k - Bz| \\ &\leq \varepsilon/6 + \varepsilon/12 = \varepsilon/4 \end{aligned} \tag{5.5}$$

by (5.3) and (5.4). These estimates together show that $r_k \geq \bar{r}/2$ for $k \geq \bar{k}_\varepsilon$.

We next choose $\delta = \delta(z,\bar{r})$ so that $|T(s)z - z| \leq \bar{r}/8$ for $s \in [0,\delta]$. By the choice of τ, there is $\bar{\delta} > 0$ such that

$$e^{a(\tau+\bar{\delta})}(\varphi(x) + (b+\varepsilon)(\tau+\bar{\delta})) \leq \beta. \tag{5.6}$$

In view of the construction of the sequence $\{t_i\}_{i\geq 0}$, we see that t_i converges to some $t \leq \tau$. In order to derive the contradiction, we put $\delta_0 = \tau + \bar{\delta} - t > 0$ and define

$$\xi = \min(\bar{r}/4(M+1), \delta, \delta_0). \tag{5.7}$$

First, it is seen that $t_k + \xi \leq \tau + \bar{\delta}$ and

$$e^{a\xi}(\varphi(x_n) + (b+\varepsilon)\xi) \leq e^{a(t_n+\xi)}(\varphi(x) + (b+\varepsilon)(t_n+\xi)) \leq \beta.$$

From (5.4), (5.6) and (5.7) we deduce

$$\begin{aligned} \sup_{s\in[0,\xi]} |T(s)x_k - x_k| &\leq \sup_{s\in[0,\xi]} (|T(s)x_k - T(s)z| + |T(s)z - z| + |z - x_k|) \\ &\leq \sup_{s\in[0,\xi]} (2|x_k - z| + |T(s)z - z|) \\ &\leq 2\bar{r}/16 + \bar{r}/8 = \bar{r}/4 \end{aligned}$$

for $k \ge \overline{k}_\varepsilon$ and $\xi(M+1) \le \bar{r}/4$. These estimates together imply

$$\xi(M+1) + \sup_{s \in [0,\xi]} |T(s)x_k - x_k| \le \bar{r}/2 \le r_k.$$

This means that $\eta_k \ge \xi$ for $k \ge \overline{k}_\varepsilon$. We now recall that $h_k = \min\{\tau - t_k, \eta_k\}$. If there is $k_0 \ge \overline{k}_\varepsilon$ such that $\eta_{k_0} \ge \tau - t_{k_0}$, then $h_{k_0} = \tau - t_{k_0}$. This implies that $\tau = t_{k_0} + h_{k_0} = t_{k_0+1}$. This is a contradiction. If $\eta_k < \tau - t_k$ for $k \ge \overline{k}_\varepsilon$, then $h_k = \eta_k \ge \xi$ for $k \ge \overline{k}_\varepsilon$. This implies that τ can be reached in a finite number of steps. This is again a contradiction. It is then concluded that τ is attained by some t_N, and the proof of Theorem 5.1 is completed. □

Using the finite sequences $\{t_i\}_{i=0}^N$ in $[0,\tau]$ and $\{x_i\}_{i=0}^N$ in $D_\beta \cap B(x,R)$ obtained for $x \in D$ by Theorem 5.1, we may define an approximate solution $u_\varepsilon : [0,\tau] \to X$ to (SP) by

$$(5.8) \quad u_\varepsilon(t) = \begin{cases} T(t-t_i)x_i + (t-t_i)Bx_i & \text{for } t \in [t_i, t_{i+1}),\ 0 \le i \le N-1 \\ T(\tau - t_{N-1})x_{N-1} + (\tau - t_{N-1})Bx_{N-1} & \text{for } t = \tau. \end{cases}$$

Then for $t \in [t_i, t_{i+1})$ and $0 \le i \le N-1$ we have

$$\begin{aligned} x_{i+1} - u_\varepsilon(t) &= [x_{i+1} - T(t_{i+1} - t_i)x_i - (t_{i+1} - t_i)Bx_i] \\ &\quad + [T(t_{i+1} - t_i)x_i - T(t - t_i)x_i] + (t_{i+1} - t)Bx_i \end{aligned}$$

and hence

$$\begin{aligned} |x_{i+1} - u_\varepsilon(t)| &\le (t_{i+1} - t_i)\varepsilon + |T(t_{i+1} - t)x_i - x_i| + (t_{i+1} - t)|Bx_i| \\ &\le (t_{i+1} - t_i)(M+1) + |T(t_{i+1} - t)x_i - x_i| \le \varepsilon \end{aligned}$$

by conditions (ii), (iv), (v) and (vi). Also, noting that $\tau = t_N$, we obtain

$$\begin{aligned} |x_N - u_\varepsilon(\tau)| &= |x_N - T(\tau - t_{N-1})x_{N-1} - (\tau - t_{N-1})Bx_{N-1}| \\ &\le (\tau - t_{N-1})\varepsilon \le \varepsilon. \end{aligned}$$

In the next section we demonstrate that for any null sequence $\{\varepsilon_n\}$ of positive numbers the sequence of the corresponding approximate solutions $\{u_{\varepsilon_n}\}$ on the interval $[0,\tau]$ converges uniformly to a mild solution of (SP) satisfying the exponential growth condition with respect to φ. To this end, one necessitates preparing a method for estimating the difference between approximate solutions. The first step for this is to establish the following theorem which may be regarded as a "coupled" subtangential condition.

In order to formulate the statement, we introduce four kinds of quantities which depend upon choices of base data $x, \hat{x} \in D$ and error bounds $\varepsilon, \hat{\varepsilon} \in (0, 1/3)$. Let a and b be constants given in condition (II) and fix any pair $x, \hat{x} \in D$ and any pair $\varepsilon, \hat{\varepsilon} \in (0, 1/3)$. First, we choose β so that

$$\beta > \max\{\varphi(x), \varphi(\hat{x})\}. \tag{5.9}$$

Next, by the continuity of B on D_β, one finds positive numbers $r = r(x, \beta, \varepsilon)$, $\hat{r} = r(\hat{x}, \beta, \hat{\varepsilon})$, $M(x, \beta, \varepsilon)$ and $M(\hat{x}, \beta, \hat{\varepsilon})$ such that

$$|By - Bx| \le \varepsilon/4 \quad \text{and} \quad |By| \le M(x, \beta, \varepsilon) \quad \text{for } y \in D_\beta \cap B(x, r), \tag{5.10}$$

$$|By - B\hat{x}| \le \hat{\varepsilon}/4 \quad \text{and} \quad |By| \le M(\hat{x}, \beta, \hat{\varepsilon}) \quad \text{for } y \in D_\beta \cap B(\hat{x}, \hat{r}). \tag{5.10$'$}$$

We then choose M so that

$$M \ge \max\{M(x, \beta, \varepsilon), M(\hat{x}, \beta, \hat{\varepsilon})\}.$$

Since the function $s \mapsto T(s)Bx$ is strongly continuous, we may assume that

$$\sup_{s \in [0, r]} |T(s)Bx - Bx| \le \varepsilon/4. \tag{5.11}$$

Also, in view of (5.9), there exists $h > 0$ such that

$$h(M+1) + \sup_{s \in [0, h]} |T(s)x - x| \le r \quad \text{and} \quad e^{ah}(\varphi(x) + (b+\varepsilon)h) \le \beta. \tag{5.12}$$

In view of this, we define $h(x, \beta, \varepsilon)$ by

$$h(x, \beta, \varepsilon) = \sup\left\{h > 0; h(M+1) + \sup_{s \in [0, h]} |T(s)x - x| \le r \text{ and } e^{ah}(\varphi(x) + (b+\varepsilon)h) \le \beta\right\}. \tag{5.13}$$

We are now in a position to state the comparison theorem.

Theorem 5.2. *Suppose that conditions* (II.1) *and* (II.2) *are satisfied. Let $x, \hat{x}$ be any pair of base data in D and $\varepsilon, \hat{\varepsilon}$ a pair of error bounds in* $(0, 1/3)$. *Let $h \in [0, h(x, \beta, \varepsilon))$, $\hat{h} \in [0, h(\hat{x}, \beta, \hat{\varepsilon}))$ and $y, \hat{y} \in D$ be chosen so that*

$$|y - T(h)x| \le h(M+1), \qquad \varphi(y) \le e^{ah}(\varphi(x) + (b+\varepsilon)h) \tag{5.14}$$

and

$$|\hat{y} - T(\hat{h})\hat{x}| \le \hat{h}(M+1), \qquad \varphi(\hat{y}) \le e^{a\hat{h}}(\varphi(\hat{x}) + (b+\hat{\varepsilon})\hat{h}). \tag{5.14$'$}$$

Then for each $\delta > 0$ and each $\eta > 0$ satisfying $h + \eta \le h(x, \beta, \varepsilon)$ and $\hat{h} + \eta \le h(\hat{x}, \beta, \hat{\varepsilon})$ there exist $z \in D_\beta \cap B(x, r)$ and $\hat{z} \in D_\beta \cap B(\hat{x}, \hat{r})$ such that

(5.15) $$|z - T(\eta)y - \eta By| < 2\eta\varepsilon, \qquad \varphi(z) \le e^{a\eta}(\varphi(y) + (b+\varepsilon)\eta),$$

(5.15)′ $$|\hat{z} - T(\eta)\hat{y} - \eta B\hat{y}| < 2\eta\hat{\varepsilon}, \qquad \varphi(\hat{z}) \le e^{a\eta}(\varphi(\hat{y}) + (b+\hat{\varepsilon})\eta),$$

and the elements $z, \hat{z}$ *satisfy*

(5.16) $$|z - \hat{z}| \le e^{\omega_\beta \eta}|y - \hat{y}| + \eta e^{\bar{\omega}_\beta \eta}(\varepsilon + \hat{\varepsilon} + \delta)$$

where $\bar{\omega}_\beta = \max\{\omega_\beta, 0\}$.

Proof. First, we note that $y \in D_\beta \cap B(x,r)$ and $\hat{y} \in D_\beta \cap B(\hat{x},\hat{r})$ since

$$|y - x| \le |y - T(h)x| + |T(h)x - x| \le h(M+1) + |T(h)x - x| \le r$$

by (5.12) and the corresponding estimate for $\hat{x}$, $\hat{y}$, $\hat{h}$ and $\hat{r}$. Let $\eta > 0$ be such that $h + \eta \le h(x,\beta,\varepsilon)$ and $\hat{h} + \eta \le h(\hat{x},\beta,\hat{\varepsilon})$.

We then demonstrate that three sequences $\{s_n\}_{n\ge0}$, $\{x_n\}_{n\ge0}$ and $\{\hat{x}_n\}_{n\ge0}$ can be inductively constructed in such a way that

(i) $s_0 = 0$, $x_0 = y$, $\hat{x}_0 = \hat{y}$;

(ii) $0 < s_{n-1} < s_n$ and $\lim_{n\to\infty} s_n = \eta$;

(iii) $|x_n - T(s_n - s_{n-1})x_{n-1} - (s_n - s_{n-1})Bx_{n-1}| \le (s_n - s_{n-1})\varepsilon$;

(iii)′ $|\hat{x}_n - T(s_n - s_{n-1})\hat{x}_{n-1} - (s_n - s_{n-1})B\hat{x}_{n-1}| \le (s_n - s_{n-1})\hat{\varepsilon}$;

(iv) $\varphi(x_n) \le e^{a(s_n - s_{n-1})}(\varphi(x_{n-1}) + (b+\varepsilon)(s_n - s_{n-1}))$;

(iv)′ $\varphi(\hat{x}_n) \le e^{a(s_n - s_{n-1})}(\varphi(\hat{x}_{n-1}) + (b+\hat{\varepsilon})(s_n - s_{n-1}))$;

(v) $e^{-\omega_\beta(s_n - s_{n-1})}|T(s_n - s_{n-1})(x_{n-1} - \hat{x}_{n-1}) + (s_n - s_{n-1})(Bx_{n-1} - B\hat{x}_{n-1})|$
$\le |x_{n-1} - \hat{x}_{n-1}| + (s_n - s_{n-1})\delta$;

(vi) $|x_n - T(s_n)x_0| \le s_n(M+1)$;

(vi)′ $|\hat{x}_n - T(s_n)\hat{x}_0| \le s_n(M+1)$;

(vii) $\varphi(x_n) \le e^{a(s_n + h)}(\varphi(x) + (s_n + h)(b+\varepsilon))$;

(vii)′ $\varphi(\hat{x}_n) \le e^{a(s_n + \hat{h})}(\varphi(\hat{x}) + (b+\hat{\varepsilon})(s_n + \hat{h}))$;

(viii) $x_n \in B(x,r) \cap D_\beta$;

(viii)′ $\hat{x}_n \in B(\hat{x},\hat{r}) \cap D_\beta$

hold for each $n \ge 0$. The estimates (iii) and (iii)′ insure the convergence of the sequences $\{x_n\}$ and $\{\hat{x}_n\}$ and the inequalities (iv) and (iv)′ lead us to the exponential growth condition for mild solutions. The estimates (v) will be used to obtain (5.16).

First, we infer from (5.14) and (5.14)′ that (vi) through (viii)′ are all valid for $n = 0$. Estimates (iii) through (v) are not formulated for $n = 0$. In the same way as in the proof of Theorem 3.1, we complete the first step of the induction argument in this sense.

We then suppose that three finite sequences $\{s_n\}_{n=0}^N$, $\{x_n\}_{n=0}^N$ and $\{\hat{x}_n\}_{n=0}^N$ have been constructed in such a way that (i), (iii) through (viii) and the corresponding (iii)′, (iv)′ and (vi)′ through (viii)′ are satisfied.

Let $\overline{h}_N$ be the supremum of the positive numbers ξ such that $s_N + \xi \le \eta$ and

$$e^{-\omega_\beta \xi}|T(\xi)(x_N - \hat{x}_N) + \xi(Bx_N - B\hat{x}_N)| \le |x_N - \hat{x}_N| + \xi\delta. \tag{5.17}$$

We then fix any $h_N \in (\overline{h}_N/2, \overline{h}_N)$ and put $s_{N+1} = s_N + h_N$. We note that $\varphi(x_N) < \beta$ and $\varphi(\hat{x}_N) < \beta$ by (vii) and (vii)′. Hence we may apply Theorem 3.1 to find x_{N+1}, $\hat{x}_{N+1} \in D$ satisfying

$$|x_{N+1} - T(s_{N+1} - s_N)x_N - (s_{N+1} - s_N)Bx_N| \le (s_{N+1} - s_N)\varepsilon, \tag{5.18}$$

$$|\hat{x}_{N+1} - T(s_{N+1} - s_N)\hat{x}_N - (s_{N+1} - s_N)B\hat{x}_N| \le (s_{N+1} - s_N)\hat{\varepsilon} \tag*{(5.18)′}$$

and

$$\varphi(x_{N+1}) \le e^{a(s_{N+1} - s_N)}(\varphi(x_N) + (b + \varepsilon)(s_{N+1} - s_N)), \tag{5.19}$$

$$\varphi(\hat{x}_{N+1}) \le e^{a(s_{N+1} - s_N)}(\varphi(\hat{x}_N) + (b + \hat{\varepsilon})(s_{N+1} - s_N)), \tag*{(5.19)′}$$

This shows that s_{N+1}, x_{N+1} and $\hat{x}_{N+1}$ are constructed without properties (iii) through (v), and that s_{N+1}, x_{N+1} and $\hat{x}_{N+1}$ satisfy (iii) through (iv)′ with $n = N + 1$. Then, letting $\xi = h_N$ in (5.17) we see that (v) is satisfied for $n = N + 1$. Our next aim is to show that the constructed s_{N+1}, x_{N+1} and $\hat{x}_{N+1}$ satisfy (vi) through (viii), respectively (vi)′ through (viii)′ for $n = N + 1$. Applying (5.10) and (5.18) we obtain

$$\begin{aligned} |x_{N+1} - T(s_{N+1})x_0| &\le (s_{N+1} - s_N)M + |x_N - T(s_N)x_0| + (s_{N+1} - s_N)\varepsilon \\ &\le s_{N+1}(M + 1) \end{aligned}$$

and, in the same way $|\hat{x}_{N+1} - T(s_{N+1})\hat{x}_0| < s_{N+1}(M + 1)$. This proves that x_{N+1} and $\hat{x}_{N+1}$ satisfy (vi) respectively (vi)′ for $n = N + 1$. Combining (vi) for $n = N + 1$ with (5.14), one obtains

$$\begin{aligned} &|x_{N+1} - T(s_{N+1} + h)x| \\ &\quad \le |x_{N+1} - T(s_{N+1})x_0| + |T(s_{N+1})x_0 - T(s_{N+1} + h)x| \\ &\quad \le (s_{N+1} + h)(M + 1) \end{aligned} \tag{5.20}$$

and, in the same way as above,

$$|\hat{x}_{N+1} - T(s_{N+1} + \hat{h})\hat{x}| \le (s_{N+1} + \hat{h})(M+1). \tag{5.21}$$

Therefore, in view of the definitions of $r(x, \beta, \varepsilon)$ and $r(\hat{x}, \beta, \hat{\varepsilon})$, we infer from (5.20) and (5.21) that

$$|x_{N+1} - x| < (s_{N+1} + h)(M+1) + |T(s_{N+1} + h)x - x| \le r(x, \beta, \varepsilon)$$

and $|\hat{x}_{N+1} - \hat{x}| < \hat{r}(\hat{x}, \beta, \hat{\varepsilon})$. This shows that x_{N+1} and $\hat{x}_{N+1}$ satisfy (viii) respectively (viii)$'$ for $n = N+1$. To show that x_{N+1} and $\hat{x}_{N+1}$ satisfy (vii) respectively (vii)$'$ for $n = N+1$, we apply (5.19) and the induction hypothesis to get

$$\begin{aligned}\varphi(x_{N+1}) &< e^{a(s_{N+1}-s_N)}(e^{a(s_N+h)}(\varphi(x) + (b+\varepsilon)(s_N + h)) + (b+\varepsilon)(s_{N+1} - s_N)) \\ &< e^{a(s_{N+1}+h)}(\varphi(x) + (b+\varepsilon)(s_{N+1} + h))\end{aligned}$$

and

$$\varphi(\hat{x}_{N+1}) \le e^{a(s_{N+1}+\hat{h})}(\varphi(\hat{x}) + (b+\hat{\varepsilon})(s_{N+1} + \hat{h})).$$

It now remains to show that $\lim_{n\to\infty} s_n = \eta$. Suppose to the contrary that $\lim_{n\to\infty} s_n = s < \eta$. Then, by Lemma 3.2, there would exist some elements z, $\hat{z} \in D$ such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} \hat{x}_n = \hat{z}$. Since D_β is closed, z, $\hat{z} \in D_\beta$. More precisely, it follows from the above relations that $\varphi(z) < \beta$ and $\varphi(\hat{z}) < \beta$. On the other hand, by the semilinear stability condition (II.2), there must exist some $\xi \in (0, \eta - s)$ such that

$$e^{-\omega_\beta \xi}|T(\xi)(z - \hat{z}) + \xi(Bz - B\hat{z})| \le |z - \hat{z}| + (1/2)\xi\delta, \tag{5.22}$$

where δ is the number employed in the estimate (v). Choose $N \ge 1$ so that $s - s_n \le \xi/2$ for each $n \ge N$. Set $\xi_n = s - s_n + \xi$. Then $s_n + \xi_n = s + \xi < \eta$ and $\bar{h}_n < 2h_n < 2(s - s_n) < \xi_n$. Hence it would follow from (5.17) that

$$e^{-\omega_\beta \xi_n}|T(\xi_n)(x_n - \hat{x}_n) + \xi_n(Bx_n - B\hat{x}_n)| > |x_n - \hat{x}_n| + \xi_n\delta$$

for $n \ge N$. Now letting $n \to \infty$ implies

$$e^{-\omega_\beta \xi}|T(\xi)(z - \hat{z}) + \xi(Bz - B\hat{z})| \ge |z - \hat{z}| + \xi\delta.$$

This contradicts (5.22) and hence $\lim_{n\to\infty} s_n = \eta$. We now demonstrate that the limits z and $\hat{z}$ are the desired elements. First, using (iv) and recalling $x_0 = y$, we have

$$\varphi(z) \le e^{a\eta}(\varphi(y) + (b+\varepsilon)\eta)$$

and, in the same way,

$$\varphi(\hat{z}) \le e^{a\eta}(\varphi(\hat{y}) + (b+\hat{\varepsilon})\eta).$$

Next, by Lemma 3.1 we see that $x_n - T(s_n)y - s_n By$ can be rewritten as

$$\sum_{k=0}^{n-1} T(s_n - s_{k+1})[x_{k+1} - T(s_{k+1} - s_k)x_k - (s_{k+1} - s_k)Bx_k]$$
$$+ \sum_{k=0}^{n-1}(s_{k+1} - s_k)T(s_n - s_{k+1})Bx_k - s_n By.$$

Hence, applying this property we have

$$\begin{aligned}
&|x_n - T(s_n)y - s_n By| \\
&\quad \le \sum_{k=0}^{n-1} |T(s_n - s_{k+1})|\,|x_{k+1} - T(s_{k+1} - s_k)x_k - (s_{k+1} - s_k)Bx_k| \\
&\qquad + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|T(s_n - s_{k+1})Bx_k - T(s_n - s_{k+1})Bx| \\
&\qquad + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|T(s_n - s_{k+1})Bx - Bx| + s_n|Bx - By| \\
&\quad \le \sum_{k=0}^{n-1}(s_{k+1} - s_k)\varepsilon + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|Bx_k - Bx| \\
&\qquad + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|T(s_n - s_{k+1})Bx - Bx| + s_n|Bx - By| \\
&\quad \le s_n\varepsilon + s_n\varepsilon/4 + s_n\varepsilon/4 + s_n\varepsilon/4.
\end{aligned}$$

In the same way as above we obtain the estimate

$$|\hat{x}_n - T(s_n)\hat{y} - B\hat{y}| \le (7\hat{\varepsilon}/4)s_n.$$

Passing to the limit as $n \to \infty$ in the above estimates, we have

$$|z - T(\eta)y - \eta By| < 2\varepsilon\eta \qquad \text{and} \qquad |\hat{z} - T(\eta)\hat{y} - \eta B\hat{y}| < 2\hat{\varepsilon}\eta.$$

The above-mentioned inequalities shows that (5.15) and (5.15)$'$ hold.

Finally, we show that the elements z and $\hat{z}$ satisfy (5.16). From (iii) and (v) one can deduce

$$|x_{k+1} - \hat{x}_{k+1}| \le |T(s_{k+1} - s_k)(x_k - \hat{x}_k) + (s_{k+1} - s_k)(Bx_k - B\hat{x}_k)| + (s_{k+1} - s_k)(\varepsilon + \hat{\varepsilon}) \le e^{\omega_\beta(s_{k+1}-s_k)}(|x_k - \hat{x}_k| + (s_{k+1} - s_k)\delta) + (s_{k+1} - s_k)(\varepsilon + \hat{\varepsilon})$$

for $0 \le k \le n$, or

$$e^{-\omega_\beta s_{k+1}}|x_{k+1} - \hat{x}_{k+1}| \le e^{-\omega_\beta s_k}|x_k - \hat{x}_k| + e^{-\omega_\beta s_k}(s_{k+1} - s_k)\delta + e^{-\omega_\beta s_{k+1}}(s_{k+1} - s_k)(\varepsilon + \hat{\varepsilon}), \tag{5.23}$$

for $0 \le k \le n$. Summing up these inequalities side by side, we obtain

$$e^{-\omega_\beta s_{n+1}}|x_{n+1} - \hat{x}_{n+1}| \le e^{-\omega_\beta s_0}|x_0 - \hat{x}_0| + \sum_{k=0}^{n} e^{-\omega_\beta s_k}(s_{k+1} - s_k)\delta + \sum_{k=0}^{n} e^{-\omega_\beta s_{k+1}}(s_{k+1} - s_k)(\varepsilon + \hat{\varepsilon}).$$

Putting $\bar{\omega}_\beta = \max\{\omega_\beta, 0\}$, we have

$$|x_{n+1} - \hat{x}_{n+1}| \le e^{\omega_\beta s_{n+1}}|y - \hat{y}| + \sum_{k=0}^{n} e^{\bar{\omega}_\beta \eta}(s_{k+1} - s_k)(\varepsilon + \hat{\varepsilon} + \delta) \le e^{\omega_\beta s_{n+1}}|y - \hat{y}| + s_{n+1}e^{\bar{\omega}_\beta \eta}(\varepsilon + \hat{\varepsilon} + \delta). \tag{5.24}$$

Passing to the limit as $n \to \infty$ in (5.24) we conclude that (5.16) holds. The proof of Theorem 5.2 is now complete. □

6. Global existence for mild solutions

In this section we discuss the construction of global mild solution to (SP). To this end, we mainly employ Theorems 4.3 and 5.2. Theorem 4.3 is not only the uniqueness theorem, but also gives the a priori estimates concerning the uniform continuity of mild solutions to (SP). Hence we may apply a standard continuation argument for local mild solutions. Also, Theorem 5.2 is understood to be a comparison theorem for subtangential conditions. Applying this result, we construct the local mild solutions as uniform limit of approximate solutions. We first establish the local existence theorem.

Theorem 6.1. *Suppose that* (II.1) *and* (II.2) *are satisfied. Let* $x \in D$, $R > 0$, $\varepsilon_0 \in (0, 1/3)$ *and* $\beta > \varphi(x)$. *Let* $M > 0$ *and* $\tau > 0$ *be such that* $|By| \le M$ *for* $y \in D_\beta \cap B(x, R)$, $\tau(M+1) + \sup_{t \in [0,\tau]} |T(t)x - x| \le R$ *and* $e^{a\tau}(\varphi(x) + (b + \varepsilon_0)\tau) < \beta$. *Then there exists a unique mild solution* $u(\cdot)$ *to* (SP) *on* $[0, \tau]$ *satisfying the initial condition* $u(0) = x$ *and the growth condition* $\varphi(u(t)) \le e^{at}(\varphi(x) + bt)$.

REMARK 6.1. It is noted that conditions (5.9), (5.10) and (5.12) are satisfied for ε, r, $M(x,\beta,\varepsilon)$, h replaced respectively by ε_0, R, M and τ.

PROOF OF THEOREM 6.1. Let $\{\varepsilon_n\}_{n\geq 1}$ be any null sequence in $(0,\varepsilon_0)$. For each ε_n we apply the argument employed in the proof of Theorem 5.1. By an induction argument, we construct decreasing seruences $\{r_i^n\}_{i=0}^{N_n-1}$, $\{\eta_i^n\}_{i=0}^{N_n-1}$ in $(0,\varepsilon_n]$, a sequence $\{t_i^n\}_{i=0}^{N_n}$ in $[0,\tau]$ and a sequence $\{x_i^n\}_{i=0}^{N_n}$ in $D_\beta \cap B(x,R)$, such that (i) through (vi) listed in the proof of Theorem 5.1 are valid for $\varepsilon = \varepsilon_n$ and $N = N_n$, and such that the partition $P_n = \{t_i^n\}_{i=0}^{N_n}$ of $[0,\tau]$ is finer than the previous partition $P_{n-1} = \{t_k^{n-1}\}_{k=0}^{N_{n-1}}$ of $[0,\tau]$.

First, one can construct $\{r_i^1\}_{i=0}^{N_1-1}$, $\{\eta_i^1\}_{i=0}^{N_1-1}$, $\{t_i^1\}_{i=0}^{N_1}$ in $[0,\tau]$ and $\{x_i^1\}_{i=0}^{N_1}$ for $\varepsilon = \varepsilon_1$, in the same way as in the proof of Theorem 5.1. Suppose that we have constructed sequences $\{t_i^n\}_{i=0}^{N_n}$ in $[0,\tau]$ and $\{x_i^n\}_{i=0}^{N_n}$. We then construct $\{t_k^{n+1}\}_{k=0}^{N_{n+1}}$ and $\{x_k^{n+1}\}_{k=0}^{N_{n+1}}$ by setting $h_k^{n+1} = \min\{\eta_k^{n+1}, t_{i+1}^n - t_k^{n+1}\}$ and $t_{k+1}^{n+1} = t_k^{n+1} + h_k^{n+1}$ provided that $t_i^n \leq t_k^{n+1} < t_{i+1}^n$. It should be noted here that h_k^{n+1} is defined by taking the minimum of η_k^{n+1} and $t_{i+1}^n - t_k^{n+1}$, instead of taking the minimum of η_k^{n+1} and $\tau - t_k^{n+1}$ (as in Theorem 5.1).

In accordance with this we define a sequence of approximate solutions $u_n(\cdot) : [0,\tau] \to X$ by

$$(6.1)\quad u_n(t) = \begin{cases} T(t-t_i^n)x_i^n + (t-t_i^n)Bx_i^n & \text{for } t \in [t_i^n, t_{i+1}^n) \\ & \text{and } 0 \leq i \leq N_n - 1 \\ T(\tau - t_{N_n-1}^n)x_{N_n-1}^n + (\tau - t_{N_n-1})Bx_{N_n-1} & \text{for } t = \tau \end{cases}.$$

Then

$$\begin{aligned} |u_n(t) - x_{i+1}^n| &\leq |T(t-t_i^n)x_i^n + (t-t_i^n)Bx_i^n - T(t_{i+1}^n - t_i^n)x_i^n - (t_{i+1}^n - t_i^n)Bx_i^n| \\ &\quad + (t_{i+1}^n - t_i^n)\varepsilon_n \\ &\leq |T(t_{i+1}^n - t)x_i^n - x_i^n| + (t_{i+1}^n - t)|Bx_i^n| + (t_{i+1}^n - t_i^n)\varepsilon_n \\ &\leq (t_{i+1}^n - t_i^n)(M+1) + |T(t_{i+1}^n - t)x_i^n - x_i^n| \leq r_i^n \leq \varepsilon_n \end{aligned}$$

for $t \in [t_i^n, t_{i+1}^n)$ by the properties (ii), (iv) and (vi). In particular, $d(u_n(t), D_\beta) \leq \varepsilon_n$ for $t \in [0,\tau]$.

We then demonstrate that the sequence $u_n(\cdot)$ is uniformly convergent on $[0,\tau]$. To this end, we apply Theorem 5.2 to estimate the difference between two approximate solutions $u_n(\cdot)$ and $u_m(\cdot)$.

Let $1 \leq n < m$, $t \in [0,\tau)$ and choose $0 \leq i \leq N_n - 1$ and $0 \leq j \leq N_m - 1$ such that $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$, or let $t = \tau$.

First, we introduce a new subdivision $\{s_l\}_{l=0}^{j+1}$ of $[0,t]$ by $s_l = t_l^m$ for $0 \leq l \leq j$ and $s_{j+1} = t$. We then apply Theorem 5.2 with $\delta = \varepsilon_m$ to construct the sequences $\{z_l\}_{l=0}^{j+1}$ and $\{\hat{z}_l\}_{l=0}^{j+1}$ satisfying $z_0 = \hat{z}_0 = x$ and (6.2) through (6.7) below.

If $s_l = t_k^n$, we put $x = x_k^n$, $\hat{x} = x_l^m$, $y = x$, $\hat{y} = \hat{x}$, $h = \hat{h} = 0$, $\eta = s_{l+1} - s_l$ in Theorem 5.2 and construct z_{l+1} and $\hat{z}_{l+1}$ satisfying

$$(6.2) \qquad |z_{l+1} - T(s_{l+1} - s_l)x_k^n - (s_{l+1} - s_l)Bx_k^n| < 2(s_{l+1} - s_l)\varepsilon_n,$$

$$|\hat{z}_{l+1} - T(s_{l+1} - s_l)x_l^m - (s_{l+1} - s_l)Bx_l^m| < 2(s_{l+1} - s_l)\varepsilon_m,$$

$$(6.3) \qquad \varphi(z_{l+1}) \le e^{a(s_{l+1}-s_l)}(\varphi(x_k^n) + (b + \varepsilon_n)(s_{l+1} - s_l)),$$

$$\varphi(\hat{z}_{l+1}) \le e^{a(s_{l+1}-s_l)}(\varphi(x_l^m) + (b + \varepsilon_m)(s_{l+1} - s_l))$$

and

$$(6.4) \qquad |z_{l+1} - \hat{z}_{l+1}| \le e^{\omega_\beta(s_{l+1}-s_l)}[|x_k^n - x_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].$$

If $s_l \in (t_k^n, t_{k+1}^n)$, we put $x = x_k^n$, $\hat{x} = x_l^m$, $y = z_l$, $\hat{y} = \hat{x}$, $h = s_l - t_k^n$, $\hat{h} = 0$ and $\eta = s_{l+1} - s_l$ in Theorem 5.2 and infer that z_{l+1} and z_{l+1} satisfy

$$(6.5) \qquad |z_{l+1} - T(s_{l+1} - s_l)z_l - (s_{l+1} - s_l)Bz_l| < 2(s_{l+1} - s_l)\varepsilon_n$$

$$|\hat{z}_{l+1} - T(s_{l+1} - s_l)x_l^m - (s_{l+1} - s_l)Bx_l^m| < 2(s_{l+1} - s_l)\varepsilon_m,$$

$$(6.6) \qquad \varphi(z_{l+1}) \le e^{a(s_{l+1}-s_l)}(\varphi(z_l) + (b + \varepsilon_n)(s_{l+1} - s_l))$$

$$\varphi(\hat{z}_{l+1}) \le e^{a(s_{l+1}-s_l)}(\varphi(x_l^m) + (b + \varepsilon_m)(s_{l+1} - s_l))$$

and

$$(6.7) \qquad |z_{l+1} - \hat{z}_{l+1}| \le e^{\omega_\beta(s_{l+1}-s_l)}[|z_l - x_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].$$

It should be noted here that in (6.5), (6.6) and (6.7) the element z_l is employed instead of the element x_k^n, and the time interval (t_k^n, t_{k+1}^n) may contain several uncommon points s_l's. We now make some comments about the use of Theorem 5.2 in the latter case. To apply this theorem in our situation, it must be verified first that z_l satisfies

$$(6.8) \qquad |z_l - T(s_l - t_k^n)x_k^n| \le (s_l - t_k^n)(M + 1)$$

and

$$(6.9) \qquad \varphi(z_l) \le e^{a(s_l - t_k^n)}(\varphi(x_k^n) + (b + \varepsilon_n)(s_l - t_k^n)) \qquad \text{for each } s_l \in (t_k^n, t_{k+1}^n),$$

which correspond to the estimates in (5.14). We now verify (6.8) and (6.9).

Suppose that $t_k^n = s_{l_0}$, and hence that s_{l_0+1} is the first uncommon point in (t_k^n, t_{k+1}^n). Then (6.2) implies

$$|z_{l_0+1} - T(s_{l_0+1} - t_k^n)x_k^n - (s_{l_0+1} - t_k^n)Bx_k^n| \le 2(s_{l_0+1} - t_k^n)\varepsilon_n,$$

and so

$$(6.10)\quad |z_{l_0+1} - T(s_{l_0+1} - t_k^n)x_k^n| \le (s_{l_0+1} - t_k^n)(M + 2\varepsilon_n) \le (s_{l_0+1} - t_k^n)(M+1),$$

which implies that $z_{l_0+1} \in D_\beta \cap B(x_k^n, r_k^n)$. Also, by (6.3), we have

$$(6.11)\qquad \varphi(z_{l_0+1}) \le e^{a(s_{l_0+1} - s_{l_0})}(\varphi(x_k^n) + (b+\varepsilon_n)(s_{l_0+1} - s_{l_0})).$$

Next let $s_l \in (t_k^n, t_{k+1}^n)$ and $l = l_0 + 2$. Then by (6.5) with $l = l_0 + 1$, condition (5.10) and the fact that $z_{l-1} \in D_\beta \cap B(x_k^n, r_k^n)$, we have

$$\begin{aligned}
&|z_l - T(s_l - t_k^n)x_k^n| \\
&\quad \le |z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| + |T(s_l - s_{l-1})z_{l-1} - T(s_l - t_k^n)x_k^n| \\
&\qquad + (s_l - s_{l-1})(|Bz_{l-1} - Bx_k^n| + |Bx_k^n|) \\
&\quad \le 2(s_l - s_{l-1})\varepsilon_n + |z_{l-1} - T(s_{l-1} - t_k^n)x_k^n| + (s_l - s_{l-1})(M + \varepsilon_n/4),
\end{aligned}$$

where (5.10) implies that $|Bz_{l-1} - Bx_k^n| \le \varepsilon_n/4$, and so that $|Bz_{l-1} - Bx_k^n| + |Bx_k^n| \le M + \varepsilon_n/4$. Hence we infer that (6.8) holds for $l = l_0 + 2$. Also, by (6.6) with $l = l_0 + 1$, we have

$$\begin{aligned}
\varphi(z_l) &\le e^{a(s_l - s_{l-1})}(\varphi(z_{l-1}) + (b+\varepsilon_n)(s_l - s_{l-1})) \\
&\le e^{a(s_l - s_{l-1})}(e^{a(s_{l-1} - t_k^n)}(\varphi(x_k^n) + (b+\varepsilon_n)(s_{l-1} - t_k^n)) + (b+\varepsilon_n)(s_l - s_{l-1})) \\
&\le e^{a(s_l - t_k^n)}(\varphi(x_k^n) + (b+\varepsilon_n)(s_l - t_k^n)).
\end{aligned}$$

This shows that (6.9) is satisfied for $l = l_0 + 2$. The case of the next uncommon s_l can be treated in the same way. Finally, we see that (6.8) and (6.9) are valid for all s_l in (t_k^n, t_{k+1}^n).

We now estimate the difference $u_m(\cdot) - u_n(\cdot)$ on $[0, \tau]$. To this end, we write

$$(6.12)\qquad |u_m(t) - u_n(t)| \le |u_m(t) - \hat{z}_{j+1}| + |\hat{z}_{j+1} - z_{j+1}| + |z_{j+1} - u_n(t)|,$$

and make an estimate of each term on the right-hand side of (6.12).

We begin by estimating the first term. We infer from the definition of $u_m(t)$, (6.2) or (6.5) with $l = j$ that

$$(6.13)\qquad |\hat{z}_{j+1} - T(t - t_j^m)x_j^m - (t - t_j^m)Bx_j^m| \le 2(t - t_j^m)\varepsilon_m.$$

We then make an estimate of the third term. If t_j^m is a common point for P_n and P_m, then (6.2) together with the definition of $u_n(t)$ implies $|z_{j+1} - u_n(t)| < 2(t - t_j^m)\varepsilon_n$, for $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$.

Next, suppose that t_j^m is not a common point for P_n and P_m. We estimate $|z_{j+1} - T(s_{j+1} - t_i^n)x_i^n - (s_{j+1} - t_i^n)Bx_i^n|$ under the assumption that $s_j =$

$t_j^m \in (t_i^n, t_{i+1}^n)$. Let $s_{j_0} = t_i^n$ and $\{s_l\}_{l=j_0+1}^{j+1}$ be a sequence constructed above. We have already constructed the elements x_i^n and z_l $(l = j_0+1, \dots, j+1)$, which correspond to s_{j_0} and s_l $(l = j_0+1, \dots, j+1)$, respectively. The application of Lemma 3.1 then implies

$$
\begin{aligned}
&|z_{j+1} - T(s_{j+1} - t_i^n)x_i^n - (s_{j+1} - t_i^n)Bx_i^n| \\
&\quad \le |z_{j_0+1} - T(s_{j_0+1} - s_{j_0})x_i^n - (s_{j_0+1} - s_{j_0})Bx_i^n| \\
&\qquad + \sum_{l=j_0+1}^{j} |z_{l+1} - T(s_{l+1} - s_l)z_l - (s_{l+1} - s_l)Bz_l| + \sum_{l=j_0}^{j} (s_{l+1} - s_l)|Bz_l - Bx_i^n| \\
&\qquad + \sum_{l=j_0}^{j} (s_{l+1} - s_l)|T(s_{j+1} - s_{l+1})Bx_i^n - Bx_i^n|.
\end{aligned}
$$

This estimate together with conditions (6.2), (6.5), (5.10) and (5.11) gives

$$
\begin{aligned}
&|z_{j+1} - T(s_{j+1} - t_i^n)x_i^n - (s_{j+1} - t_i^n)Bx_i^n| \\
&\quad \le 2\sum_{l=j_0}^{j} (s_{l+1} - s_l)\varepsilon_n + 2\sum_{l=j_0}^{j} (s_{l+1} - s_l)(\varepsilon_n/4) \\
&\quad \le 3\varepsilon_n(s_{j+1} - t_i^n).
\end{aligned}
$$

Thus, we obtain

$$
|z_{j+1} - u_n(t)| \le 3(s_{j+1} - t_i^n)\varepsilon_n \tag{6.14}
$$

whether or not t_j^m is a common point of P_n and P_m.

We next make an estimate of the second term on the right-hand side of (6.12). For this purpose we apply (6.2) and the property (iv) in Theorem 5.1 to get the estimate

$$
\begin{aligned}
|\hat{z}_l - x_l^m| &\le |\hat{z}_l - T(s_l - s_{l-1})x_{l-1}^m - (s_l - s_{l-1})Bx_{l-1}^m| \\
&\quad + |x_l^m - T(s_l - s_{l-1})x_{l-1}^m - (s_l - s_{l-1})Bx_{l-1}^m| \\
&\le 3(s_l - s_{l-1})\varepsilon_m.
\end{aligned} \tag{6.15}
$$

Suppose that $[t_k^n, t_{k+1}^n] = [s_{l_0}, s_{l_1}]$. In the same way as in the derivation of (6.15), one obtains

$$
\begin{aligned}
|z_{l_1} - x_{k+1}^n| &\le |z_{l_1} - T(s_{l_1} - t_k^n)x_k^n - (s_{l_1} - t_k^n)Bx_k^n| \\
&\quad + |x_{k+1}^n - T(s_{l_1} - t_k^n)x_k^n - (s_{l_1} - t_k^n)Bx_k^n| \le 4(t_{k+1}^n - t_k^n)\varepsilon_n.
\end{aligned} \tag{6.16}
$$

Applying (6.15) and (6.16) we have the following estimates.

If s_l is a common point t_k^n of P_n and P_m, then we apply (6.4) to get

$$|z_{l+1} - \hat{z}_{l+1}| \le e^{\omega_\beta(s_{l+1}-s_l)}[|x_k^n - z_l| + |z_l - \hat{z}_l| + |\hat{z}_l - x_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)]$$

$$\le e^{\omega_\beta(s_{l+1}-s_l)}[4(t_k^n - t_{k-1}^n)\varepsilon_n + |z_l - \hat{z}_l|]$$

$$+ e^{\omega_\beta(s_{l+1}-s_l)}[3(s_l - s_{l-1})\varepsilon_m + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)];$$

hence we obtain

$$(6.17) \qquad e^{-\omega_\beta s_{l+1}}|z_{l+1} - \hat{z}_{l+1}|$$

$$\le e^{-\omega_\beta s_l}|z_l - \hat{z}_l| + e^{-\omega_\beta s_l}[4(t_k^n - t_{k-1}^n)\varepsilon_n + 3(s_l - s_{l-1})\varepsilon_m$$

$$+ (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].$$

If s_l is not a common point for P_n and P_m, then we use (6.7) to obtain

$$|z_{l+1} - \hat{z}_{l+1}| \le e^{\omega_\beta(s_{l+1}-s_l)}[|z_l - \hat{z}_l| + |\hat{z}_l - x_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)]$$

$$\le e^{\omega_\beta(s_{l+1}-s_l)}[|z_l - \hat{z}_l| + 3(s_l - s_{l-1})\varepsilon_m + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)];$$

hence we get

$$(6.18) \qquad e^{-\omega_\beta s_{l+1}}|z_{l+1} - \hat{z}_{l+1}| \le e^{-\omega_\beta s_l}|z_l - \hat{z}_l|$$

$$+ e^{-\omega_\beta s_l}[3(s_l - s_{l-1})\varepsilon_m + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].$$

We then sum up the inequalities (6.17) and (6.18) with respect to $l = 1, \ldots, j$. We also use the inequality

$$|z_1 - \hat{z}_1| \le e^{\omega_\beta(s_1-s_0)}(s_1 - s_0)(\varepsilon_n + 2\varepsilon_m),$$

which follows from (6.4), and we obtain

$$e^{-\omega_\beta s_{j+1}}|z_{j+1} - \hat{z}_{j+1}| \le \sum_{l=0}^{j} e^{-\omega_\beta s_l}(s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m) + 3\sum_{l=1}^{j} e^{-\omega_\beta s_l}(s_l - s_{l-1})\varepsilon_m$$

$$+ 4\sum_{l=1}^{i} e^{-\omega_\beta t_l^n}(t_l^n - t_{l-1}^n)\varepsilon_n.$$

This means that

$$(6.19)\qquad |z_{j+1} - \hat{z}_{j+1}| \le \sum_{l=0}^{j} e^{\omega_\beta(s_{j+1}-s_l)}(s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)$$
$$+ 3\sum_{l=1}^{j} e^{\omega_\beta(s_{j+1}-s_l)}(s_l - s_{l-1})\varepsilon_m$$
$$+ 4\sum_{l=1}^{i} e^{\omega_\beta(s_{j+1}-t_l^n)}(t_l^n - t_{l-1}^n)\varepsilon_n$$
$$\le e^{\omega_\beta t}[t(\varepsilon_n + 2\varepsilon_m) + 3\varepsilon_m t_j^m + 4\varepsilon_n t_i^n],$$

and the desired estimate for the second term on the right-hand side is obtained.

We now combine the estimates (6.13), (6.14) and (6.19) for the first, third and the second term of the right-hand side of (6.12) to deduce that $u_m(t) - u_n(t)$ is estimated as

$$(6.20)\qquad |u_m(t) - u_n(t)| \le 2\varepsilon_m(t - t_j^m) + 3\varepsilon_n(t - t_i^n)$$
$$+ e^{\omega_\beta t}[(\varepsilon_n + 2\varepsilon_m)t + 3\varepsilon_m t_j^m + 4\varepsilon_n t_{i+1}^n]$$
$$\le 5te^{\omega_\beta t}(\varepsilon_n + \varepsilon_m) \le 5\tau e^{\omega_\beta \tau}(\varepsilon_n + \varepsilon_m)$$

This means that the sequence $\{u_n(\cdot)\}$ of the approximate solutions converges uniformly on $[0, \tau]$ to some X-valued function $u(\cdot)$ on $[0, \tau]$. Since $d(u_n(t), D_\beta) \le \varepsilon_n$ as mentioned after the definition of $u_n(\cdot)$, it follows that $u(t) \in D_\beta$ for each $t \in [0, \tau]$.

The limit function $u(\cdot)$ so obtained on $[0, \tau]$ gives the desired mild solution to (SP) on $[0, \tau]$. To verify this we define a step function

$$(6.21)\qquad \gamma_n(t) = \begin{cases} t_i^n & \text{for } t \in [t_i^n, t_{i+1}^n),\ 0 \le i \le N_n - 1 \\ t_{N_n-1}^n & \text{for } t = \tau \end{cases},$$

and an X-valued function

$$(6.22)\qquad v_n(t) = T(t)x + \int_0^t T(t-s)Bu_n(\gamma_n(s))ds \qquad \text{for } t \in [0, \tau].$$

In view of the definition of $u_n(\cdot)$ and $\gamma_n(\cdot)$, it is easily seen that the function $v_n(\cdot)$ is strongly continuous on $[0, \tau]$.

If $t \in [t_i^n, t_{i+1}^n)$, then

$$
\begin{aligned}
&u_n(t) - v_n(t) \\
&\quad = T(t - t_i^n)x_i^n + (t - t_i^n)Bx_i^n - T(t)x - \int_0^t T(t-s)B(u_n(\gamma_n(s)))ds \\
&\quad = T(t - t_i^n)x_i^n + (t - t_i^n)Bx_i^n - \sum_{k=0}^{i-1} \int_{t_k^n}^{t_{k+1}^n} T(t-s)Bx_k^n\, ds \\
&\qquad - \int_{t_i}^{t} T(t-s)Bx_i^n\, ds - T(t)x \\
&\quad = \sum_{k=0}^{i-1} T(t - t_{k+1}^n)[x_{k+1}^n - T(t_{k+1}^n - t_k^n)x_k^n - (t_{k+1}^n - t_k^n)Bx_k^n] \\
&\qquad - \sum_{k=0}^{i-1} \int_{t_k^n}^{t_{k+1}^n} [T(t-s)Bx_k^n - T(t - t_{k+1}^n)Bx_k^n]ds \\
&\qquad - \int_{t_i^n}^{t} (T(t-s)Bx_i^n - Bx_i^n)ds.
\end{aligned}
$$

Applying the properties (iv) through (vi) of Theorem 5.1 possessed by the double sequence $\{x_n^k\}$, we have

$$
\begin{aligned}
|u_n(t) - v_n(t)| &\le \sum_{k=0}^{i-1} (t_{k+1}^n - t_k^n)\varepsilon_n + \sum_{k=0}^{i-1} (t_{k+1}^n - t_k^n)\varepsilon_n/4 + (t - t_i^n)\varepsilon_n/4 \\
&< 5/4t\varepsilon_n.
\end{aligned}
$$

It is also shown in the same way that the above estimate holds for $t = \tau$. Therefore the function $v_n(\cdot)$ converges uniformly on $[0, \tau]$ to $u(\cdot)$ and it follows that $u(\cdot)$ is strongly continuous on $[0, \tau]$. On the other hand, we have

$$
\begin{aligned}
|u_n(\gamma_n(t)) - u_n(t)| &= |x_i^n - T(t - t_i^n)x_i^n + (t - t_i^n)Bx_i^n| \\
&\le (t - t_i^n)M + |T(t - t_i^n)x_i^n - x_i^n| \le \varepsilon_n
\end{aligned}
$$

for $t \in [t_i^n, t_{i+1}^n)$, $i = 0, \dots N_n - 1$. This estimate is also valid for $t = \tau$ since $\gamma(\tau) = t_{N_n-1}^n$. Therefore, it follows that

$$
u_n(\gamma_n(\cdot)) \to u(\cdot) \qquad \text{as } n \to \infty
$$

and the convergence is uniform on $[0, \tau]$. Since $u_n(\gamma_n(t))$ and $u(t)$ belong to D_β for each $t \in [0, \tau]$, the continuity of B on D_β asserts that

$$
Bu_n(\gamma_n(t)) \to Bu(t) \qquad \text{as } n \to \infty \text{ uniformly on } [0, \tau].
$$

One can now pass to the limit as $n \to \infty$ in (6.22) to conclude that

$$u(t) = T(t)x + \int_0^t T(t-s)Bu(s)ds$$

holds for $t \in [0, \tau]$. Also, we have

$$\varphi(u_n(\gamma_n(t))) = \varphi(x_i^n) \le e^{at_i^n}(\varphi(x) + (b + \varepsilon_n)t_i^n),$$

for $t \in [t_i^n, t_{i+1}^n)$, $i = 0, \dots, N_n - 1$ and for $t = \tau$. Letting $n \to \infty$ in the above estimate and applying the lower semicontinuity of φ, we get

$$\varphi(u(t)) \le \varliminf_{n\to\infty} \varphi(u_n(\gamma_n(t))) \le e^{at}(\varphi(x) + bt) \qquad \text{for } t \in [0, \tau].$$

This concludes that the limit function $u(\cdot)$ gives a unique mild solution to (SP) on $[0, \tau]$. The proof of Theorem 6.1 is now complete. □

We are now in a position to state our global existence theorem.

Theorem 6.2. *Suppose that a semilinear operator $A + B$ satisfies the explicit subtangential condition* (II.1) *and semilinear stability condition* (II.2). *Then for each x in D there exists a unique global mild solution $u(\cdot) \equiv u(\cdot; x)$ to* (SP) *on* $[0, \infty)$.

Proof. Let $x \in D$. Then $x \in D_\beta$ for some $\beta > 0$ and Theorem 6.1 asserts via the standard continuation argument that one finds a nonextensible mild solution $u(\cdot) = u(\cdot, x) : [0, \tau_{\max}) \to X$ satisfying the exponential growth condition $\varphi(u(t)) \le e^{at}(\varphi(x) + bt)$ for $t \in [0, \tau_{\max})$. Suppose that $\tau_{\max} < \infty$. Then, by Theorem 4.3, we would have

$$|u(t+h) - u(t)| \le e^{\omega_\beta t}|u(h) - x| \tag{6.23}$$

for $\beta > e^{a\tau_{\max}}(\alpha + b\tau_{\max})$, $\alpha \ge \max\{\varphi(x), \varphi(u(h))\}$, and $h \in (0, \tau_{\max} - t)$. It now follows from (6.23) that there would exist a limit $\lim_{t\uparrow\tau_{\max}} u(t) = w$ in D. Now the application of Theorem 6.1 implies that there must exist a unique local mild solution $\bar{u}(t) : [\tau_{\max}, \bar{\tau}) \to X$ to the (nonautonomous) semilinear problem

$$u'(t) = (A + B)u(t), \quad t > \tau_{\max}; \qquad u(\tau_{\max}) = w \tag{SP$'$}$$

subject to the growth condition $\varphi(u(t)) \le e^{a(t-\tau_{\max})}(\varphi(w) + b(t - \tau_{\max}))$ for $t \in [\tau_{\max}, \bar{\tau})$.

We then define a new function $\tilde{u}(\cdot) : [0, \bar{\tau}) \to X$ by

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [0, \tau_{\max}) \\ \bar{u}(t), & t \in [\tau_{\max}, \bar{\tau}). \end{cases}$$

It is easily seen that the strongly continuous function $\tilde{u}(\cdot)$ on $[0, \tau]$ gives a mild solution to (SP) satisfying the exponential growth condition with respect to φ. This contradicts the maximality of $u(\cdot)$, and so we must have $\tau_{\max} = \infty$. The proof is now complete. □

In view of Theorem 6.2, we may demonstrate that condition (II) implies the assertion (I) in our main result, Theorem 1. In fact, given $x \in D$, let $u(\cdot; x)$ be the associated mild solution to (SP) on $[0, \infty)$ given by Theorem 6.2. For any fixed $t \geq 0$, we define an operator $S(t)$ from D into itself by $S(t)x = u(t; x)$, $x \in D$. Then the family $S = \{S(t); t \geq 0\}$ of the solution operators to (SP) forms a semigroup on D satisfying conditions (I.1) and (I.2). This shows that (II) implies (I). Consequently, the proof of Theorem 1 in the nonconvex case is now complete.

7. Semigroups in the convex case and groups of nonlinear operators

In this section we treat a characterization theorem for nonlinear semigroups, Theorem 1, in the convex case and also discuss generation and characterization of groups of nonlinear operators. Under the convexity assumptions for the domain D and the functional φ we consider conditions (I) through (V) which are stated in Theorem 1.

Theorem 7.1. *If D and φ are convex, then conditions* (I) *through* (V) *are equivalent to each other.*

Proof. It is obvious that (III) implies (IV). To show that (IV) implies (V), let $x \in D$ and $\{\varepsilon_n\}$ any null sequence of positive numbers. Given $\alpha > 0$ with $x \in D_\alpha$, we may take a null sequence $\{\lambda_n\}$ of positive numbers, $x_n \in D(A) \cap D$ and z_n with $|z_n| < \varepsilon_n$ under condition (IV.3), which states that for $\alpha > 0$ and $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\alpha, \varepsilon)$ such that for $\lambda \in (0, \lambda_0)$ and $x \in D_\alpha$ there exist $x_\lambda \in D(A) \cap D$ and $z_\lambda \in X$ satisfying $|z_\lambda| < \varepsilon$,

$$x_\lambda - \lambda(A+B)x_\lambda = x + \lambda z_\lambda \qquad \text{and} \qquad \varphi(x_\lambda) \leq (1-\lambda a)^{-1}(\varphi(x) + (b+\varepsilon)\lambda).$$

Accordingly, $(1/\lambda_n)|x_n - \lambda_n(A+B)x_n - x| = |z_{\lambda_n}| \leq \varepsilon_n \to 0$ as $n \to \infty$. Also,

$$(1/\lambda_n)(\varphi(x_n) - \varphi(x)) \leq [a\varphi(x) + (b+\varepsilon_n)]/(1-\lambda_n a),$$

and hence

$$\varlimsup_{n\to\infty} (1/\lambda_n)(\varphi(x_n) - \varphi(x)) \leq a\varphi(x) + b.$$

Let $\varepsilon > 0$ and $y \in D(A) \cap D$ be such that $|x - y| < \varepsilon/2$. Since y and $\{x_n\}_{n\geq 1}$ lie in D_β for some $\beta > \alpha$, we have

$$|x_n - x| \le (1 - \lambda_n\omega_\beta)^{-1}[|x_n - \lambda_n(A+B)x_n - y - \lambda_n(A+B)y|] + |x-y|$$
$$\le (1 - \lambda_n\omega_\beta)^{-1}[|x_n - \lambda_n(A+B)x_n - x| + |x-y| + \lambda_n|(A+B)y|]$$
$$+ |y-x|,$$

and so $\overline{\lim}_{n\to\infty} |x_n - x| \le 2|y-x| < \varepsilon$.

Since ε was arbitrary, it follows that $\overline{\lim}_{n\to\infty} |x_n - x| = 0$. Thus we conclude that (IV) implies (V). The implication from (V) to (I) is verified by applying known generation results given for instance in [12]. See also [6], [8], and [15]. Notice that no convexity assumptions are required for the proof. As for the proof of the fact that (III) implies (I), we first recall that (I) and (II) are equivalent, and so that the semilinear operator $A+B$ is strongly quasidissipative. One can then apply Theorem 2.4 in [11] and follow the proof of Theorem 3.1 in [11] line by line to derive (III). This completes the proof. □

We end this paper with a characterization theorem for nonlinear operator groups. As described in [3], it is possible to formulate generation theorems for nonlinear groups defined as below.

Let D be a convex subset of X and φ a l.s.c. and convex functional on X such that $D \subset D(\varphi)$. A one-parameter family $G = \{G(t); t \in \mathbf{R}\}$ of operators from D into itself is called a locally Lipschitzian group on D with respect to φ, if it satisfies the following three conditions:

(G1) $G(t+s)x = G(t)G(s)x$, $G(0)x = x$ for $x \in D$ and $s, t \in \mathbf{R}$.

(G2) For $x \in D$, $G(\cdot)x \in C(\mathbf{R}; X)$.

(G3) For each $\alpha > 0$ and $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbf{R}$ such that

$$|G(t)x - G(t)y| \le e^{\omega|t|}|v - w|$$

for $x, y \in D_\alpha$ and $t \in [-\tau, \tau]$.

Suppose now that $A+B$ and $-A-B$ generate locally Lipschitzian semigroups S_+ and S_- in the sense of Theorem 1. Then we have

$$(d^+/dt)[S_+(t)S_-(t)x] = 0, \qquad \text{for each } t \ge 0 \text{ and } x \in D. \tag{7.1}$$

This can be proved in the same way as in [3]. Note that the assumption of local Lipschitz continuity for B and $-B$ is not necessary to derive this identity. Using (7.1), one defines a nonlinear group G on D by setting

$$G(t) = \begin{cases} S_+(t) & \text{for } t \ge 0, \\ S_-(-t) & \text{for } t < 0. \end{cases}$$

In view of this, one applies Theorem 1 to establish the following theorem for the characterization of nonlinear operator groups of locally Lipschitz operators with respect to the functional φ. See also [3].

Theorem 7.2. *Let $a, b \geq 0$, A a linear operator in X such that $\pm A$ satisfy condition* (A) *and let B be a nonlinear operator on D such that B satisfies condition* (B). *Let φ be a l.s.c. functional on X with $D \subset D(\varphi)$ and denote by $G_A = \{G_A(t)\}$ the (C_0)-group generated by A. Then the following statements are equivalent:*

(I) *There is a nonlinear group $G = \{G(t); t \in \mathbf{R}\}$ of locally Lipschitz operators on D satisfying the properties given below:*

(I.1) *$G(t)x = G_A(t)x + \int_0^t G_A(t-s)BG(s)x\,ds$ for $t \in \mathbf{R}$ and $x \in D$.*

(I.2) *For each $\alpha > 0$ and $\tau > 0$ there is $\omega_1 = \omega_1(\alpha, \tau) \in \mathbf{R}$ such that*

$$|G(t)x - G(t)y| \leq e^{\omega_1(\alpha,\tau)|t|}|x - y|$$

for each $x, y \in D_\alpha$.

(I.3) *$\varphi(G(t)x) \leq e^{a|t|}(\varphi(x) + b|t|)$ for $t \in \mathbf{R}$ and $x \in D$.*

(II) *The following subtangential condition and semilinear stability condition are satisfied:*

(II.1) *For $x \in D$ and $\varepsilon > 0$ there exist $(h_1, x_{h_1}) \in (0, \varepsilon] \times D$ and $(h_2, x_{h_2}) \in [-\varepsilon, 0) \times D$ such that*

$$(1/h_i)|G_A(h_i)x + h_i Bx - x_{h_i}| \leq \varepsilon, \qquad \varphi(x_{h_i}) \leq e^{a|h_i|}(\varphi(x) + (b+\varepsilon)h_i) \qquad i = 1, 2.$$

(II.2) *For each $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\varliminf_{h \to 0}(1/|h|)[|G_A(h)(x-y) + h(Bx - By)| - |x - y|] \leq \omega_\alpha |x - y|$$

for each $x, y \in D_\alpha$.

Moreover, if D and φ are both convex, then the above statements aree-quivalent to the following:

(III) *The following denseness condition, quasidissipativity condition and implicit subtangential condition are satisfied:*

(III.1) *$D(A) \cap D$ is dense in D.*

(III.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbf{R}$ such that*

$$\langle (A+B)x - (A+B)y, x - y \rangle_i \leq \omega_\alpha |x - y|^2$$

$$\langle (A+B)x - (A+B)y, x - y \rangle_s \geq -\omega_\alpha |x - y|^2$$

for $x, y \in D_\alpha \cap D(A)$.

(III.3) *To $\alpha > 0$ and $\varepsilon > 0$ there corresponds $\lambda_0 = \lambda_0(\alpha, \varepsilon) > 0$ and for $v \in D_\alpha$ and $\lambda \in \mathbf{R}$ with $|\lambda| < \lambda_0(\alpha)$ there exist $v_\lambda \in D(A) \cap D$ and $z_\lambda \in X$ such that $|z_\lambda| < \varepsilon$*

$$v_\lambda - \lambda(A+B)v_\lambda = v + \lambda z_\lambda,$$

$$\varphi(v_\lambda) \leq (1 - |\lambda|a)^{-1}(\varphi(v) + (b+\varepsilon)|\lambda|).$$

It should be noted here again that convexity conditions for the domain D and functional φ are not required in the implication from (III) to (I). This result can be extended to the case where several lower semicontinuous functional $\varphi_1, \cdots, \varphi_n$ on X are used and those extensions are applicable to various quasilinear equations. For typical applications to the generalized KdV equation, see the forthcoming paper [2].

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