BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Tomul LVII (LXI), Fasc. 3, 2011
Secţia
CONSTRUCŢII DE MAŞINI

# ANALYTICAL AND GRAPHICAL SOLUTIONS TO PROBLEMS IN DESCRIPTIVE GEOMETRY INVOLVING PLANES AND LINES

BY

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**Abstract.** Several common situations which occur when solving problems in descriptive geometry, including finding the intersection of two planes passing through points with coordinates of a certain form and finding the intersections between particular lines and planes are investigated through analytical and graphical methods.

**Key words:** planes, lines, traces, analytic geometry, descriptive geometry.

## 1. The statement of the problem

Of concern in this paper is the following problem, encompassing several situations commonly occuring in concrete applications of descriptive geometry.

*Given the points A(100,10,30), B(60,50,10), C(60,20,40), M(120,0,80), N(60,80,0),* $R_x$*(10,0,0), I(80,y,z), find:*

*a) the coordinates of* $I$ *knowing that this point is situated on the (AB) line;*

*b) the traces of the plane [P] defined by the lines (CI) and (AB);*

*c) the intersection between the planes [P] and [Q], where [Q] is a plane parallel to (Ox) which has traces at distance 25 and elevation 65, respectively;*

*d) the intersection point* $K$ *between the line* $(AB)$ *and the plane* $[Q]$.

We shall solve this problem first by means of analytic geometry, briefly discussing the main theoretical points and the formulae concerned, and then by means of descriptive geometry.

## 2. The analytical approach

We first discuss several properties of a plane $[Q]$ which is parallel to $(Ox)$ and passes through $Q_y(0,d,0)$ of distance $d$ and $Q_z(0,0,e)$ of elevation $e$, respectively. Namely, we shall find the equation of $[Q]$ and determine its intersection with given lines and planes.

### 2.1. *The equation of [Q] plane*

Since two vectors parallel to $[Q]$ are $\vec{i}$ and $\overrightarrow{Q_yQ_z} = -d\vec{j} + e\vec{k}$, the vector $\overrightarrow{n_Q} = \vec{i} \times \overrightarrow{Q_yQ_z} = -d\vec{k} - e\vec{j}$ is normal to $[Q]$. The equation of $[Q]$ is then $-dz - ey + D = 0$, the constant $D$ being determined through the condition that $Q_y$ belongs to $[Q]$. Consequently, the equation of $[Q]$ can be written as:

$$[Q]: ey + dz - de = 0. \tag{1}$$

### 2.2. *The intersections of $[Q]$ with lines and planes*

We find now the intersection of a line determined by two of its points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and the plane $[Q]$. One sees that the parametric equation of the line $(AB)$ is $\dfrac{x - x_A}{x_B - x_A} = \dfrac{y - y_A}{y_B - y_A} = \dfrac{z - z_A}{z_B - z_A} = t$, that is [1], [5]:

$$(AB): x = x_A + t(x_B - x_A),\ y = y_A + t(y_B - y_A),\ z = z_A + t(z_B - z_A), \tag{2}$$

$t$ being a real parameter. To find the intersection between $(AB)$ and $[Q]$, we solve the system consisting in Eqs (1) and (2). By substituting $y$ and $z$ given by Eq (2) into Eq (1), we find that:

$$t = \frac{de - ey_A - dz_A}{e(y_B - y_A) + d(z_B - z_A)}. \tag{3}$$

Consequently, the point of intersection $K$ between $(AB)$ and $[Q]$ has coordinates:

$$x_K = x_A + \frac{de - ey_A - dz_A}{e(y_B - y_A) + d(z_B - z_A)}(x_B - x_A), \tag{4}$$

(5) $$y_K = y_A + \frac{de - ey_A - dz_A}{e(y_B - y_A) + d(z_B - z_A)}(y_B - y_A),$$

(6) $$z_K = z_A + \frac{de - ey_A - dz_A}{e(y_B - y_A) + d(z_B - z_A)}(z_B - z_A).$$

We find now the intersection between the $[Q]$ plane and a $[P]$ plane determined by $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$. This line can be thought as being determined by the point $K$ above, the intersection between $(AB)$ and $[Q]$, and the point $H$, the intersection between $(AC)$ and $[Q]$. By similarity with Eqs (4)-(6), it is seen that the coordinates of $H$ are given by:

(7) $$x_H = x_A + \frac{de - ey_A - dz_A}{e(y_C - y_A) + d(z_C - z_A)}(x_C - x_A),$$

(8) $$y_H = y_A + \frac{de - ey_A - dz_A}{e(y_C - y_A) + d(z_C - z_A)}(y_C - y_A),$$

(9) $$z_H = z_A + \frac{de - ey_A - dz_A}{e(y_C - y_A) + d(z_C - z_A)}(z_C - z_A).$$

Then the intersection $(KH) \equiv (D)$ between $[P]$ and $[Q]$ has the parametric equation:

(10) $$\frac{x - x_K}{x_H - x_K} = \frac{y - y_K}{y_H - y_K} = \frac{z - z_K}{z_H - z_K} = t\,, t \in \mathbf{R}\,,$$

$x_K, y_K, z_K$ and $x_H, y_H, z_H$ being given by Eqs (4)-(6) and (7)-(9).

**2.3.** *The traces of a plane [P]*

In what follows, we shall determine the traces of a plane, determined by three of its points, on the coordinate planes. It can be shown [2] that the traces of a plane $[P]$ determined by $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$ have the following equations:

(11) $$(P_h): x\Delta_x + y\Delta_y = \Delta; \quad z = 0\,,$$

(12) $$(P_l): y\Delta_y + z\Delta_z = \Delta; \quad x = 0\,,$$

(13) $$(P_v): z\Delta_z + x\Delta_x = \Delta; \quad y = 0.$$

Also, [P] intersects the coordinate axes in the points $P_x$, $P_y$, $P_z$ given by:

$$P_x = P_x\left(\frac{\Delta}{\Delta_x},0,0\right),\ P_y = P_y\left(0,\frac{\Delta}{\Delta_y},0\right),P_z = P_z\left(0,0,\frac{\Delta}{\Delta_z}\right), \tag{14}$$

$$\Delta_x = \begin{vmatrix} y_A & z_A & 1 \\ y_B & z_B & 1 \\ y_C & z_C & 1 \end{vmatrix},\ \Delta_y = \begin{vmatrix} z_A & x_A & 1 \\ z_B & x_B & 1 \\ z_C & x_C & 1 \end{vmatrix},\ \Delta_z = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \tag{15}$$

$$\Delta = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}. \tag{16}$$

We are now ready to solve the problem stated in Section 1.

**Solution**

a) Since $I$ is situated on the line $(AB)$, its coordinates $x_I, y_I, z_I$ verify Eq (2) for a value of $t$ which is to be determined. From the first part of Eq (2), it follows that $t = \frac{1}{2}$ and consequently that $y_I = 30$ and $z_I = 20$.

b) The plane $[P]$ can be thought as being determined by $A, B$ and $C$. Consequently, since $\Delta = 108000$, $\Delta_x = 600$, $\Delta_y = 1200$ and $\Delta_z = 1200$, it follows then from Eq (14) that the intersections of $[P]$ with the coordinate axes are $P_x(180,0,0)$, $P_y(0,90,0)$, $P_z(0,0,90)$. It also follows from Eqs (11)-(13) that the trace of the plane $[P]$ on $(xOy)$ is $(P_h): 2x + y = 90;\ z = 0$, the trace of $[P]$ on $(xOz)$ is $(P_v): 2x + z = 90;\ y = 0$ and the trace of $[P]$ on $(yOz)$ is $(P_l): y + z = 90;\ x = 0.$

c) Using Eqs (4)-(6) and Eqs (7)-(9), it follows that $x_H = 90$, $y_H = \frac{25}{2}$, $z_H = \frac{65}{2}$ and $x_K = \frac{670}{7}$, $y_K = \frac{100}{7}$, $z_K = \frac{195}{7}$. From Eq (10), one may deduce that the intersection between the plane $[P]$ and the plane $[Q]$ is the line $(KH)$ which has the parametric equation:

(17) $$\frac{x-\frac{670}{7}}{-16}=\frac{y-\frac{100}{7}}{-5}=\frac{z-\frac{195}{7}}{13}=t,\quad t\in\mathbf{R}.$$

For $z=0$, it follows that $t=-\frac{15}{7}$ and consequently the intersection between $(KH)$ and the plane $(xOy)$ is the point $H_2(130,25,0)$. Similarly, for $y=0$, it follows that $t=\frac{20}{7}$ and the intersection between $(KH)$ and the plane $(xOz)$ is the point $V_2(50,0,65)$.

d) As seen above, $x_K=\frac{670}{7}$, $y_K=\frac{100}{7}$, $z_K=\frac{195}{7}$.

## 3. Descriptive Geometry approach

a) Because the point $I$ is lying on the straight line $(AB)$, its projections are located on the corresponding projections of the straight line (Fig. 1) [4]. Hence, if a point $I\in(AB)$, it follows that $i\in(ab)$, $i'\in(a'b')$ and $i''\in(a''b'')$.

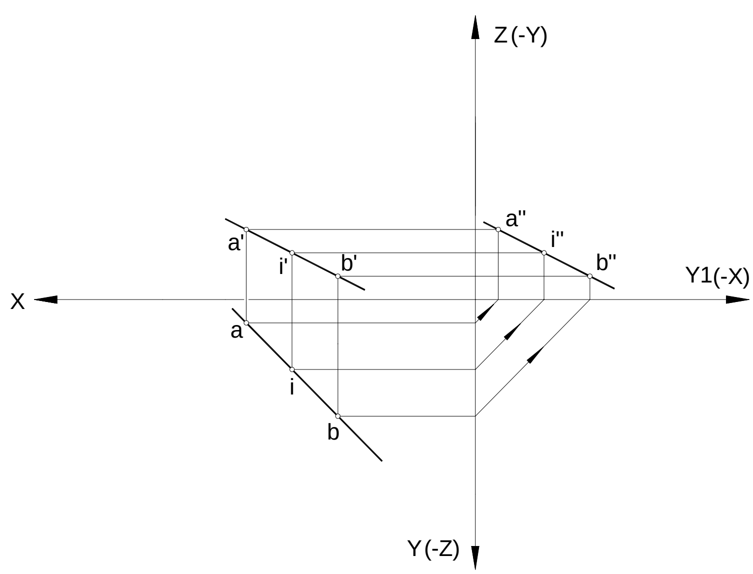


Fig. 1 – Projections of the point I [3]

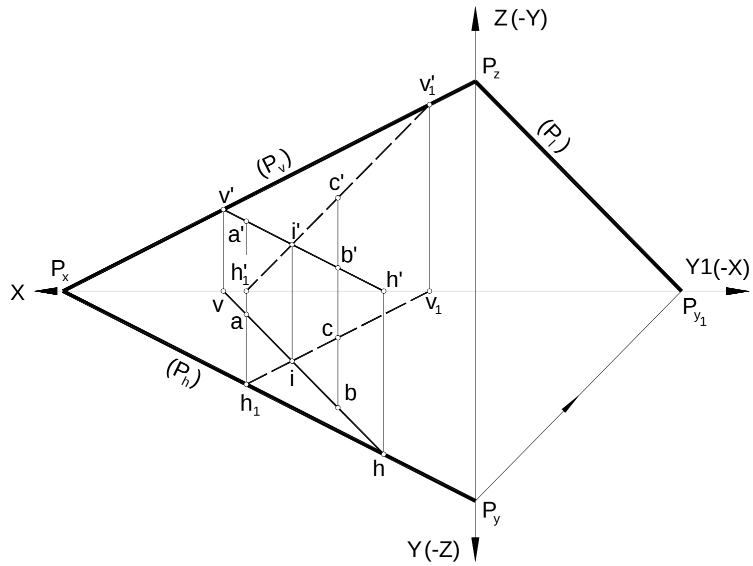


Fig. 2 – Traces of the plane [P]

b) We determine first the projections $v$ , $v_1$ and $h'$ , $h_1'$ , by extending the projections *(ab), (ci)* and *(a'b')*, *(c'i')*, respectively, up to the (*Ox*) axis, and then, by using the projecting lines, we find the traces of given straight lines: *h* on *(ab)*, $h_1$ on *(ci)* and *v'* on *(a'b')*, $v_1'$ on *(c'i')*, respectively (Fig. 2).

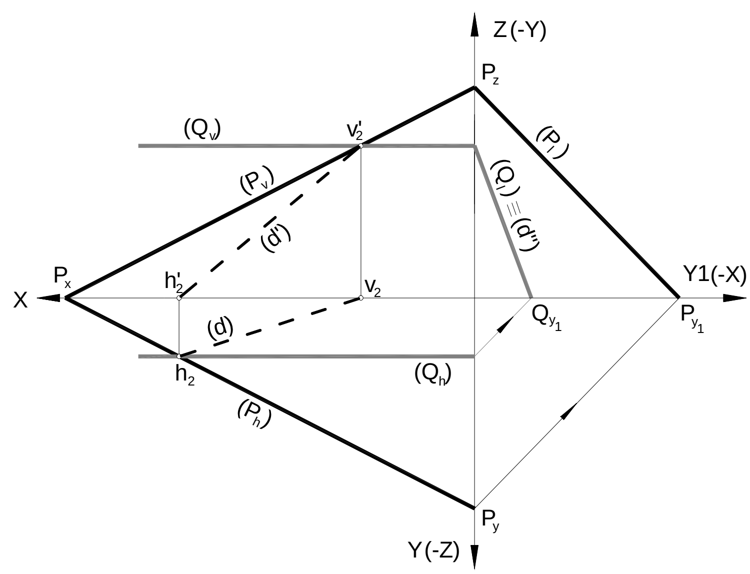


Fig. 3 – (D)(d,d',d''), the intersection line of planes [P] and [Q]

c) Two concurrent planes $[P]$ and $[Q]$ intersect along a straight line *(D)(d,d',d")* (Fig. 3), whose traces lie at the intersection of corresponding traces of both planes.

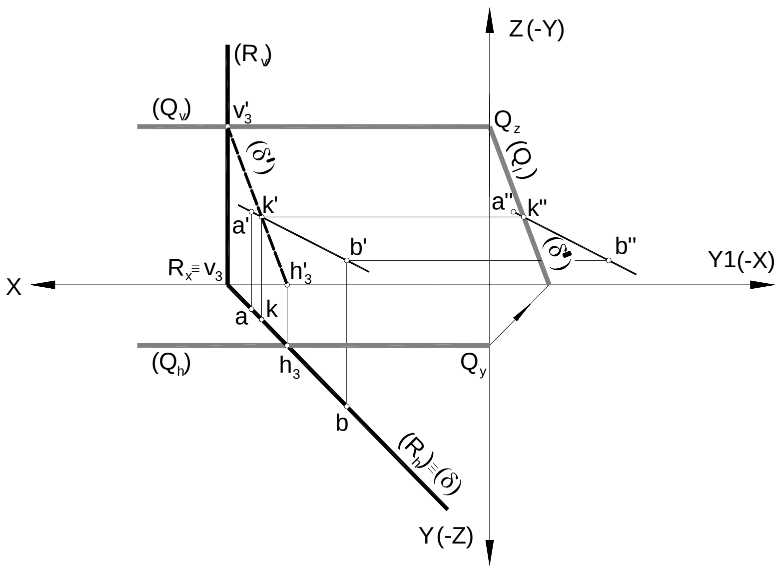


Fig. 4 – K, the intersection point of (AB)-[Q] intersection

d) We draw an auxiliary cutting plane $[R]$ which contains the given straight line and we find the line of intersection of the cutting plane $[R]$ and the given plane $[Q]$ (Fig. 4). The required point $K(k,k',k'')$ is at the intersection of the straight lines $(AB)(ab,a'b',a''b'')$ and $(\Delta)(\delta,\delta',\delta'')$, common to both planes $[P]$ and $[R]$.

## 4. Conclusions

As we can see from a comparative view of both solutions presented above, the one employing Descriptive Geometry offers a quick and elegant reasoning which appeals to one's intuition, while the approach based on Analytic Geometry, although more computational, offers general formulas which can be applied to a large class of problems which are reductible to the ones discussed in this paper.

*22.01.2011*

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## R E F E R E N C E S

1. C ă r ă u ş u A., *Vector algebra, analytic & differential geometry*. Ed. PIM, Iaşi, **2003.**

2. G e o r g e s c u P., H î n c u G., *Finding the traces of a given plane: analytically and through graphical constructions*, Proceedings of ICEGD2011 *Sustainable Eco-Design*, 2-4 june 2011, Iaşi, Bul. Inst. Polit. Iaşi, **LVII** (**LXI**), Facultatea de Construcţii de Maşini, (**2011**).

3. H î n c u G., *Infografică. AutoCAD*, Ed. Societăţii Academice "Matei Teiu Botez", Iaşi, **2009.**

4. H î n c u G., *Geometrie descriptivă. Curs şi aplicaţii*, Ed. PIM, Iaşi, **2009.**

5. * * * *Mica Enciclopedie Matematică*, Ed. Tehnică, Bucureşti, **1985**.

## SOLUŢII ANALITICE ŞI GRAFICE ALE UNOR PROBLEME DIN GEOMETRIA DESCRIPTIVĂ ÎN CARE INTERVIN PLANE ŞI DREPTE

(Rezumat)

Lucrarea de faţă abordează câteva probleme de intersecţie ale unor drepte şi plane particulare prin metodele geometriei analitice şi respectiv ale geometriei descriptive. Un studiu comparativ al celor două metode atestă faptul că raţionamentul bazat pe geometrie descriptivă este mai intuitiv, mai rapid, în vreme ce raţionamentul analitic, deşi mai laborios, poate fi adaptat pentru rezolvarea unei clase largi de probleme înrudite.