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ON THE IMPULSIVE CONTROL OF A N-PREY ONE-PREDATOR FOOD WEB MODEL

BY

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Abstract. The dynamics of a n-prey one-predator food web model which is subject to impulsive controls is studied from the viewpoint of finding sufficient conditions for the extinction of prey and for prey and predator permanence. Using the Floquet theory of impulsively perturbed systems of ordinary differential equations, the local and global stability of the so-called prey-free periodic solution are obtained in terms of integral conditions with biological significance. The permanence and partial permanence of the system are also investigated using similar conditions.

Key words: impulsive controls, stability, permanence, integrated pest management

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1. Introduction

Lately, it has been widely documented that the persistent use of pesticides is prone to having undesirable long-term side effects, due to the accumulation of residual chemicals in the various links of the human food chain (see, for instance, Alavanja, Hoppin and Kamel [1]). The overexposure to pesticides often leads to chronical health problems, being also associated with a general decrease in the biodiversity of the environment, since non-target beneficial organisms are sometimes less resistant to pesticides than pests are. Also, if pests have a quick reproductive rate, the overuse of pesticides may lead to a selection of pesticide-resistant pest varieties. Here, by pest we shall understand any organism which is detrimental to human health or to crops, for example insects such as locusts or spiders, pathogens, weeds or even mammals such as mice or other rodents.

Integrated pest management (IPM) has consequently emerged as an environmentally friendly approach to pest control, based on the use of a large palette of alternative techniques (biological, mechanical, and not only) to pest control, pesticides being used only when they are deemed an absolute necessity. See, for instance, Kogan [6] for a survey on the aims and techniques of IPM. One of the usual approaches to biological pest control is to release natural predators of the pest to be contained, in the form of an innoculative or inundative release. Another approach is the release of infective pest individuals, with the purpose of establishing the endemicity of a pest disease, on the grounds that infected pests are less likely to reproduce or to damage the environment. Sometimes, different pests of the same crop or different subspecies of the same pest need to be regulated simultaneously, their variable traits being in need to be considered.

Our aim is to describe an IPM strategy devised to control the spread of several competing pest species through the release of their natural predators and through pesticide use. The immediate jumps in the sizes of the predator species after each predator release as well the quick action of many modern pesticides motivate the use of a model with impulsive discontinuities. It is assumed that the amount of predator individuals which are released each time is constant and that as a result of pesticide use fixed proportions of pest and predator species are removed from the environment.

Since our IPM strategy relies on the release of predators, we shall briefly review several aspects of predator-prey interactions. A rather simple, yet useful, model framework for the understanding of predator-prey interactions is the following model

$$(1) \qquad \begin{cases} x'(t) = x(t)f(x(t)) - y(t)F(x(t), y(t)) - d_x x(t), \\ y'(t) = y(t)G(x(t), y(t)) - d_y y(t). \end{cases}$$

In the above, $x = x(t)$ denotes the density of the prey population and $y = y(t)$ denotes the density of the predator population. The function $f = f(x)$ represents the per capita growth rate of the prey population in the absence of predation, while $F = F(x, y)$ is the predator's functional response, that is, the number of prey individuals consumed per unit area and unit time by a single predator, or the dependence of the prey consumption rate upon the density of the prey. The functional response can be expressed as $F(x, y) = xF_1(x, y)$, where $F_1(x, y)$ represents the per capita death rate of prey due to predation by a a single predator. Accordingly, $yF_1(x, y)$ represents the per capita death rate of prey due to predation by all predators. The function $G = G(x, y)$ is the predator's numerical response, that is, the per capita growth rate of the predator population. In many cases, $G(x, y) = eF(x, y)$, where e is a conversion coefficient, as the growth of the predator class directly depends on the surplus of energy acquired through prey consumption. The constants d_x and d_y are the natural mortalities of the prey and predator, respectively. If the mortality of prey is mostly due to predation, then d_x can be neglected. Also, at least formally, the natural mortality term $d_x x$ can be absorbed into the growth term $xf(x)$.

2. The mathematical model and its biological well-posedness

The previously mentioned biological considerations have lead us to the formulation of the following model which describes a strategy for the control of n competing prey (pest) species via the periodic release of a common predator and pesticide spraying

$$
\text{(S)}\quad \begin{cases}
x_i'(t) = x_i(t)\left[r_i - \displaystyle\sum_{j=1}^{n} a_{ij}x_j(t)\right] & t \neq (n+l-1)T,\ t \neq nT; \\
\qquad\qquad - x_i(t)y(t)\varphi_i(\mathbf{x}(t), y(t)), & 1 \leq i \leq n \\
y'(t) = \displaystyle\sum_{i=1}^{n} e_i x_i(t)y(t)\varphi_i(\mathbf{x}(t), y(t)) & t \neq (n+l-1)T,\ t \neq nT; \\
\qquad\qquad - dy(t), & \\
\Delta x_i(t) = -\delta_i x_i(t), & t = (n+l-1)T,\ 1 \leq i \leq n; \\
\Delta y(t) = -\delta y(t), & t = (n+l-1)T; \\
\Delta x_i(t) = 0, & t = nT,\ 1 \leq i \leq n; \\
\Delta y(t) = \mu, & t = nT.
\end{cases}
$$

In the above model, $x_i = x_i(t)$ represents the density of the (prey) species i, while $y = y(t)$ represents the density of the predator species and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the vector of all prey densities. Also, (S) is understood to be a competition model in which the constants a_{ii} are used to describe the effects of intraspecific competition on species i, while a_{ij}, $i \neq j$, describes the effects of interspecific competition between species i and j on species i. The constant r_i is used to denote the intrinsic birth rates of species i in the absence of predation and competition and e_i represents the conversion efficiency of prey consumed from species i into newborn predators.

Regarding the effects of the biological and chemical controls, $T > 0$ is the periodicity of both impulsive perturbations, while $0 < l < 1$ is a parameter used to describe the time lag lT between predator release and pesticide spraying, which are not simultaneous. Similarly, $\Delta\psi(t) = \psi(t+) - \psi(t)$ for $\psi \in \{x_i, y\}$, $1 \leq i \leq n$ and $t > 0$ represent the instantaneous jumps in the prey and predator population sizes after the use of controls. Each time the pesticides are applied, fixed proportions δ_i of prey species i, $1 \leq i \leq n$, and δ of predator species, $0 \leq \delta_i < 1$ for all $1 \leq i \leq n$ and $0 \leq \delta < 1$, are removed from the environment. Also, μ is the constant amount of predators which are released each time.

The prey death rates $\varphi_i : [0,\infty)^{n+1} \to [0,\infty)$, $\varphi_i \in C([0,\infty)^{n+1})$, $1 \leq i \leq n$, are assumed to satisfy the following assumptions.

- (H1) For all $1 \leq i \leq n$, $x_j \to \varphi_i(\mathbf{x}, y)$ is decreasing for $x_k \in [0, \infty)$, $k \neq j$ and $y \in [0, \infty)$, $1 \leq j \leq n$, and $y \to \varphi_i(\mathbf{x}, y)$ is decreasing for $\mathbf{x} \in [0, \infty)^n$.
- (H2) For all $1 \leq i \leq n$, $x_j \to x_j\varphi_i(\mathbf{x}, y)$ is increasing for $x_k \in [0, \infty)$, $k \neq j$ and $y \in [0, \infty)$, $1 \leq j \leq n$, and $y \to y\varphi_i(\mathbf{x}, y)$ is increasing for $\mathbf{x} \in [0, \infty)^n$.

Hypotheses (H1) and (H2) are satisfied, for instance, if the functional response of the predators species are all of Beddington-deAngelis type, for which $\varphi_i(\mathbf{x}, y) = \frac{b_i}{1+\sum_{j=1}^n k_i^j x_j + a_i y}$, or of Ivlev type, for which $\varphi_i(\mathbf{x}, y) = \frac{a_i(1-e^{-k_i x_i})}{x_i}$, $x_i \neq 0$, $\varphi_i(0, y) = a_i k_i$. In this regard, our results encompass the corresponding findings in Baek [2], Yu et. al. [9] and Georgescu [4], where the dynamics of impulsively perturbed two-prey one-predator is discussed via similar arguments (although [2] also discusses the effect of seasonal perturbations, our framework can be adapted to obtain the findings therein). This investigation attempts to extend the results presented in [4] by means of allowing for an arbitrary number of predator species and for more general predator functional responses than those employed in [4]. For further results on IPM strategies relying on the use of impulsive biological and chemical controls, see Nundloll, Mailleret and Grognard [7], Wang, Wang and Lin [8].

We shall now discuss the well-posedness of (S) and state a few auxiliary results which will be of use in what follows. First, it is possible to prove on the lines of [4, Lemmas 3.2 and 3.3] that the Cauchy problem with strictly positive initial data is well-posed for (S), that is, solutions $(\mathbf{x}, y)$ of (S) starting with strictly positive initial data are defined on the whole $\mathbb{R}_+$ and are strictly positive and bounded, with a boundedness constant not depending on the initial data.

We now introduce a few basic results regarding the Floquet theory of impulsive and periodic systems of ordinary differential equations which will be used in the next section to discuss the local stability of the so-called prey-free periodic solution. Let us consider the impulsive linear system

$$\begin{cases} X'(t) = A(t)X(t), & t \neq \tau_k,\ t \in \mathbb{R}; \\ \Delta X = B_k X, & t = \tau_k,\ \tau_k < \tau_{k+1},\ k \in \mathbb{Z}. \end{cases} \tag{2}$$

under the following hypotheses.

- (A1) $A(\cdot) \in PC(\mathbb{R}, M_n(\mathbb{R}))$ and there is $T > 0$ such that $A(t+T) = A(t)$ for all $t \geq 0$.
- (A2) $B_k \in M_n(\mathbb{R})$, $\det(I_n + B_k) \neq 0$ for $k \in \mathbb{Z}$.
- (A3) There is $q \in \mathbb{N}^*$ such that $B_{k+q} = B_k$, $\tau_{k+q} = \tau_k + T$ for $k \in \mathbb{Z}$.

Let now $\Phi(t)$ be a fundamental matrix of (2). Then there is a unique nonsingular matrix $\mathcal{M} \in M_n(\mathbb{R})$ such that $\Phi(t+T) = \Phi(t)\mathcal{M}$ for all $t \in \mathbb{R}$, which is called the monodromy matrix of (2) corresponding to Φ. All monodromy matrices of (2) corresponding to different Φ's, being similar, have the same eigenvalues λ_1, λ_2, … , λ_n, which are called the Floquet multipliers of (2). These eigenvalues determine whether $\Phi(t)$ shrinks or expands with the time, as seen in the following stability result, where by elementary divisors of a square matrix we understand the characteristic polynomials of its Jordan blocks.

Lemma 1. ([3]) *Suppose that conditions (A1)-(A3) hold. Then*

(i) *The system* (2) *is stable if and only if all Floquet multipliers* λ_k, $1 \le k \le n$ *satisfy* $|\lambda_k| \le 1$ *and if* $|\lambda_k| = 1$, *then to* λ_k *there corresponds a simple elementary divisor.*

(ii) *The system* (2) *is asymptotically stable if and only if all Floquet multipliers* $\lambda_k, 1 \le k \le n$ *satisfy* $|\lambda_k| < 1$.

(iii) *The system* (2) *is unstable if there is a Floquet multiplier* λ_k *such that* $|\lambda_k| > 1$.

It is easy to see that due to the pulsed supply of predators which occurs with periodicity T, the predators are never in danger of extinction. Since there is no such supply of prey (pests), as there is no logical reason to be, since pests need to be removed, not raised, a possible outcome of the model is the extinction of all prey species. When all prey species are extinct, the dynamics of the remaining predator population is given by the solution of the subsystem

$$(3)\qquad \begin{cases} y'(t) = -dy(t), & t \neq (n+l-1)T,\ t \neq nT \\ \Delta y(t) = -\delta y(t), & t = (n+l-1)T; \\ \Delta y(t) = \mu, & t = nT; \\ y(0+) = y_0. \end{cases}$$

It is seen that the system constructed with the first three equations in (3) has a periodic solution which attracts all solutions of (3) starting with strictly positive y_0, as observed in the following Lemma (Lemma 4.2 of Georgescu and Moroşanu [5]).

Lemma 2. ([5]) *The system constructed with the first three equations in* (3) *has a* T*-periodic solution* y_d^*. *With this notation, the following properties are satisfied.*

(i) $\int_0^T y_d^*(t)dt = \frac{\mu}{1-e^{-dT}(1-\delta)}\left[(1-e^{-dlT}) + (1-\delta)(e^{-dlT} - e^{-dT})\right]$.

(ii) $\lim\limits_{t\to\infty} |y(t) - y_d^*(t)| = 0$ *for all solutions* $y(t)$ *of* (3) *starting with strictly positive* y_0.

(iii) $\sup\limits_{t\ge 0} \left|y_d^*(t) - y_{\tilde{d}}^*(t)\right| \le f_2(d, \tilde{d}; T, a, \delta)$, *with* $\lim\limits_{\tilde{d}\to d} f_2(\tilde{d}, d; T, a, \delta) = 0$.

The notation y_d^*, which emphasizes the dependence of the solutions of (3) on d (as opposed to the dependence on δ and μ) has been chosen since systems of type (3) occur throughout this paper with different d's but with the same δ and μ. Also, one may explicitely compute y_d^*, in the form

$$(4)\qquad y_d^* = \begin{cases} \frac{\mu}{1-e^{-dT}(1-\delta)} e^{-d(t-nT)}, & t \in (nT, (n+l)T] \\ \frac{\mu}{1-e^{-dT}(1-\delta)} e^{-d(t-nT)}(1-\delta), & t \in ((n+l)T, (n+1)T]. \end{cases}$$

When all prey species are extinct, the system consisting of the first $n+1$ equations of (S) has a periodic solution $E^* = (\mathbf{0}, y_d^*)$, which will be called in what follows the prey-free periodic solution. Here, $\mathbf{0} = (0, 0, \ldots, 0)$ is the null n-dimensional vector. Although in concrete situations the desired outcome of an IPM is not necessarily the complete extinction of all pests, as this may be unfeasible from a practical point of view, or may damage the ecosystem, but the stabilization of their total population size under an economically significant threshold called the economic injury level (EIL), it is not a large stretch of imagination to interpret the success of our IPM strategy in terms of the stability properties of E^*. That is, if E^* is globally asymptotically stable, then the pests can be eradicated irrespective of their initial population sizes, while if E^* is only locally asymptotically stable, then the pests can be eradicated only in favorable circumstances.

3. The local and global stability results

We shall now give sufficient conditions for the local and global stability of E^*. In what follows, we shall denote $\frac{\mathbf{r}}{\mathbf{a}} = (\frac{r_1}{a_{11}}, \frac{r_2}{a_{22}}, \ldots, \frac{r_n}{a_{nn}})$ and $\boldsymbol{\varepsilon} = (\varepsilon, \varepsilon, \ldots, \varepsilon)$, the latter being also a n-dimensional vector. We shall also denote

$$P_i(a_1, a_2, \ldots, a_n, a_{n+1}) = a_{n+1}\varphi_i(a_1, a_2, \ldots, a_n, a_{n+1}), \quad 1 \leq i \leq n.$$

Theorem 1. *The prey-free periodic solution E^* is locally asymptotically stable if*

$$\int_0^T P_i(\mathbf{0}, y_d^*(t))dt - \ln(1-\delta_i) > r_i T, \quad \textit{for all } 1 \leq i \leq n. \tag{5}$$

Further, E^* is globally asymptotically stable if

$$\int_0^T P_i(\frac{\mathbf{r}}{\mathbf{a}}, y_d^*(t))dt - \ln(1-\delta_i) > r_i T, \quad \textit{for all } 1 \leq i \leq n. \tag{6}$$

Proof. To discuss the local stability of E^*, we shall use the method of small amplitude perturbations. To this purpose, let us denote

$$x_i(t) = u_i(t),\ 1 \leq i \leq n, \quad y(t) = v(t) + y_d^*(t),$$

where u_i, $1 \leq i \leq n$, and v are understood to be small amplitude perturbations. The linearization of (S) is then given by

$$\left\{\begin{array}{ll} u_i'(t) = u_i(t)\left[r_i - P_i(\mathbf{0}, y_d^*(t))\right], & t \neq (n+l-1)T,\ t \neq nT, \\ & 1 \leq i \leq n; \\ v'(t) = \displaystyle\sum_{i=1}^n e_i u_i(t) P_i(\mathbf{0}, y_d^*(t)) & t \neq (n+l-1)T,\ t \neq nT; \\ \quad - dv(t), & \\ \Delta u_i(t) = -\delta_i u_i(t), & t = (n+l-1)T,\ 1 \leq i \leq n; \\ \Delta v(t) = -\delta v(t), & t = (n+l-1)T; \\ \Delta u_i(t) = \Delta v(t) = 0, & t = nT,\ 1 \leq i \leq n. \end{array}\right. \tag{7}$$

We need now study the stability of the null solution of (7). Let Φ_L be a fundamental matrix of the differential system constructed with the first $n+1$ equations of (7). Consequently, Φ_L satisfies

$$\frac{d\Phi_L}{dt}(t) = A(t)\Phi_L(t),$$

where

$$A(t) = \begin{pmatrix} r_1 - P_1(\mathbf{0}, y_d^*(t)) & 0 & \dots & 0 & 0 \\ 0 & r_2 - P_2(\mathbf{0}, y_d^*(t)) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & r_n - P_n(\mathbf{0}, y_d^*(t)) & 0 \\ e_1 P_1(\mathbf{0}, y_d^*(t)) & e_2 P_2(\mathbf{0}, y_d^*(t)) & \dots & e_n P_n(\mathbf{0}, y_d^*(t)) & -d \end{pmatrix}.$$

Here and in what follows we have used the notations

$$P_i(a_1, a_2, \dots, a_n, a_{n+1}) = a_{n+1}\varphi_i(a_1, a_2, \dots, a_n), \quad 1 \le i \le n.$$

A fundamental matrix Φ_L of (7) which satisfies $\Phi_L(0) = I_{n+1}$ is a lower triangular matrix with

$$(\Phi_L(t))_{i,i} = e^{\int_0^t r_i - P_i(\mathbf{0}, y_d^*(s))ds}, \quad 1 \le i \le n,$$
$$(\Phi_L(t))_{n+1,n+1} = e^{-dt}.$$

Since

$$u_i(t+) = (1 - \delta_i)u_i(t), \quad v(t+) = (1 - \delta)v(t),$$

for $t = (n + l - 1)T+$, and

$$u_i(t+) = u_i(t), \quad v(t+) = v(t),$$

for $t = nT$, the monodromy matrix

$$\mathcal{M} = \operatorname{diag}[1 - \delta_1, 1 - \delta_2, \dots, 1 - \delta_n, 1 - \delta]\Phi_L(T) \tag{8}$$

is also lower diagonal. Its eigenvalues, found on the main diagonal, are

$$\lambda_i = (1 - \delta_i)e^{\int_0^T r_i - P_i(\mathbf{0}, y_d^*(t))dt} > 0; \quad 1 \le i \le n$$
$$\lambda_{n+1} = (1 - \delta)e^{-dT}.$$

As $\lambda_{n+1} \in (0, 1)$ and $\lambda_i > 0$, $1 \le i \le n$, it follws that E^* is locally asymptotically stable provided that (5) holds.

We shall now prove that E^* is globally asymptotically stable provided that (6) holds. To this purpose, let us choose $\varepsilon > 0$ such that

$$\xi_i = (1 - \delta_i)e^{\int_0^T r_i - P_i\left(\frac{\mathbf{r}}{\mathbf{a}} + \boldsymbol{\varepsilon}, y_d^*(t) - \varepsilon\right)dt} < 1 \quad \text{for all } 1 \le i \le n.$$

Since $y'(t) \ge -dy(t)$, it follows from the comparison lemma for systems of impulsively perturbed ordinary differential inequalities (see [3]) that $y(t) \ge y_1(t)$ for all $t \ge 0$, where y_1 satisfies

$$\begin{cases} y_1'(t) = -dy_1(t), & t \ne nT,\ t \ne (n + l - 1)T; \\ \Delta y_1(t) = -\delta y_1(t), & t = (n + l - 1)T; \\ \Delta y_1(t) = \mu, & t = nT; \\ y_1(0+) = y(0+). \end{cases} \tag{9}$$

It then follows from the properties of y_d^* listed in the Lemma above that $y(t) \ge y_d^*(t) - \varepsilon$ for t large enough; for the sake of simplicity, let us suppose that $y(t) \ge$

$y_d^*(t)-\varepsilon$ for all $t\geq 0$. Let now $1\leq i\leq n$. Similarly, since $x_i'(t)\leq x_i(t)(r_i-a_{ii}x_i(t))$, it follows that $x_i(t)\leq x_{1i}(t)$ for all $t\geq 0$, where x_{1i} satisfies

$$\begin{cases} x_{1i}'(t)=x_{1i}(t)(r_i-a_{ii}x_{1i}(t)), & t\neq(n+l-1)T;\\ \Delta x_{1i}(t)=-\delta_i x_{1i}(t), & t=(n+l-1)T;\\ x_{1i}(0+)=x_i(0+). \end{cases} \tag{10}$$

Since $\limsup\limits_{t\to\infty} x_{1i}(t)\leq \frac{r_i}{a_{ii}}$, it then follows that $x_i(t)\leq \frac{r_i}{a_{ii}}+\varepsilon$ for t large enough; for the sake of simplicity, let us suppose that $x_i(t)\leq \frac{r_i}{a_{ii}}+\varepsilon$ for all $t>0$. By the monotonicity assumptions on φ_i, it is seen that

$$\begin{cases} x_i'(t)\leq x_i(t)\left[r_i-P_i\left(\dfrac{\mathbf{r}}{\mathbf{a}}+\boldsymbol{\varepsilon},y_d^*(t)-\varepsilon\right)\right], & t\neq(n+l-1)T;\\ x_i(t+)=(1-\delta_i)x_i(t) & t=(n+l-1)T. \end{cases}$$

By integrating the above inequality on $((n+l-1)T,(n+l)T]$, it is seen that

$$x_i((n+l)T)\leq x_i((n+l-1)T)(1-\delta_i)e^{\int_0^T r_i-P_i\left(\frac{\mathbf{r}}{\mathbf{a}}+\boldsymbol{\varepsilon},y_d^*(t)-\varepsilon\right)dt},$$

that is, $x_i((n+l)T)\leq x_i((n+l-1)T)\xi_i$. Then $x_i((n+l)T)\leq x_i(lT)\xi_i^n$ and consequently

$$x_i((n+l)T)\to 0 \quad \text{for } n\to\infty. \tag{11}$$

Since

$$0<x_i(t)<x_i((n+l-1)T)e^{r_iT} \text{ for } t\in((n+l-1)T,(n+l)T]$$

it follows from (11) that $x_i(t)\to 0$ for $t\to\infty$.

We now prove that $y(t)-y_d^*(t)\to 0$ as $t\to\infty$. To this purpose, let us denote

$$d_{\varepsilon'}=d-\sum_{i=1}^n e_i\varepsilon'\varphi_i(0,\ldots,\varepsilon',\ldots,0,0)$$

and let us choose ε' such that $d_{\varepsilon'}>0$. Since $x_i(t)\to 0$ for $t\to\infty$, it follows that there is $\tilde{T}>0$ such that $0<x_i(t)<\varepsilon'$ for $t\geq\tilde{T}$ and $1\leq i\leq n$; without loss of generality, we may suppose that

$$0<x_i(t)<\varepsilon' \quad \text{for } t\geq 0 \text{ and } 1\leq i\leq n.$$

One then has

$$-dy(t)\leq y'(t)\leq -d_{\varepsilon'}y(t)$$

for $t\neq(n+l-1)T$, $t\neq nT$. Let us denote

$$\tilde{y}^*=y^*_{d_{\varepsilon'}}.$$

Using a comparison argument, it follows that $y_1(t)\leq y(t)\leq y_2(t)$ and $y_1(t)-y_d^*(t)\to 0$, $y_2(t)-\tilde{y}^*(t)\to 0$ as $t\to\infty$, where y_1 is as defined in (9) and y_2 is defined through a system similar to (9) with d replaced by $d_{\varepsilon'}$.

Let now $\varepsilon_1>0$. It follows that

$$y^*(t)-\varepsilon_1\leq y(t)\leq\tilde{y}^*(t)+\varepsilon_1$$

for t large enough. Since $\sup\limits_{t\in[0,T]} |y_d^*(t) - \tilde{y}^*(t)| \to 0$ for $\varepsilon' \to 0$ and ε_1 is arbitrary, it follows that $y(t) - y_d^*(t) \to 0$ as $t \to \infty$, which ends the proof of the global stability result. □

4. The permanence of the system

We shall now study the permanence of (S). To this purpose, we introduce the following definition.

Definition 1. *The system* (S) *is said to be permanent (uniformly persistent) if there are* m*,* $M > 0$ *such that for each solution of* (S) *with strictly positive initial data* $x(0)$*,* $y(0)$*,* $z(0)$*, it follows that there is* $T_0 > 0$ *such that* $m \le x(t), y(t), z(t) \le M$ *for all* $t \ge T_0$*. Here,* T_0 *may depend upon the initial data, but* m *and* M *do not.*

In biological terms, if (S) is permanent, then the pests and the predator will coexist forever without facing extinction or growing indefinitely. Obviously, the permanence of (S) excludes any kind of stability of the prey-free periodic solution and it is associated with a failure of our IPM strategy

Theorem 2. *The system* (S) *is permanent provided that*

$$d > \sum_{j\neq i} e_j \frac{r_j}{a_{jj}} \varphi_j(0, \ldots, \frac{r_j}{a_{jj}}, \ldots, 0, 0), \quad \textit{for all } 1 \le i \le n. \tag{12}$$

and

$$\int_0^T P_i(\mathbf{0}, y_d^*(t))dt - \ln(1-\delta_i) < \left[r_i - \sum_{j\neq i} a_{ij} \frac{r_j}{a_{jj}} \right] T, \quad \textit{for all } 1 \le i \le n. \tag{13}$$

Proof. Suppose that $(\mathbf{x}(\cdot), y(\cdot))$ is a solution of (S) which starts with strictly positive initial data $\mathbf{x}(0)$, $y(0)$. As previously noted, there is a constant $M > 0$ not depending on the initial data such that $x_i(t) \le M$, $y(t) \le M$ for $t \ge 0$ and all $1 \le i \le n$. Also, as done above, we note that $y(t) > y^*(t) - \varepsilon'$ for t large enough, for all $\varepsilon' \in (0, \frac{\mu e^{-dT}(1-\delta)}{1-e^{-dT}(1-\delta)})$. Consequently, $y(t) \ge m_y$ for t large enough, where

$$m_y = \frac{\mu e^{-dT}(1-\delta)}{1 - e^{-dT}(1-\delta)} - \varepsilon'$$

and ε' is fixed as indicated above.

We then only need to find $\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n > 0$ such that $x_i(t) \ge \overline{m}_i$ for t large enough and all $1 \le i \le n$. First, let us choose $m_1, m_2, \ldots, m_n$ and ε small enough, so that

$$m_i < \frac{r_i}{a_{ii}} + \varepsilon, \tag{14}$$

$$\begin{aligned} d > {} & e_i m_i \varphi_i(0, \ldots, m_i, \ldots, 0, 0) \\ & + \sum_{j\neq i} e_j \frac{r_j}{a_{jj}} \varphi_j(0, \ldots, \frac{r_j}{a_{jj}} + \varepsilon, \ldots, 0, 0), \end{aligned} \tag{15}$$

for all $1 \leq i \leq n$ and

$$\theta_i = (1-\delta_i)e^{\int_0^T r_i - a_{ii}m_i - \sum_{j\neq i} a_{ij}\left(\frac{r_j}{a_{jj}}+\varepsilon\right) - P_i(0, y_{1i}^*(t))dt} > 1$$

for all $1 \leq i \leq n$, where

$$y_{1i}^* = y^*_{d - e_i m_i \varphi_i(0,\ldots,m_i,\ldots,0) - \sum_{j\neq i} e_j \frac{r_j}{a_{jj}}\varphi_j(0,\ldots,\frac{r_j}{a_{jj}}+\varepsilon,\ldots,0)}.$$

We first prove that there are $t_1, t_2, \ldots, t_n > 0$ such that $x_i(t_i) \geq m_i$ for $1 \leq i \leq n$. We shall argue by contradiction. Suppose that $x_1(t) \leq m_1$ for all $t > 0$.

Let us choose $\eta_1 > 0$ such that

$$\theta_1' = (1-\delta_1)e^{\int_0^T r_1 - a_{11}m_1 - \sum_{j\neq 1} a_{1j}\left(\frac{r_j}{a_{jj}}+\varepsilon\right) - P_i(\mathbf{0}, y_{11}^*(t)+\eta_1)dt} > 1.$$

One then has

$$y'(t) \leq -y(t)\Bigg[d - e_i m_i \varphi_i(0,\ldots,m_i,\ldots,0,0) + \sum_{j\neq i} e_j \frac{r_j}{a_{jj}}\varphi_j(0,\ldots,\frac{r_j}{a_{jj}}+\varepsilon,\ldots,0,0)\Bigg]$$

for $t \neq (n+l-1)T$, $t \neq nT$, while

$$y(t+) = (1-\delta)y(t) \quad \text{for } t = (n+l-1)T, \quad y(t+) = y(t)+\mu \quad \text{for } t = nT.$$

Consequently, there is $n_1 \in \mathbb{N}$ such that $y(t) \leq y_1^* + \eta_1$ for $t \geq n_1 T$. One then has

$$x_1'(t) \geq x_1(t)\left[r_1 - a_{11}m_1 - \sum_{j\neq 1} a_{1j}\left(\frac{r_j}{a_{jj}}+\varepsilon\right) - P_i(\mathbf{0}, y_{11}^*(t)+\eta_1)\right]$$

for $t \neq (n+l)T$, $t \geq n_1 T$, while

$$x_1(t+) = (1-\delta_1)x_1(t) \quad \text{for } t = (n+l-1)T.$$

By integrating the above on $((n+l-1)T, (n+l)T]$, $n \geq n_1 + 1$, one sees that

$$x_1((n+l)T) \geq x_1((n+l-1)T)\theta_1'. \tag{16}$$

Consequently, $x_1((n+k+l)T) \geq {\theta_1'}^k x((n+l)T) \to \infty$ as $k \to \infty$, which is a contradiction, since $x_1(t) \leq m_1$ for all $t \geq 0$. As a result, there is $t_1 > 0$ such that $x_1(t_1) > m_1$. The same argument can be used to establish the existence of $t_2, \ldots, t_n$.

If $x_1(t) \geq m_1$ for all $t \geq t_1$, there is nothing left to prove. Otherwise, $x_1(t) < m_1$ for some $t > t_1$. Let us denote $s_1 = \inf\{t > t_1; x_1(t) < m_1\}$. If $s_1 \neq (n+l-1)T$, then $x_1(s_1) = m_1$. If $s_1 = (n+l-1)T$, then not necessarily $x_1(s_1) = m_1$, but $x_1(s_1) \in \left[m_1, \frac{m_1}{1-\delta_1}\right]$. Note that it is not possible that $x_1(s) < m_1$ for all $s > s_1$, so necessary $x_1(t) \geq m_1$ for some $t \geq s_1$. By a similar argument, one may construct a sequence $(s_n)_{n\geq 2}$ such that

(1) $x_1(s_{2k+1}) \in [m_1, \frac{m_1}{1-\delta_1}]$, $x_1(s_{2k}) = m_1$.

(2) $x_1(s) \leq m_1$ for $s \in (s_{2k+1}, s_{2k+2})$.
(3) $x_1(s) > m_1$ for $s \in (s_{2k}, s_{2k+1})$,

showcasing the fact that x_1 oscillates about m_1.

We now show that $\overline{T} = \sup\{s_{2k} - s_{2k-1}; k \in \mathbb{N}^*\} < \infty$. Suppose that this is not the case. Then there is $(k_j)_{j\geq 1}$ such that $s_{2k_j} - s_{2k_j-1} > jT$. Consequently, in a way similar to the derivation of (16), it follows that

$$x_1(s_{2k}) \geq x_1(s_{2k-1}){\theta'_1}^{j+2} e^{-2r_1T},$$

which is a contradiction, as ${\theta'_1}^{j+2} \to \infty$ for $j \to \infty$ and $x_1(s_{2k}) = m_1$. It then follows that

$$x'_1(t) \geq x_1(t)\left[r_1 - a_{11}m_1 - \sum_{j\neq 1} a_{1j}\left(\frac{r_j}{a_{jj}} + \varepsilon\right) - P_i(\mathbf{0}, M)\right]$$

for $t \in (s_{2k+1}, s_{2k+2})$, and consequently

$$x(s) \geq \tilde{m}_1 \quad \text{for } s \in (s_{2k+1}, s_{2k+2}),$$

where

$$\tilde{m}_1 = m_1 e^{\left[r_1 - a_{11}m_1 - \sum_{j\neq 1} a_{1j}\left(\frac{r_j}{a_{jj}} + \varepsilon\right) - P_i(\mathbf{0},M)\right]T}.$$

Putting $\overline{m}_1 = \min(\tilde{m}_1, m_1)$, it follows that $x_1(t) \geq \overline{m}_1$ for t large enough, so the x_1-population is permanent. By a similar argument, one may prove that the other prey populations are permanent, which ends the proof of the permanence result. □

From the proofs of Theorems 1 and 2, it can also be seen that the following partial permanence result holds.

Corollary 1. *Let* $(\mathbf{x}(\cdot), y(\cdot))$ *be a solution of* (S) *starting with strictly positive initial data and let* I *be a subset of* $\{1, 2, \ldots, n\}$*. Then the predator and all prey species* i *for which* $i \in I$ *are permanent, while the other prey species tend to extinction provided that*

$$d > \sum_{j\neq i} e_j \frac{r_j}{a_{jj}} \varphi_j(0, \ldots, \frac{r_j}{a_{jj}}, \ldots, 0, 0), \quad i \in I.$$

and

$$\int_0^T P_i(\mathbf{0}, y_d^*(t))dt - \ln(1 - \delta_i) < \left[r_i - \sum_{j\neq i} a_{ij}\frac{r_j}{a_{jj}}\right]T, \quad i \in I,$$

$$\int_0^T P_i(\frac{\mathbf{r}}{\mathbf{a}}, y_d^*(t))dt - \ln(1 - \delta_i) > r_i T, \quad i \notin I.$$

5. Biological significance and concluding remarks

Let us rewrite the equation describing the dynamics of the i-th pest species as

$$x_i'(t) = x_i(t)\left[r_i - \sum_{j=1}^{n} a_{ij}x_j(t) - y(t)\varphi_i(\mathbf{x}(t), y(t))\right].$$

It is seen that the integral $\int_0^T P_i(\mathbf{0}, y_d^*(t))dt$, equal to $\int_0^T y_d^*\varphi_i(\mathbf{0}, y_d^*(t))dt$, approximates the per capita loss of pest species i in a period T due to predation when this species approaches extinction and r_iT approximates the per capita growth of the same species in the same conditions. The term $-\ln(1-\delta_i)$ represents the per capita loss of pest species i due to pesticide spraying.

Consequently, when all prey populations approach extinction and the local stability condition (5) holds, their size losses in a period exceeds their growths in the same amount of time and they cannot escape extinction. The significance of (6) is similar, but this time, in order to be able to drive to extinction the pest species under any circumstances, the predation should be larger even at higher pest densities, where saturation occurs.

As far as the significance of the permanence condition is concerned, condition (12) expresses the fact that the death rate of the predator class is large enough, so that the size of the predator class does not grow too large to impair upon the survival of the prey species. Also, condition (13) asserts that predation on species i is weak even in unfavorable circumstances, when the other prey species, competing with species i, reach the carrying capacity of the environment.

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ASUPRA CONTROLULUI IMPULSIV AL UNUI LANŢ TROFIC CU N SPECII DE PRADĂ ŞI O SPECIE DE RĂPITOR

(Rezumat)

Lucrarea de faţă studiază dinamica unui lanţ trofic cu n specii de pradă şi o specie de răpitor , care este supus la metode de control de tip impulsiv, din punct de vedere al găsirii unor condiţii suficiente pentru extincţia prăzii şi pentru permanenţa prăzii şi a răpitorului. Cu ajutorul teoriei Floquet a sistemelor de ecuaţii diferenţiale ordinare perturbate impulsiv, stabilitatea locală şi globală a aşa-numitei soluţii periodice libere de pradă sunt obţinute în termenii unor condiţii integrale cu semnificaţie biologică. Permanenţa totală şi parţială a sistemului sunt de asemenea investigate folosind condiţii similare.