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Impulsive perturbations of a three-trophic prey-dependent food chain system

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Abstract

The dynamics of an impulsively controlled three-trophic food chain system with general nonlinear functional responses for the intermediate consumer and the top predator are analyzed using the Floquet theory and comparison techniques. It is assumed that the impulsive controls act in a periodic fashion, the constant impulse (the biological control) and the proportional impulses (the chemical controls) acting with the same period, but not simultaneously. Sufficient conditions for the global stability of resource and intermediate consumer-free periodic solution and of the intermediate consumer-free periodic solution are established, the latter corresponding to the success of the integrated pest management strategy from which our food chain system arises. In this regard, it is seen that, theoretically speaking, the control strategy can be always made to succeed globally if proper pesticides are employed, while as far as the biological control is concerned, its global effectiveness can also be reached provided that the top predator is voracious enough or the (constant) number of top predators released each time is large enough or the release period is small enough. Some situations which lead to chaotic behavior of the system are also investigated by means of numerical simulations.

Keywords: Simple food chain; Integrated pest management; Impulsive perturbations; Global stability; Chaos

1. Introduction

Classical two-species continuous time models have constituted for a long time the main tool used to investigate the interactions between ecological populations (see, for instance, [1–3]). However, as seen from Poincaré's theorem, such models have only two behavior patterns, that is, they approach either a limit cycle or an equilibrium point and consequently fail to capture the complex behavior of some natural ecosystems. Further, other shortcomings of certain two-species models have also been pointed out. These are the paradox of enrichment [4], which states that an increase in the carrying capacity of the environment in a Lotka–Volterra model will cause an increase in the size of the predator class at equilibrium, but not in that of the prey class, and the paradox of biological control [5], which states that the low prey equilibrium densities of a Lotka–Volterra model are inherently unstable. Other authors

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have also criticised ratio-dependent type models, which were introduced as a replacement of Lotka–Volterra models (see [6,7]).

Consequently, another paradigm started to prevail, that is, the idea that the behavior of a complex system can be understood only when mutual interactions between a larger number of species are considered in a single model [8]. As a result, complex behavior, in the form of stable equilibria, limit cycles, multiple attractors and chaos, has been observed in three- or more-species models [9] and it has also been noted that the dynamical outcome may depend on the initial population sizes, which is more in line with the results of field experiments and observations.

To understand the dynamical behavior of ecological communities, one should start by tracing their food webs and quantifying the strength of the respective interspecies interactions. It has been observed by Hastings and Powell in [10] and by Klebanoff and Hastings in [11] that since food webs often describe a net of nonlinear predator–prey interactions, there is a natural tendency of food webs to oscillate and chaos may ultimately arise when two or more predator–prey subsystems oscillate with incommensurate frequencies. McCann and Yodzis [12] mention that the parameter values chosen by Hastings and Powell in [10] may be biologically unfeasible, but the conclusions obtained in [10] are valid, and indicate biologically reasonable sets of parameter values which also produce chaos. They also provide comments about which biological conditions (metabolic types) favor the apparition of chaos. Six natural types of food web configurations are studied in McCann et al. [13] and it is also found that the dominance of strong consumer–resource interactions may generate cyclic dynamics when the frequencies of oscillation are commensurate, respectively chaotic dynamics when the frequencies of oscillation are incommensurate, while the dominance of weak coupling between interactions may dampen the total oscillation of the system, together with other biological factors, such as omnivory and food-chain-predation mechanisms. See also Bascompte et al. [14].

The so-called simple food chain, which is studied in our paper, is a tri-trophic food chain which appears when a top predator P feeds on an intermediate consumer C, which in turn feeds on a resource R. In this model, neither the intermediate consumer nor the top predator feeds on other resources and nutrient recycling is not accounted for. The qualitative behavior of the simple food chain model with Holling type II functional responses for both the top predator and the intermediate consumer, that is, for $g_i(x) = \frac{a_i x}{1+b_i x}$, $i \in \{1, 2\}$, has been studied in detail by Hastings and Powell in [10] and by Klebanoff and Hastings in [11]. See also [15,16]. In these papers, it has been found that the model may exhibit chaotic behavior in the neighborhood of the intermediate consumer-free equilibrium and it has also been observed that the clearance rate of the intermediate consumer b_1 is a key parameter for the stability of the model. A thorough analysis of the simple food chain model with ratio-dependent functional response for both the top predator and the intermediate consumer has been performed in Hsu et al. [17]. Particularly, a tristability situation has been observed, in which different solutions tend to the origin, intermediate consumer-free equilibrium and positive equilibrium, respectively, for the same set of parameters and a discussion of the feasibility of the biological control has also been provided. Chaotic-looking solutions have also been found to exist for certain parameter values.

Simple food chain models may naturally appear as a result of integrated pest management strategies. As it has been noted that the abuse of pesticides has undesirable long-term environmental consequences, in order to regulate pest populations, use is often made of different methods which are specifically suited to the target pest and minimize the harmful effects on the environment or on nontarget organisms.

Biological control is defined as the reduction of pest populations by using their natural enemies (see [18]). While an approach to biological control (importation) relies on the import of exotic natural enemies of the pests, other approaches (augmentation and conservation) rely on supplementing or manipulating the existing natural enemies in order to enhance their effectiveness and on modifying the environment, respectively. A way to achieve augmentation is to release pest pathogens or infected pests with the purpose of generating an epidemic in the pest population, on the grounds that infected pests generally cause less environmental damage, another one being to breed natural predators of the pest in laboratories and to release them periodically in the ecosystem. Consequently, in our food chain model R is the resource to be protected, C is the pest which should be regulated and P is a natural predator of the pest which is augmented by means of periodic release of laboratory-bred individuals.

Adequate efficiency is attained when at least one of these approaches is combined with the responsible use of chemical controls (pesticide spraying) and the use of mechanical accessories, such as pest barriers and pest traps, in the form of an integrated pest management strategy. Note that the integrated pest management strategy is considered successful when the pest population is reduced under certain economically significant levels, rather than when the pest population is totally eradicated, as the latter might be economically or logistically unfeasible, or it might be potentially damaging to the environment. In this regard, the economic injury level (EIL) is defined in Stern et al. [19]

as the amount of pest injury which will justify the cost of using controls or the lowest pest density which causes economic damage.

Due to the inherent discontinuity of human activities (that is, pesticides cannot be sprayed all year round but only during certain periods of the year), a natural choice is to use discrete impulsive controls rather than continuous controls for our pest management strategy. In this regard, the effect of impulsive perturbations on the simple food chain model has been studied by Zhang and Chen [20] assuming linear responses for the top predator and the intermediate consumer, by Zhang and Chen in [21] assuming Holling type II functional responses, by Zhang et al. in [22] assuming Holling type IV (or simplified Monod–Haldane) functional responses and by Zhang et al. in [23] and Zhang and Chen in [24] assuming Beddington–DeAngelis functional responses. In all these papers, only the case of a constant impulsive perturbation has been considered. See also [25–27] for related results regarding the impulsive control of predator–prey systems and Georgescu and Moroşanu [28] for the discussion of an integrated pest management strategy involving biological and chemical impulsive controls. State-dependent impulsive control strategies are investigated in Meng et al. [29] and Jiao et al. [30], while a different approach to the control of a model related to ours, based on the use of Pontryagin's principle, is employed in Apreutesei [31].

2. The model

The abundance and interaction of resource, intermediate consumer and top predator populations may be expressed in terms of their biomass per spatial unit. In this regard, let $x(t)$, $y(t)$, $z(t)$ be the biomass per spatial unit of the resource, intermediate consumer and top predator, respectively. As previously mentioned, we assume that the top predator feeds on the intermediate consumer only and in turn the intermediate consumer feeds on the resource only, while the nutrient recycling is not accounted for.

The functional responses of the intermediate consumer and of the top predator are denoted by the nonlinear smooth functions g_1, g_2, depending only on the resource biomass density and on the intermediate consumer biomass density, respectively, and satisfying a few assumptions which will be outlined below. Due to the assumption above, our model may be called, following the terminology given in Arditi and Ginzburg [32] or in Huisman and DeBoer [33], prey dependent, as opposed to a model in which the functional responses of the predators are functions of the prey-to-predator ratios, which is called ratio dependent (or, more generally, predator dependent, when the dependence of the functional responses upon the characteristics of the predator classes takes a more complicated form). It is supposed that in the absence of predation from the intermediate consumer, the resource grows according to a logistic growth with intrinsic growth rate r and carrying capacity r/a.

The processes of resource conversion into intermediate consumer biomass and of intermediate consumer into top predator biomass, respectively, are characterized by constant conversion rates k_1 and k_2. The death rates d_1 and d_2 of the intermediate consumer and of the top predator, respectively, are also assumed to be constant.

It is assumed that top predators are bred in laboratories and subsequently released in an impulsive and periodic fashion of period T, in a fixed number μ each time. It is also assumed that pesticides are sprayed in an impulsive and periodic fashion, with the same period as that of the action of releasing top predators, but at different moments. As a result of pesticide spraying, fixed proportions δ_1, δ_2, δ_3 of the resource, intermediate consumer and top predator biomass are degraded each time.

On the basis of the above assumptions, we may formulate the following impulsively perturbed model

$$
\text{(S)}\quad \begin{cases} x'(t) = x(t)[r - ax(t)] - g_1(x(t))y(t), & t \neq (n+l-1)T,\ t \neq nT; \\ y'(t) = k_1 g_1(x(t))y(t) - g_2(y(t))z(t) - d_1 y(t), & t \neq (n+l-1)T,\ t \neq nT; \\ z'(t) = k_2 g_2(y(t))z(t) - d_2 z(t), & t \neq (n+l-1)T,\ t \neq nT; \\ \Delta x(t) = -\delta_1 x(t), & t = (n+l-1)T; \\ \Delta y(t) = -\delta_2 y(t), & t = (n+l-1)T; \\ \Delta z(t) = -\delta_3 x(t), & t = (n+l-1)T; \\ \Delta x(t) = 0, & t = nT; \\ \Delta y(t) = 0, & t = nT; \\ \Delta z(t) = \mu, & t = nT. \end{cases}
$$

Here, $T > 0$, $0 < l < 1$, $\Delta\varphi(t) = \varphi(t+) - \varphi(t)$ for $\varphi \in \{x, y, z\}$ and $t > 0$, $0 \leq \delta_1, \delta_2, \delta_3 < 1$, $n \in \mathbb{N}^*$. The functions g_1, g_2 are assumed to satisfy the following assumptions.

(G) g_i is of class C^1 on $\mathbb{R}_+$, $g_i(0) = 0$, increasing and such that $x \mapsto g_i(x)/x$ is decreasing on $\mathbb{R}_+$, $\left|g_i'(x)\right| \leq L_i$ for $x \in \mathbb{R}_+$, $i \in \{1, 2\}$, where $L_1, L_2 \geq 0$.

Note that hypothesis (G) is satisfied if functions g_1, g_2 represent Holling type II functional responses, that is, $g_i(x) = \frac{a_i x}{1+b_i x}$, $i \in \{1, 2\}$, in which a_i, $i \in \{1, 2\}$ are the search rates of the resource and of the intermediate consumer, respectively, and b_i, $i \in \{1, 2\}$, represent the corresponding clearance rates, that is, search rates multiplied by the (supposedly constant) handling time. Also, the above-mentioned constants L_1 and L_2 can be taken as globally Lipschitz constants for g_1, g_2, respectively.

Impulsive perturbations of our three-trophic food chain model have also been considered by Zhang and Chen in [21], in the form of the periodic constant impulsive perturbations of the top predator only (that is, no second group of conditions in (S)), with particular Holling type II functional responses for the intermediate consumer and for the top predator. In [21], the local asymptotic stability of the intermediate consumer-extinction periodic solution is established, provided that the impulsive period T is small enough, and it is also shown that the resource and intermediate consumer-free periodic solution is unstable.

By using similar techniques, that is, the Floquet theory of impulsively perturbed systems of ordinary differential equations and comparison techniques, we are also able to prove further global stability results for both the intermediate consumer-free periodic solution and the resource and intermediate consumer-free periodic solution, under appropriate conditions, the former result representing a sufficient condition for the success of our pest control strategy. Note that, due to the impulsive top predator release of constant strength, our controlled system does not exhibit the domino effect, characteristic to the unperturbed food chain system, that is, if one species dies out then all the species at higher-trophic levels die out as well (although the extinction of the resource will attract the extinction of the intermediate consumer, of course). Due to the proportional impulsive perturbations at $t = (n+l-1)T$, $n \in \mathbb{N}^*$, the resource and intermediate consumer-free periodic solution is no longer unstable for any values of the parameters involved, as it is the case when only constant impulsive perturbations of z are employed, and the existence of a threshold parameter which controls its stability is also established. Also, our food chain system may be interpreted as the nonlinear coupling of two predator–prey subsystems (intermediate consumer–resource and top predator–intermediate consumer) through the mediation of the intermediate consumer, while the impulsive perturbations induce commensurate oscillations, as they act with the same period T. It is therefore expected that the system will display an oscillatory behavior, tending to a (impulsively perturbed) limit cycle of period T for an important portion of the parameter space, corresponding to impulsive and periodic perturbations with significant strength.

3. Preliminaries

In this section we shall introduce a few definitions and notations together with a few auxiliary results relating to comparison methods and the Floquet theory for impulsively perturbed systems of ordinary differential equations. The biological well-posedness of the Cauchy problem associated to our system (S) for strictly positive initial data will also be established.

Let us denote by $f = (f_1, f_2, f_3)$ the mapping defined by the right-hand sides of the first three equations in (S). Let also $\mathcal{V}_0$ be the set of functions $V : \mathbb{R}_+ \times \mathbb{R}_+^3 \to \mathbb{R}_+$ which are locally Lipschitz in the second variable, continuous on $((n+l-1)T, nT] \times \mathbb{R}_+^3$ and on $(nT, (n+l)T] \times \mathbb{R}_+^3$ and for which the limits $\lim_{(t,y)\to((n+l-1)T+,x)} V(t, y) = V((n+l-1)T+, x)$ and $\lim_{(t,y)\to(nT+,x)} V(t, y) = V(nT+, x)$ exist and are finite for $x \in \mathbb{R}_+^3$ and $n \in \mathbb{N}^*$.

For $V \in \mathcal{V}_0$, we define the upper-right Dini derivative of V with respect to the system (S) at $(t, x) \in ((n+l-1)T, nT) \times \mathbb{R}_+^3$ or $(nT, (n+l)T) \times \mathbb{R}_+^3$ by

$$D^+V(t, x) = \limsup_{h\downarrow 0} \frac{1}{h} \left[V(t+h, x+hf(t, x)) - V(t, x)\right].$$

We now indicate a comparison result for solutions of impulsive differential inequalities which allows us to estimate the values of the solutions of (S). We suppose that $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following hypotheses.

(H) h is continuous on $((n+l-1)T, nT] \times \mathbb{R}_+$ and on $(nT, (n+l)T] \times \mathbb{R}_+$ and the limits $\lim_{(t,y)\to((n+l-1)T+,x)} h(t, y) = h((n+l-1)T+, x)$, $\lim_{(t,y)\to(nT+,x)} h(t, y) = h(nT+, x)$ exist and are finite for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}^*$.

Lemma 3.1 (*[34]*). *Let $V \in \mathcal{V}_0$ and assume that*

$$\begin{cases} D^+V(t,x(t)) \le h(t,V(t,x(t))), & t \ne (n+l-1)T, nT; \\ V(t,x(t+)) \le \psi_n^1(V(t,x(t))), & t = (n+l-1)T; \\ V(t,x(t+)) \le \psi_n^2(V(t,x(t))), & t = nT, \end{cases} \tag{3.1}$$

where $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies (H) *and $\psi_n^1, \psi_n^2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing for all $n \in \mathbb{N}^*$. Let $r(t)$ be the maximal solution of the impulsive Cauchy problem*

$$\begin{cases} u'(t) = h(t,u(t)), & t \ne (n+l-1)T, nT; \\ u(t+) = \psi_n^1(u(t)), & t = (n+l-1)T; \\ u(t+) = \psi_n^2(u(t)), & t = nT; \\ u(0+) = u_0 \end{cases} \tag{3.2}$$

defined on $[0,\infty)$. Then $V(0+,x_0) \le u_0$ implies that $V(t,x(t)) \le r(t)$ for all $t \ge 0$, where $x(t)$ is an arbitrary solution of (3.1).

Note that under appropriate regularity conditions the Cauchy problem (3.2) has a unique solution and in that case r becomes the unique solution of (3.2). We now indicate a result which provides estimations for the solution of a system of differential inequalities.

Lemma 3.2 (*[34]*). *Let the function $u \in PC^1(\mathbb{R}_+,\mathbb{R})$ satisfy the inequalities*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} \le (\ge) p(t)u(t) + f(t), & t \ne \tau_k, t > 0; \\ u(\tau_k+) \le (\ge) d_k u(\tau_k) + h_k, & k \ge 0; \\ u(0+) \le (\ge) u_0, \end{cases} \tag{3.3}$$

where $p, f \in PC(\mathbb{R}_+,\mathbb{R})$ and $d_k \ge 0$, h_k and u_0 are constants and $(\tau_k)_{k\ge 0}$ is a strictly increasing sequence of positive real numbers. Then, for $t > 0$,

$$u(t) \le (\ge) u_0 \left(\prod_{0<\tau_k<t} d_k\right) \mathrm{e}^{\int_0^t p(s)\mathrm{d}s} + \int_0^t \left(\prod_{0\le\tau_k<t} d_k\right) \mathrm{e}^{\int_s^t p(\tau)\mathrm{d}\tau} f(s)\mathrm{d}s + \sum_{0<\tau_k<t} \left(\prod_{\tau_k<\tau_j<t} d_j\right) \mathrm{e}^{\int_{\tau_k}^t p(\tau)\mathrm{d}\tau} h_k.$$

In the above, by $PC(\mathbb{R}_+,\mathbb{R})$ ($PC^1(\mathbb{R}_+,\mathbb{R})$) is meant the class of real piecewise continuous (real piecewise continuously differentiable) functions defined on $\mathbb{R}_+$. For other results on impulsive differential equations, see Bainov and Simeonov [34]. Using Lemma 3.2, it is now possible to prove that the Cauchy problem with strictly positive initial data is well-posed for our system (S), that is, solutions (x, y, z) starting with strictly positive initial data remain strictly positive and bounded on their whole domains.

Lemma 3.3. *The positive orthant $(\mathbb{R}_+^*)^3$ is an invariant region for the system* (S).

Proof. Let us consider $(x, y, z) : [0, T_0) \to \mathbb{R}^3$ a saturated solution for (S) which starts with strictly positive $x(0)$, $y(0)$, $z(0)$. Under our assumptions (G), it is easy to see that

$$\begin{cases} x'(t) \ge x(t)\left[r - ax(t) - g_1'(0)y(t)\right], & 0 \le t < T_0, t \ne (n+l-1)T, nT; \\ y'(t) \ge y(t)\left[k_1 g_1(x(t)) - g_2'(0)z(t) - d_1\right], \\ z'(t) = z(t)\left[k_2 g_2(y(t)) - d_2\right] \end{cases}$$

as long as the solution remains positive. It then follows from Lemma 3.2 that

$$\begin{cases} x(t) \ge x(0)\,(1-\delta_1)^{\left[\frac{t+(1-l)T}{T}\right]}\, \mathrm{e}^{\int_0^t p_1(s)\mathrm{d}s}, & 0 \le t < T_0; \\ y(t) \ge y(0)\,(1-\delta_2)^{\left[\frac{t+(1-l)T}{T}\right]}\, \mathrm{e}^{\int_0^t p_2(s)\mathrm{d}s}, \\ z(t) \ge z(0)\,(1-\delta_3)^{\left[\frac{t+(1-l)T}{T}\right]}\, \mathrm{e}^{\int_0^t p_3(s)\mathrm{d}s}, \end{cases}$$

where

$$\begin{cases} p_1(t) = r - ax(t) - g_1'(0)y(t); \\ p_2(t) = k_1 g(x(t)) - g_2'(0)z(t) - d_1; \\ p_3(t) = k_2 g_2(y(t)) - d_2, \end{cases}$$

that is, x, y, z remain strictly positive on $[0, T_0)$. □

Also, using Lemma 3.2, it is possible to show that all solutions of (S) starting in $(\mathbb{R}_+^*)^3$ remain bounded and are actually defined on the whole $\mathbb{R}_+$.

Lemma 3.4. *All solutions* $(x(\cdot), y(\cdot), z(\cdot))$ *of* (S) *with initial data* $(x(\cdot), y(\cdot), z(\cdot)) \in (\mathbb{R}_+^*)^3$ *are bounded and defined on* $\mathbb{R}_+$.

Proof. Let us consider a solution $(x(\cdot), y(\cdot), z(\cdot))$ of (S) starting with strictly positive $x(0)$, $y(0)$, $z(0)$ and define $u_1 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$u_1(t) = x(t) + \frac{1}{k_1} y(t) + \frac{1}{k_1 k_2} z(t), \quad t \geq 0.$$

One then has

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = x(t)[r - ax(t)] - \frac{d_1}{k_1} y(t) - \frac{d_2}{k_1 k_2} z(t), \quad t > 0, t \neq (n + l - 1)T, t \neq nT. \tag{3.4}$$

Let us denote $D = \min(d_1, d_2)$. It follows that

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} + Du_1 \leq x(t)\left[r + D - ax(t)\right], \quad t > 0, t \neq (n + l - 1)T, t \neq nT. \tag{3.5}$$

As the right-hand side of (3.5) is bounded from above by $C = (r + D)^2/(4a)$, it follows that

$$\frac{\mathrm{d}u_1}{\mathrm{d}t}(t) + Du_1(t) \leq C, \quad t > 0, t \neq (n + l - 1)T, t \neq nT,$$

together with

$$u_1((n + l - 1)T+) \leq (1 - \delta)u_1((n + l - 1)T)$$

and

$$u_1(nT+) = u_1(nT) + \frac{\mu}{k_1 k_2},$$

where $\delta = \min(\delta_1, \delta_2, \delta_3)$. By Lemma 3.2, it follows that

$$\begin{aligned} u_1(t) \leq\ & u_1(0+) \left[\prod_{0 < (n+l-1)T < t} (1 - \delta) \right] \mathrm{e}^{-Dt} \\ & + C \int_0^t \left[\prod_{s \leq (n+l-1)T < t} (1 - \delta) \right] \mathrm{e}^{-D(t-s)} \mathrm{d}s + \sum_{0 < nT < t} \frac{\mu}{k_1 k_2} \mathrm{e}^{-D(t-nT)}, \quad t > 0, \end{aligned} \tag{3.6}$$

which yields

$$u_1(t) \leq u_1(0+)\mathrm{e}^{-Dt} + \frac{C(1 - \mathrm{e}^{-Dt})}{D} + \frac{\mu}{k_1 k_2} \frac{\mathrm{e}^{DT}}{\mathrm{e}^{DT} - 1}, \quad t > 0, \tag{3.7}$$

and since the limit of the right-hand side of (3.7) for $t \to \infty$ is

$$L = \frac{C}{D} + \frac{\mu}{k_1 k_2} \frac{\mathrm{e}^{DT}}{\mathrm{e}^{DT} - 1} < \infty,$$

it easily follows that u_1 is bounded on its domain. Consequently, x, y, z are bounded and it follows by an easy continuability argument that they are defined on the whole $\mathbb{R}_+$. □

It is also important to note that by a very similar procedure one may obtain that

$$u_2(t) \le u_2(0+)\mathrm{e}^{-d_1 t} + \frac{C_1}{d_1}\left(1 - \mathrm{e}^{-d_1 t}\right),$$

where $u_2(t) = x(t) + (1/k_1)y(t)$ and $C_1 = (r + d_1)^2/(4a)$. Consequently,

$$x(t) + \frac{1}{k_1}y(t) \le \left[x(0+) + \frac{1}{k_1}y(0+)\right] + \frac{(r+d_1)^2}{4ad_1} \quad \text{for } t > 0.$$

At this point, it is useful to note that from the above it may be seen that y is bounded, with boundedness constant

$$B = k_1\left[x(0+) + \frac{1}{k_1}y(0+) + \frac{(r+d_1)^2}{4ad_1}\right]. \tag{3.8}$$

Of course, this boundedness constant is not necessarily optimal, but it is important to note that it is T- and μ-independent. In the following, we shall also be interested in finding an ultimate boundedness constant for y, that is, a boundedness constant for y after the transient effects of the initial data are eliminated, rather than a boundedness constant; the former can be much smaller.

We now introduce a few basic results regarding the Floquet theory of impulsive systems of ordinary differential equations. Let us consider the system

$$\begin{cases} X'(t) = A(t)X(t), & t \ne \tau_k, t \in \mathbb{R}; \\ \Delta X = B_k X, & t = \tau_k, \tau_k < \tau_{k+1}, k \in \mathbb{Z}, \end{cases} \tag{3.9}$$

under the following hypotheses.

(H1) $A(\cdot) \in PC(\mathbb{R}, M_n(\mathbb{R}))$ and there is $T > 0$ such that $A(t + T) = A(t)$ for all $t \ge 0$.
(H2) $B_k \in M_n(\mathbb{R})$, $\det(I_n + B_k) \ne 0$ for $k \in \mathbb{Z}$.
(H3) There is $q \in \mathbb{N}^*$ such that $B_{k+q} = B_k$, $\tau_{k+q} = \tau_k + T$ for $k \in \mathbb{Z}$.

Let $\Phi(t)$ be a fundamental matrix of (3.9). Then there is a unique nonsingular matrix $M \in M_n(\mathbb{R})$ such that $\Phi(t + T) = \Phi(t)M$ for all $t \in \mathbb{R}$, which is called the monodromy matrix of (3.9) corresponding to Φ. Actually, all monodromy matrices of (3.9) are similar and consequently they have the same eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are called the Floquet multipliers of (3.9). Under these hypotheses, the following result holds.

Lemma 3.5 (*[34]*). *Suppose that conditions* (H1)–(H3) *hold. Then*

1. *The system (3.9) is stable if and only if all Floquet multipliers λ_k, $1 \le k \le n$ satisfy $|\lambda_k| \le 1$ and if $|\lambda_k| = 1$, then to λ_k there corresponds a simple elementary divisor.*
2. *The system (3.9) is asymptotically stable if and only if all Floquet multipliers λ_k, $1 \le k \le n$ satisfy $|\lambda_k| < 1$.*
3. *The system (3.9) is unstable if there is a Floquet multiplier λ_k such that $|\lambda_k| > 1$.*

Note that we shall be able to use Lemma 3.5 in our settings even though we also employ the constant impulse $\Delta z = \mu$ apparently not covered by our Lemma 3.5. This happens since we shall actually study the stability of certain periodic discontinuous solutions by means of the small amplitude perturbations method and the above-mentioned jump condition disappears after we reduce our problem to the stability of the null solution for certain systems, using the proper change of variables which involves discontinuous functions.

4. Periodically forced subsystems

When the intermediate consumer y is eradicated, it is easy to see that the equations in (S) decouple and we are led to consider the properties of the subsystems

$$(\mathrm{RS};\, x) \quad \begin{cases} x'(t) = x(t)[r - ax(t)], & t \ne nT, (n + l - 1)T; \\ \Delta x(t) = -\delta_1 x(t), & t = (n + l - 1)T; \\ \Delta x(t) = 0, & t = nT; \\ x(0+) = x_0, \end{cases}$$

and

$$(\mathrm{RS};z)\quad \begin{cases} z'(t) = -d_2 z(t), & t \neq nT, (n+l-1)T; \\ \Delta z(t) = -\delta_3 z(t), & t = (n+l-1)T; \\ \Delta z(t) = \mu, & t = nT; \\ z(0+) = z_0, \end{cases}$$

which describe the dynamics of the resource and of the top predator, respectively, in the absence of the intermediate consumer. First, it is seen that if a certain inequality which characterizes resource productivity is satisfied, then the system formed with the first three equations in (RS; x) has a periodic solution, to which all solutions of (RS; x) starting with strictly positive x_0 tend as $t \to \infty$, while if the opposite inequality is satisfied, then all solutions of (RS; x) tend to 0 as $t \to \infty$. The above-mentioned periodic solution will be labeled as x_r^* rather than, for instance, x_{r,a,δ_1}^*, as systems of type (RS; x) will always occur with the same a and δ_1, but sometimes with different r's, so there is no danger of confusion.

Lemma 4.1. *The following statements hold.*

1. *Suppose that* $\ln(1-\delta_1) + rT > 0$. *Then the system formed with the first three equations in* (RS; x) *has a periodic solution* x_r^*. *With this notation, the following properties are satisfied.*
   (a) $\int_0^T x_r^*(t)\mathrm{d}t = (1/a)\,[\ln(1-\delta_1) + rT]$.
   (b) $\lim_{t\to\infty} \left|x(t) - x_r^*(t)\right| = 0$ *for all solutions* $x(t)$ *of* (RS; x) *starting with strictly positive* x_0.
   (c) $\sup_{t\geq 0} \left|x_{r_1}^*(t) - x_{r_2}^*(t)\right| \leq f_1(r_1, r_2; T, a, \delta_1)$, *with* $\lim_{r_1\to r_2} f_1(r_1, r_2; T, a, \delta_1) = 0$.
2. *Suppose that* $\ln(1-\delta_1) + rT \leq 0$. *Then* $\lim_{t\to\infty} x(t) = 0$ *for all solutions of* (RS; x).

Proof. First, it is easy to see that

$$u(t) = \frac{(r/a)u(t_0)\mathrm{e}^{r(t-t_0)}}{(r/a) + u(t_0)\left(\mathrm{e}^{r(t-t_0)} - 1\right)}, \quad (n+l-1)T \leq t_0 < t \leq (n+l)T \tag{4.1}$$

for any solution u of the first equation in (RS; x), and so

$$u((n+l)T) = \frac{(r/a)u((n+l-1)T+)\mathrm{e}^{rT}}{r/a + u((n+l-1)T+)\left(\mathrm{e}^{rT} - 1\right)}.$$

Then

$$u((n+l)T+) = (1-\delta_1)\frac{(r/a)u((n+l-1)T+)\mathrm{e}^{rT}}{(r/a) + u((n+l-1)T+)\left(\mathrm{e}^{rT} - 1\right)} \tag{4.2}$$

for any solution u of the first equation in (RS; x). Now suppose that $\ln(1-\delta_1) + rT > 0$. By the periodicity requirement, it follows that

$$x_r^*((n+l-1)T+) = \frac{(r/a)\left[(1-\delta_1)\mathrm{e}^{rT} - 1\right]}{\mathrm{e}^{rT} - 1}$$

and so

$$x_r^*(lT+) = \frac{(r/a)\left[(1-\delta_1)\mathrm{e}^{rT} - 1\right]}{\mathrm{e}^{rT} - 1}.$$

Obviously, as $\ln(1-\delta_1) + rT > 0$, the periodic solution x_r^* does indeed exist, is unique and strictly positive. Actually, it may be seen that

$$x_r^*(t) = \frac{(r/a)A\mathrm{e}^{r(t-(n+l-1)T)}}{1 + A\left(\mathrm{e}^{r(t-(n+l-1)T)} - 1\right)}, \quad (n+l-1)T < t \leq (n+l)T, \tag{4.3}$$

where

$$A = \frac{(1-\delta_1)\mathrm{e}^{rT} - 1}{\mathrm{e}^{rT} - 1}. \tag{4.4}$$

Also, since

$$\frac{x_r^{*\prime}(t)}{x_r^*(t)} = r - ax_r^*(t), \quad t \in ((n+l-1)T, (n+l)T],$$

it follows that

$$\ln\left(x_r^*(t_2)\right) - \ln\left(x_r^*(t_1)\right) = \int_{t_1}^{t_2} \left[r - ax_r^*(s)\right] \mathrm{d}s; \quad (n+l-1)T < t_1 \le t_2 \le (n+l)T$$

and so

$$\ln\left(x_r^*((n+l)T)\right) - \ln\left(x_r^*((n+l-1)T+)\right) = \int_{(n+l-1)T}^{(n+l)T} \left[r - ax_r^*(s)\right] \mathrm{d}s.$$

By the periodicity of x_r^*, it follows that

$$-\ln(1-\delta_1) = \int_0^T \left[r - ax_r^*(s)\right] \mathrm{d}s,$$

from which the first assertion follows. Now let $x(t)$ be a solution of (RS; x) starting with strictly positive initial data. We shall prove that $\lim_{t\to\infty} \left|x(t) - x_r^*(t)\right| = 0$.

If $x(lT+) = x_r^*(lT+)$, then obviously $x \equiv x_r^*$. Now suppose that $x(lT+) > x_r^*(lT+)$; if the reverse inequality is satisfied one can devise a similar argument to obtain the conclusion mentioned above.

Let us denote $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f(x) = (1-\delta_1)\frac{(r/a)x\mathrm{e}^{rT}}{(r/a) + x(\mathrm{e}^{rT}-1)}.$$

It is then seen that $x \mapsto f(x)$ is strictly increasing on $\mathbb{R}_+$, while $x \mapsto f(x)/x$ is strictly decreasing on $\mathbb{R}_+$. By (4.1), it is also seen that

$$x_r^*((l+1)T+) = f(x_r^*(lT+)), \qquad x((l+1)T+) = f(x(lT+))$$

and by the periodicity of x_r^* it is seen that $x_r^*((l+1)T+) = x_r^*(lT+)$. It follows that

$$x((l+1)T+) = f(x(lT+)) > f(x_r^*(lT+)) = x_r^*(lT+),$$

since f is strictly increasing on $\mathbb{R}_+$. Also,

$$x((l+1)T+) = f(x(lT+)) = \frac{f(x(lT+))}{x(lT+)}x(lT+) < x(lT+),$$

since $x(lT+) > x_r^*(lT+)$, $f(x_r^*(lT+)) = x_r^*(lT+)$ and $x \mapsto f(x)/x$ is strictly decreasing on $\mathbb{R}_+$.

Similarly, by an induction argument,

$$\begin{aligned} x((n+l+1)T+) &= f(x((n+l)T+)) > f(x_r^*((n+l)T+)) = f(x_r^*(lT+)) \\ &= x_r^*(lT+) \end{aligned}$$

and

$$\begin{aligned} x((n+l+1)T+) &= f(x((n+l)T+)) = \frac{f(x((n+l)T+))}{x((n+l)T+)}x((n+l)T+) \\ &< x((n+l)T+). \end{aligned}$$

One then obtains that $(x((n+l)T+))_{n\ge 0}$ is monotonically decreasing and bounded from below by $x_r^*(lT+)$, so it is convergent to some $w_1 > 0$. Also,

$$x((n+l+1)T+) - x((n+l)T+) = f(x((n+l)T+)) - x((n+l)T+) \to 0 \quad \text{as } n \to \infty.$$

From the above, it follows that $f(w_1) = w_1$, and so $w_1 = x_r^*(lT+)$, since the equation $f(t) = t$ has a single strictly positive solution. It then follows that $(x((n+l)T+))_{n\ge 0} \to x_r^*(lT+)$ for $n \to \infty$. Also, by (4.1), one may prove that

$$\left|x(t) - x_r^*(t)\right| \le \mathrm{e}^{rT}\left|x((n+l)T+) - x_r^*((n+l)T+)\right|, \quad \text{for } t \in ((n+l)T, (n+l+1)T],$$

from which the second assertion follows. The remaining assertion can be proved by direct computation, making use of the explicit representation formula (4.3) and of the T-periodicity of $x^*_{r_1}$ and $x^*_{r_2}$. In fact, one may obtain that

$$\left|x^*_{r_1}(t) - x^*_{r_2}(t)\right| \le |r_1 - r_2|\left[\frac{1}{a} + \frac{(1-A)^2}{a}\right] + \frac{A(1-A)}{a}\left(r_1 \mathrm{e}^{r_2 T} - r_2 \mathrm{e}^{r_1 T}\right) \quad \text{for } t \ge 0.$$

Now suppose that $\ln(1-\delta_1) + rT \le 0$. Again, by (4.2), it is seen that

$$\begin{aligned} x((n+l)T+) &= (1-\delta_1)\frac{(r/a)x((n+l-1)T+)\mathrm{e}^{rT}}{(r/a) + x((n+l-1)T+)\left(\mathrm{e}^{rT}-1\right)} \\ &= x((n+l-1)T+)\frac{(r/a)(1-\delta_1)\mathrm{e}^{rT}}{(r/a) + x((n+l-1)T+)\left(\mathrm{e}^{rT}-1\right)} \\ &\le x((n+l-1)T+), \end{aligned}$$

as $(1-\delta_1)\mathrm{e}^{rT} \le 1$. It then follows that $(x((n+l-1)T+))_{n\ge 0}$ is monotonically decreasing and bounded from below by 0, so it converges to some $w_2 \ge 0$. Since $x((n+l)T+) = f(x((n+l-1)T+))$, it follows that $f(w_2) = w_2$. Then

$$w_2 = w_2\frac{(r/a)(1-\delta_1)\mathrm{e}^{rT}}{(r/a) + w_2\left(\mathrm{e}^{rT}-1\right)}$$

and, since $\ln(1-\delta_1) + rT \le 0$, it easily follows that $w_2 = 0$. By (4.1), it also follows that

$$x(t) \le x((n+l-1)T+)\mathrm{e}^{rT} \quad \text{for } t \in ((n+l-1)T, (n+l)T]$$

and so $\lim_{t\to\infty} x(t) = 0$. □

We now suggest an approximate interpretation of the hypotheses in Lemma 4.1. Let us suppose that x approaches 0 in (RS; x). Then rT approximates the total growth (per unit biomass) of the resource biomass in a period, while $\ln(1-\delta_1)$ is a correction term which accounts for the loss of resource biomass (per unit biomass) due to pesticide spraying. If the total growth rT does not exceed the loss $\ln(1-\delta_1)$, there is a net loss of resource biomass when x approaches 0 and so the resource biomass $x(t)$ tends to 0 as $t \to \infty$, while if $\ln(1-\delta_1) + rT > 0$, there is a net gain of resource biomass when x approaches 0 which prevents the extinction of the resource x.

Secondly, it is seen that the system formed with the first three equations in (RS; z) has a periodic solution to which all solutions of (RS; z) starting with strictly positive z_0 tend as $t \to \infty$, irrespective of the sign of $\ln(1-\delta_1) + rT$. This happens since the survival of the top predator is assured by the periodic impulse μ and does not depend upon the survival or extinction of the resource, although the persistence level is, of course, indirectly affected. Again, this solution will be labeled as $z^*_{d_2}$, for reasons similar to those outlined above.

Lemma 4.2. *The system formed with the first three equations in* (RS; z) *has a periodic solution* $z^*_{d_2}$. *With this notation, the following properties are satisfied.*

1. $\int_0^T z^*_{d_2}(t)\mathrm{d}t = \frac{\mu}{1-\mathrm{e}^{-d_2 T}(1-\delta_3)}\left[(1-\mathrm{e}^{-d_2 lT}) + (1-\delta_3)(\mathrm{e}^{-d_2 lT} - \mathrm{e}^{-d_2 T})\right]$.
2. $\lim_{t\to\infty}\left|z(t) - z^*_{d_2}(t)\right| = 0$ *for all solutions* $z(t)$ *of* (RS; z) *starting with strictly positive* z_0.
3. $\sup_{t\ge 0}\left|z^*_{d_2}(t) - z^*_{\tilde{d}_2}(t)\right| \le f_2(d_2, \tilde{d}_2; T, a, \delta_3)$, *with* $\lim_{\tilde{d}_2\to d_2} f_2(\tilde{d}_2, d_2; T, a, \delta_3) = 0$.

Proof. First, it is easy to see that

$$u(t) = \mathrm{e}^{-d_2(t-t_0)}u(t_0) \quad t, t_0 \in ((n+l-1)T, nT) \text{ or } (nT, (n+l+1)T) \tag{4.5}$$

for any solution u of the first equation in (RS; z) and so

$$\begin{aligned} z^*_{d_2}((n+1)T+) &= z^*_{d_2}((n+1)T) + \mu \\ &= \mathrm{e}^{-d_2(1-l)T} z^*_{d_2}((n+l)T+) + \mu \\ &= \mathrm{e}^{-d_2(1-l)T}(1-\delta_3) z^*_{d_2}((n+l)T) + \mu \\ &= \mathrm{e}^{-d_2(1-l)T}(1-\delta_3) e^{-d_2 lT} z^*_{d_2}(nT+) + \mu \\ &= \mathrm{e}^{-d_2 T}(1-\delta_3) z^*_{d_2}(nT+) + \mu. \end{aligned}$$

By the periodicity requirement, it follows that

$$z^*_{d_2}(nT+) = \mathrm{e}^{-d_2T}(1-\delta_3)z^*_{d_2}(nT+) + \mu$$

and so

$$z^*_{d_2}(0+) = \frac{\mu}{1-\mathrm{e}^{-d_2T}(1-\delta_3)}. \tag{4.6}$$

Obviously, by (4.6), the periodic solution searched for does indeed exist, is unique and strictly positive. Actually, it may be seen that

$$z^*_{d_2} = \begin{cases} \dfrac{\mu}{1-\mathrm{e}^{-d_2T}(1-\delta_3)}\mathrm{e}^{-d_2(t-nT)}, & t \in (nT, (n+l)T] \\ \dfrac{\mu}{1-\mathrm{e}^{-d_2T}(1-\delta_3)}\mathrm{e}^{-d_2(t-nT)}(1-\delta_3), & t \in ((n+l)T, (n+1)T]. \end{cases} \tag{4.7}$$

The first assertion follows then by direct computation.

Now let z be a solution of (RS; z) with strictly positive initial data. We shall prove that $\lim_{t\to\infty}\left|z(t) - z^*_{d_2}(t)\right| = 0$. It is seen that $z - z^*_{d_2}$ verifies the system

$$\begin{cases} (z-z^*_{d_2})'(t) = -d_2(z-z^*_{d_2})(t), & t \neq (n+l-1)T, t \neq nT; \\ \Delta(z-z^*_{d_2})(t) = -\delta_3(z-z^*_{d_2})(t), & t = (n+l-1)T; \\ \Delta(z-z^*_{d_2})(t) = 0, & t = nT. \end{cases}$$

Consequently,

$$z(t) - z^*_{d_2}(t) = \begin{cases} \mathrm{e}^{-d_2(t-(n-1)T)} & \left(z(0+) - \dfrac{\mu}{1-\mathrm{e}^{-d_2T}(1-\delta_3)}\right)(1-\delta_3)^{n-1}, \\ \quad t \in ((n-1)T, (n+l-1)T]; & \\ \mathrm{e}^{-d_2(t-(n-1)T)} & \left(z(0+) - \dfrac{\mu}{1-\mathrm{e}^{-d_2T}(1-\delta_3)}\right)(1-\delta_3)^{n}, \\ \quad t \in ((n+l-1)T, nT]; & \end{cases} \tag{4.8}$$

from which the second assertion follows. The third assertion can be proved by direct computation, as done for Lemma 4.1. □

5. Local stability results: A Floquet analysis

In this section we study the local stability of the resource and intermediate consumer-free periodic solution $(0, 0, z^*_{d_2}(t))$ and of the intermediate consumer-free periodic solution $(x^*_r(t), 0, z^*_{d_2}(t))$ by means of the Floquet theory, supposing that the productivity condition for the resource $\ln(1-\delta_1) + rT > 0$ is satisfied. In this sense, it will be seen that the local stability of the intermediate consumer-free periodic solution is governed by a threshold-like condition expressed in terms of an integral involving the periodic solutions x^*_r and $z^*_{d_2}$ introduced in the previous section, while the resource and intermediate consumer-free periodic solution is always unstable.

Theorem 5.1. *Suppose that* $\ln(1-\delta_1) + rT > 0$. *The following properties hold.*

1. *The resource and intermediate consumer-free periodic solution* $(0, 0, z^*_{d_2}(t))$ *is unstable.*
2. *The intermediate consumer-free periodic solution* $(x^*_r(t), 0, z^*_{d_2}(t))$ *is locally asymptotically stable provided that*

$$\ln(1-\delta_2) + \int_0^T \left[k_1 g_1(x^*_r(s)) - g_2'(0)z^*_{d_2}(s) - d_1\right] \mathrm{d}s < 0 \tag{5.1}$$

and unstable provided that the reverse inequality holds.

Proof. To study the stability of the resource and intermediate consumer-free periodic solution $(0, 0, z^*_{d_2}(t))$, let us denote

$$x(t) = u(t), \qquad y(t) = v(t), \qquad z(t) = w(t) + z^*_{d_2}(t), \tag{5.2}$$

u, v and w being understood as small amplitude perturbations. Substituting (5.2) into the first three equations of (S), one obtains

$$\begin{cases} u'(t) = u(t)\left[r - au(t)\right] - g_1(u(t))v(t) \\ v'(t) = k_1 g_1(u(t))v(t) - g_2(v(t))\left[w(t) + z^*_{d_2}(t)\right] - d_1 v(t) \\ w'(t) = k_2 g_2(v(t))\left[w(t) + z^*_{d_2}(t)\right] - d_2 w(t). \end{cases} \tag{5.3}$$

The corresponding linearization of (5.3) at $(0, 0, 0)$ is

$$\begin{cases} u'(t) = ru(t) \\ v'(t) = -\left[g_2'(0)z^*_{d_2}(t) + d_1\right]v(t) \\ w'(t) = k_2 g_2'(0)z^*_{d_2}(t)v(t) - d_2 w(t) \end{cases} \tag{5.4}$$

and so a fundamental matrix of (5.4) is

$$\Phi^1_L(t) = \begin{pmatrix} \mathrm{e}^{rt} & 0 & 0 \\ 0 & \mathrm{e}^{-\int_0^t \left[g_2'(0)z^*_{d_2}(s) + d_1\right]\mathrm{d}s} & 0 \\ 0 & \left(\int_0^t k_2 g_2'(0)z^*_{d_2}(s)\mathrm{e}^{-\int_0^s \left[g_2'(0)z^*_{d_2}(\tau) + d_1\right]\mathrm{d}\tau}\mathrm{d}s\right)\mathrm{e}^{-d_2 t} & \mathrm{e}^{-d_2 t} \end{pmatrix}. \tag{5.5}$$

The linearization of the jump conditions at $(n + l - 1)T$ reads as

$$\begin{cases} \Delta u(t) = -\delta_1 u(t), \quad t = (n + l - 1)T; \\ \Delta v(t) = -\delta_2 v(t), \\ \Delta w(t) = -\delta_3 w(t), \end{cases} \tag{5.6}$$

while the linearization of the jump conditions at nT reads as

$$\begin{cases} \Delta u(t) = 0, \quad t = nT; \\ \Delta v(t) = 0, \\ \Delta w(t) = 0. \end{cases} \tag{5.7}$$

Consequently, the local stability of the resource and intermediate consumer-free periodic solution $(0, 0, z^*_{d_2}(t))$ can be analyzed by studying the eigenvalues of the monodromy matrix

$$M_1 = \begin{pmatrix} 1 - \delta_1 & 0 & 0 \\ 0 & 1 - \delta_2 & 0 \\ 0 & 0 & 1 - \delta_3 \end{pmatrix} \Phi^1_L(T).$$

Since the eigenvalues of M_1 are

$$\lambda_1 = (1 - \delta_1)\mathrm{e}^{rT}, \qquad \lambda_2 = (1 - \delta_2)\mathrm{e}^{-\int_0^T \left[g_2'(0)z^*_{d_2}(s) + d_1\right]\mathrm{d}s}, \qquad \lambda_3 = (1 - \delta_3)\mathrm{e}^{-d_2 T}$$

and $\lambda_1 > 1$, it follows that the resource and intermediate consumer-free periodic solution $(0, 0, z^*_{d_2}(t))$ is unstable, with a one-dimensional unstable manifold.

We now study the stability of the intermediate consumer-free periodic solution $(x^*_r(t), 0, z^*_{d_2}(t))$. Let us denote

$$x(t) = u(t) + x^*_r(t), \qquad y(t) = v(t), \qquad z(t) = w(t) + z^*_{d_2}(t), \tag{5.8}$$

u, v, w being understood again as small amplitude perturbations. Substituting (5.8) into the first three equations of (S), one obtains

$$\begin{cases} u'(t) = u(t)\left[r - a(u(t) + x^*_r(t))\right] - g_1(u(t) + x^*_r(t))v(t) \\ v'(t) = k_1 g_1(u(t) + x^*_r(t))v(t) - g_2(v(t))(w(t) + z^*_{d_2}(t)) - d_1 v(t) \\ w'(t) = k_2 g_2(v(t))(w(t) + z^*_{d_2}(t)) - d_2 w(t). \end{cases} \tag{5.9}$$

The corresponding linearization of (5.9) at $(0, 0, 0)$ is

$$\begin{cases} u'(t) = u(t)\left[r - ax^*_r(t)\right] - g_1(x^*_r(t))v(t) \\ v'(t) = \left[k_1 g_1(x^*_r(t)) - g_2'(0)z^*_{d_2}(t) - d_1\right]v(t) \\ w'(t) = k_2 g_2'(0)z^*_{d_2}(t)v(t) - d_2 w(t). \end{cases} \tag{5.10}$$

Let us define

$$\varphi : \mathbb{R}_+ \to \mathbb{R}, \qquad \varphi(t) = \int_0^t \left[r - a x_r^*(s)\right] \mathrm{d}s,$$

$$\psi : \mathbb{R}_+ \to \mathbb{R}, \qquad \psi(t) = \int_0^t \left[k_1 g_1(x_r^*(s)) - g_2'(0) z_{d_2}^*(s) - d_1\right] \mathrm{d}s.$$

Then a fundamental matrix of (5.10) is

$$\Phi_L^2(t) = \begin{pmatrix} \mathrm{e}^{\varphi(t)} & -\mathrm{e}^{\varphi(t)} \int_0^t g_1(x_r^*(s)) \mathrm{e}^{\psi(s)-\varphi(s)} \mathrm{d}s & 0 \\ 0 & \mathrm{e}^{\psi(t)} & 0 \\ 0 & \mathrm{e}^{-d_2 t} \int_0^t k_2 g_2'(0) z_{d_2}^*(s) \mathrm{e}^{d_2 s + \psi(s)} \mathrm{d}s & \mathrm{e}^{-d_2 t} \end{pmatrix}.$$

The linearization of the jump conditions at $(n + l - 1)T$ and nT gives again (5.6) and (5.7). Consequently, the local stability of the intermediate consumer-free periodic solution $(x_r^*(t), 0, z_{d_2}^*(t))$ can be analyzed by studying the eigenvalues of the monodromy matrix

$$M_2 = \begin{pmatrix} 1-\delta_1 & 0 & 0 \\ 0 & 1-\delta_2 & 0 \\ 0 & 0 & 1-\delta_3 \end{pmatrix} \Phi_L^2(T).$$

It is seen that the eigenvalues of M_2 are

$$\lambda_1 = (1-\delta_1)\mathrm{e}^{\varphi(T)}, \qquad \lambda_2 = (1-\delta_2)\mathrm{e}^{\psi(T)}, \qquad \lambda_3 = (1-\delta_3)\mathrm{e}^{-d_2 T}.$$

It is obvious that $0 < \lambda_3 < 1$. Also, $\lambda_1 = 1$, from Lemma 4.1. If (5.1) is satisfied, then $0 < \lambda_2 < 1$ and $\lambda_1 = 1$ is a simple eigenvalue, which implies that $(x_r^*(t), 0, z_{d_2}^*(t))$ is stable. If the reverse of (5.1) is satisfied, then $\lambda_2 > 1$ and $(x_r^*(t), 0, z_{d_2}^*(t))$ is unstable. Finally, noting that $\int_0^T g_1(x_r^*(s))\mathrm{e}^{\psi(s)-\varphi(s)}\mathrm{d}s > 0$, since the integrand is strictly positive, one sees that if

$$\ln(1-\delta_2) + \int_0^T \left[k_1 g_1(x_r^*(s)) - g_2'(0) z_{d_2}^*(s) - d_1\right] \mathrm{d}s = 0, \tag{5.11}$$

then $(x_r^*(t), 0, z_{d_2}^*(t))$ is again unstable, since $\lambda = 1$ is an eigenvalue of multiplicity 2 and its elementary divisor is not simple. □

Note that the meaning of condition (5.1) is completely similar to that of condition $\ln(1-\delta_1) + rT < 0$, but applied to the dynamics of y this time. Namely, suppose that y approaches 0. Then $\int_0^T \left[k_1 g_1(x_r^*(s)) - g_2'(0) z_{d_2}^*(s) - d_1\right] \mathrm{d}s$ approximates the total growth (per unit biomass) of the intermediate consumer biomass in a period (note that $\lim_{t\to 0}(g(t)/t) = g_2'(0)$), while $\ln(1-\delta_2)$ is a correction term which accounts for the loss of intermediate consumer biomass (per unit biomass) due to pesticide spraying. If the total growth exceeds the loss $\ln(1-\delta_2)$, then there is a net gain of consumer biomass when y approaches 0 which prevents the extinction of the intermediate consumer, while if the loss $\ln(1-\delta_2)$ exceeds the total growth, there is a net loss of consumer biomass when y approaches 0 and so $y(t)$ tends to 0 as $t \to \infty$. Also, condition $\ln(1-\delta_1) + rT > 0$ ensures the instability of the resource and intermediate consumer-free periodic solution, since it prevents the extinction of the resource.

Since g_1 and g_2 are general functional responses, we have to state our stability condition (5.1) in terms of the periodic solutions x_r^* and $z_{d_2}^*$ rather than in a more explicit form. Actually, this form may make more sense even when the particular forms of g_1 and g_2 are known (for instance, when g_1, g_2 are Holling type II functional responses), as the resulting explicit inequalities are rather cumbersome and their interpretations are not transparent.

6. Global stability results

In this section, we perform a global stability analysis of the resource and intermediate consumer-free periodic solution $(0, 0, z_{d_2}^*(t))$ and of the intermediate consumer-free periodic solution $(x_r^*(t), 0, z_{d_2}^*(t))$, respectively.

Theorem 6.1. *The following statements hold.*

1. *Suppose that* $\ln(1-\delta_1)+rT \le 0$. *Then the resource and intermediate consumer-free periodic solution* $(0, 0, z^*_{d_2}(t))$ *is globally asymptotically stable.*
2. *Suppose that* $\ln(1-\delta_1)+rT > 0$. *Then the intermediate consumer-free periodic solution* $(x^*_r(t), 0, z^*_{d_2}(t))$ *is globally asymptotically stable provided that*

$$\ln(1-\delta_2)+\int_0^T \left[k_1 g_1(x^*_r(s)) - c_{g_2} z^*_{d_2}(s) - d_1\right] \mathrm{d}s < 0, \tag{6.1}$$

where

$$c_{g_2} = \inf_{0\le u\le M_y} g_2'(u),$$

M_y *being an ultimate boundedness constant for* y.

Proof. Suppose first that $\ln(1-\delta_1)+rT \le 0$. Let $\varepsilon_1 > 0$ such that $k_1 g_1(\varepsilon_1) < d_1$ (this is always possible since $\lim_{\varepsilon\to 0} g_1(\varepsilon) = 0$) and let also $\eta = (1-\delta_1)\mathrm{e}^{(k_1 g_1(\varepsilon_1)-d_1)T}$. Note that $0 < \eta < 1$. It is seen that

$$x'(t) = x(t)\,[r - ax(t)] - g_1(x(t))y(t) \le x(t)\,[r - ax(t)]$$

and so, by Lemma 3.1, $x(t) \le \tilde{x}(t)$ for $t \ge 0$, where $\tilde{x}$ is the solution of (RS; x) with the same initial data at $0+$ as x. As any such solution $\tilde{x}$ tends to 0 for $t \to \infty$, by Lemma 4.1, x tends to 0 as well and there is $T_1 > 0$ such that $x(t) \le \varepsilon_1$ for $t \ge T_1$. For the sake of simplicity, we suppose that $x(t) \le \varepsilon_1$ for all $t > 0$. One then obtains that

$$\begin{aligned} y'(t) &= k_1 g_1(x(t))y(t) - g_2(y(t))z(t) - d_1 y(t) \\ &\le y(t)\,[k_1 g_1(\varepsilon_1) - d_1], \quad t \ne (n+l-1)T. \end{aligned}$$

By integrating the above inequality on $((n+l-1)T, (n+l)T]$, one obtains

$$\ln\left(y((n+l)T)\right) - \ln\left(y((n+l-1)T+)\right) \le (k_1 g_1(\varepsilon_1) - d_1)T \quad \text{for } n \ge 1$$

and so

$$\ln\left(y((n+l)T)\right) - \ln\left(y((n+l-1)T)\right) - \ln(1-\delta_1) \le (k_1 g_1(\varepsilon_1) - d_1)T \quad \text{for } n \ge 1.$$

It then follows that

$$y((n+l)T) \le y((n+l-1)T)\eta$$

and consequently

$$y((n+l)T) \le y(lT)\eta^n,$$

which implies that $y((n+l)T) \to 0$ as $n \to \infty$. Also,

$$y(t) \le y((n+l-1)T+)\mathrm{e}^{(k_1 g_1(\varepsilon_1)-d_1)(t-(n+l-1)T)}, \quad t \in ((n+l-1)T, (n+l)T]$$

which implies that

$$y(t) \le y((n+l-1)T+), \quad t \in ((n+l-1)T, (n+l)T]$$

and consequently $y(t) \to 0$ as $t \to \infty$.

We finish by proving that $z(t) - z^*_{d_2}(t) \to 0$ as $t \to \infty$. To this purpose, let $0 < \varepsilon_2 < d_2/(k_2 L_2)$. Since $y(t) \to 0$ as $t \to \infty$, there is some $T_2 > 0$ such that $y(t) \le \varepsilon_2$ for all $t \ge T_2$. For the sake of simplicity, we suppose that $y(t) \le \varepsilon_2$ for all $t > 0$.

It follows that

$$\begin{aligned} z'(t) &= k_2 g_2(y(t))z(t) - d_2 z(t) \\ &\le k_2 L_2 y(t) z(t) - d_2 z(t) \\ &\le -(d_2 - k_2 L_2 \varepsilon_2) z(t), \quad t \ne (n+l-1)T, t \ne nT. \end{aligned}$$

Consequently, one infers from Lemma 3.1 that

$$\tilde{z}_1(t) \le z(t) \le \tilde{z}_2(t),$$

where $\tilde{z}_1$ is the solution of (RS; z) with the same initial data at $0+$ as z and $\tilde{z}_2$ is the solution of (RS; z) with d_2 changed into $d_2 - k_2 L_2 \varepsilon_2$ and the same initial data at $0+$ as z.

As these solutions become close to $z^*_{d_2}(t)$, respectively to $z^*_{d_2-k_2L_2\varepsilon_2}(t)$ as $t \to \infty$, by Lemma 4.2, it follows that, for t large enough,

$$z^*_{d_2}(t) - \varepsilon_2 \le z(t) \le z^*_{d_2-k_2L_2\varepsilon_2}(t) + \varepsilon_2$$

and the conclusion follows from Lemma 4.2. The first assertion is now established.

Now suppose that $\ln(1-\delta_1) + rT > 0$. We first show that $y(t) \to 0$ as $t \to \infty$. To this purpose, choose $\varepsilon_3 > 0$ such that

$$\ln(1-\delta_2) + \int_0^T \left[k_1 g_1(x^*_r(s) + \varepsilon_3) - c_{g_2}(z^*_{d_2}(s) - \varepsilon_3) - d_1\right] \mathrm{d}s < 0.$$

This choice is obviously feasible, as $\left|g_1(x^*_r(s) + \varepsilon_3) - g_1(x^*_r(s))\right| \le L_1\varepsilon_3$ and (6.1) is satisfied. Let us also denote

$$\xi = (1-\delta_2)\mathrm{e}^{\int_0^T \left[k_1 g_1(x^*_r(s)+\varepsilon_3) - c_{g_2}(z^*_{d_2}(s)-\varepsilon_3) - d_1\right]\mathrm{d}s}$$

and observe that $0 < \xi < 1$.

It is seen that

$$x'(t) = x(t)\left[r - ax(t)\right] - g_1(x(t))y(t) \le x(t)\left[r - ax(t)\right],$$

and so, by Lemma 3.1, $x(t) \le \tilde{x}(t)$ for $t \ge 0$, where $\tilde{x}$ is the solution of (RS; x) with the same initial data at $0+$ as x. As any such solution becomes close to $x^*_r(t)$ for $t \to \infty$, by Lemma 4.1, there is some $T_3 > 0$ such that $x(t) \le x^*_r(t) + \varepsilon_3$ for $t \ge T_3$. For the sake of simplicity, we suppose that $x(t) \le x^*_r(t) + \varepsilon_3$ for all $t > 0$.

Also,

$$z'(t) = k_2 g_2(y(t))z(t) - d_2 z(t) \ge -d_2 z(t),$$

and so, by Lemma 3.1, $z(t) \ge \tilde{z}(t)$ for $t \ge 0$, where $\tilde{z}$ is the solution of (RS; z) with the same initial data at $0+$ as z. As any such solution becomes close to $z^*_{d_2}(t)$ for $t \to \infty$, by Lemma 4.2, there is some $T_4 > 0$ such that $z(t) \ge z^*_{d_2}(t) - \varepsilon_3$ for $t \ge T_4$. For the sake of simplicity, we suppose that $z(t) \ge z^*_{d_2}(t) - \varepsilon_3$ for all $t > 0$.

Since $y(t)$ is ultimately bounded, there is $T_5 > 0$ such that $y(t) \le M_y$ for all $t \ge T_5$, where M_y is an ultimate boundedness constant for y. For the sake of simplicity, we suppose that $y(t) \le M_y$ for all $t > 0$. Also, note that in this situation $g_2(y(t)) \ge c_{g_2} y(t)$ for $t \ge 0$. One then obtains that

$$\begin{aligned} y'(t) &= k_1 g_1(x(t))y(t) - g_2(y(t))z(t) - d_1 y(t) \\ &\le y(t)\left[k_1 g_1(x^*_r(t) + \varepsilon_3) - c_{g_2}(z^*_{d_2}(t) - \varepsilon_3) - d_1\right], \quad t \ne (n+l-1)T, \end{aligned}$$

and it consequently follows that

$$\frac{y'(t)}{y(t)} \le k_1 g_1(x^*_r(t) + \varepsilon_3) - c_{g_2}(z^*_{d_2}(t) - \varepsilon_3) - d_1, \quad t \ne (n+l-1)T.$$

By integrating the above inequality on $((n+l-1)T, (n+l)T]$, one obtains

$$\ln(y((n+l)T)) - \ln(y((n+l-1)T+)) \le \int_{(n+l-1)T}^{(n+l)T} \left[k_1 g_1(x^*_r(t) + \varepsilon_3) - c_{g_2}(z^*_{d_2}(t) - \varepsilon_3) - d_1\right] \mathrm{d}t$$

and so

$$\ln(y((n+l)T)) - \ln(y((n+l-1)T)) - \ln(1-\delta_2) \le \int_0^T \left[k_1 g_1(x^*_r(t) + \varepsilon_3) - c_{g_2}(z^*_{d_2}(t) - \varepsilon_3) - d_1\right] \mathrm{d}t$$

by periodicity. It then follows that

$$y((n+l)T) \le y((n+l-1)T)\xi$$

and consequently

$$y((n+l)T) \le y(lT)\xi^n,$$

which implies that $y((n+l)T) \to 0$ as $n \to \infty$. Also

$$\frac{y'(t)}{y(t)} \le k_1 g_1(x(t)) \le \tilde{k},$$

$\tilde{k}$ being a suitable boundedness constant, so

$$y(t) \le y((n+l-1)T+)\mathrm{e}^{\tilde{k}(t-(n+l-1)T)}, \quad t \in ((n+l-1)T, (n+l)T]$$

which implies that

$$y(t) \le (1-\delta_2)y((n+l-1)T)\mathrm{e}^{\tilde{k}T}, \quad t \in ((n+l-1)T, (n+l)T],$$

and consequently $y(t) \to 0$ as $t \to \infty$. We now prove that $x(t) - x_r^*(t) \to 0$ as $t \to \infty$. To this purpose, let $0 < \varepsilon_4 \le r/L_1$. Since $y(t) \to 0$ as $t \to \infty$, there is $T_6 > 0$ such that $y(t) < \varepsilon_4$ for $t \ge T_6$. For the sake of simplicity, we suppose that $y(t) < \varepsilon_4$ for all $t > 0$.

It follows that

$$\begin{aligned} x'(t) &= x(t)\,[r - ax(t)] - g_1(x(t))y(t) \\ &= x(t)\left[r - \frac{g_1(x(t))}{x(t)}y(t) - ax(t)\right] \\ &\ge x(t)\,[(r - L_1\varepsilon_4) - ax(t)] \end{aligned}$$

for $t \ne (n+l-1)T$, $t \ne nT$. Consequently, one infers from Lemma 3.1 that

$$\tilde{x}_1(t) \le x(t) \le \tilde{x}_2(t),$$

where $\tilde{x}_2$ is the solution of (RS; x) with the same initial data at $0+$ as x and $\tilde{x}_1$ is the solution of (RS; x) with r changed into $r - L_1\varepsilon_4$ and the same initial data at $0+$ as x.

As these solutions become close to $x_r^*(t)$, respectively to $x_{r-L_1\varepsilon_4}^*(t)$ as $t \to \infty$, by Lemma 4.1, it follows that, for t large enough,

$$x_{r-L_1\varepsilon_4}^*(t) - \varepsilon_4 \le x(t) \le x_r^*(t) + \varepsilon_4$$

and the conclusions now follow again from Lemma 4.1. To prove that $z(t) - z_{d_2}^*(t) \to 0$ as $t \to \infty$, we may proceed as done for the proof of the first assertion. The second assertion is now established. □

Note that condition (6.1) has a somewhat theoretical value and is only sufficient for the global asymptotic stability of the intermediate consumer-free periodic solution. One may not expect, though, an integral condition of type (6.1) to be threshold-like (to be necessary as well). This happens since (S) has to inherit, at least partially, the chaotic behavior of the unperturbed system, which is attained for a certain window in the parameter space, as noted in Klebanoff and Hastings [11]. At this point, the availability of a good estimate of the ultimate boundedness constant for y or of c_{g_2} is crucial. In this regard, if one considers the case in which g_2 is a Holling type II functional response, $g_2(y) = (a_2 y)/(1 + b_2 y)$, then $g_2'(y) = a_2/(1 + b_2 y)^2$ and then $\inf_{y\in\mathbb{R}_+} g_2'(u) = 0$. Consequently, if no good estimations for the ultimate boundedness constant are available and B is large, then the only sensible way to ensure the validity of (6.1) is to assume that

$$\ln(1-\delta_2) + \int_0^T \left[k_1 g_1(x_r^*(s)) - d_1\right] \mathrm{d}s < 0,$$

but this is a rather crude estimation, which ensures the extinction of the intermediate consumer even if no top predator is present.

Note also that, at least formally, both the local stability condition (5.1) and the global stability condition (6.1) display a significant dependence on the functional response g_2 of the top predator, with a dominance on the dependence on a_2.

7. Biological interpretations of the stability results

From Theorem 6.1, we note that if the pesticide is not selective enough, that is, if δ_1 is large enough to make $\ln(1-\delta_1)+rT$ negative, or, in other words, if the pesticide has a significant negative effect on the growth of the resource biomass, then the resource and intermediate consumer-free periodic solution is globally asymptotically stable, which means that our control strategy fails. Alternatively, this means that a nonselective pesticide should not be applied very often (T should be large) in order to avoid resource extinction. Of course, this may have a negative impact on the overall success of the integrated pest management strategy.

From Theorem 6.1, it is seen that, theoretically speaking, our control strategy can be always made to succeed globally by the use of proper pesticides, provided that δ_1 is small enough, in order to have the inequality $\ln(1-\delta_1)+rT>0$ satisfied, and δ_2 is large enough to have (6.1) satisfied, for any given top predator functional response g_2. Also, it is seen that an aggressive ($g_2'(0)$ large enough) top predator may stabilize an otherwise unstable intermediate consumer-free periodic solution, at least locally (see (5.1)). In order to stabilize the intermediate consumer-free periodic solution globally, the top predator should be aggressive enough, even at large intermediate consumer densities, when saturation effects are supposed to appear, so that

$$\inf_{0\le u\le M_y} g_2'(u) > \frac{\ln(1-\delta_2)+\int_0^T \left[k_1 g_1(x_r^*(s))-d_1\right]\mathrm{d}s}{\int_0^T z_{d_2}^*(s)\mathrm{d}s}.$$

If g_2 is a Holling type II functional response (see above) or a Ivlev functional response ($g_2(x)=k(1-\mathrm{e}^{-bx})$), which are convex regarded as functions of x, then the above reduces to

$$g_2'(M_y) > \frac{\ln(1-\delta_2)+\int_0^T \left[k_1 g_1(x_r^*(s))-d_1\right]\mathrm{d}s}{\int_0^T z_{d_2}^*(s)\mathrm{d}s}.$$

Note that $\int_0^T k_1 g_1(x_r^*(s))\mathrm{d}s$ and $\int_0^T z_{d_2}^*(s)\mathrm{d}s$ are g_2-independent.

Since $\lim_{\mu\to\infty}\int_0^T z_{d_2}^*(s)\mathrm{d}s=+\infty$, from Lemma 4.2, and x_r^* does not depend upon μ, it is seen from Theorem 6.1 that the intermediate consumer-free periodic solution can be stabilized globally by means of increasing μ alone. Note that B, the global boundedness constant for y which is indicated in (3.8) and which may also serve as an ultimate boundedness constant for y, is μ-independent.

Also,

$$\int_0^T k_1 g_1(x_r^*(s))\mathrm{d}s < k_1 L_1 \int_0^T x_r^*(s)\mathrm{d}s = k_1 L_1 (1/a)\left(\ln(1-\delta_1)+rT\right),$$

from Lemma 4.1, so

$$\limsup_{T\downarrow -(\ln(1-\delta_1))/r} \int_0^T k_1 g_1(x_r^*(s))\mathrm{d}s \le 0.$$

As

$$\liminf_{T\downarrow -(\ln(1-\delta_1))/r} c_{g_2}\int_0^T (z_{d_2}^*(s)+d_1)\mathrm{d}s > 0$$

from Lemma 4.2 and c_{g_2} is T-independent, it is seen from Theorem 6.1 that the intermediate consumer-free periodic solution can also be globally stabilized by means of decreasing T alone, in such a way that $\ln(1-\delta_1)+rT$ remains strictly positive.

However, as mentioned in the Introduction, our purpose is to drive the intermediate consumer population under the economic injury level rather than eradicate it completely, so our pest management strategy may be considered successful even in situations in which (6.1) is not satisfied, provided that the intermediate consumer population stabilizes under the economic injury level.

Accepting $(1/T)\int_0^T f(t)\mathrm{d}t$ as an averaging measure for the oscillations of a periodic and positive function f of period T (an average level of persistence, that is), it is seen from Lemma 4.2 that an increase in μ causes an increase

in the average level of $z^*_{d_2}$, while from Lemma 4.1 it is seen that an increase in μ has no effect on the average level of x^*_r.

From Lemma 4.1, it may also be observed that an increase in the carrying capacity of the environment (a decrease of a while keeping r constant, that is) causes an increase in the average level of x^*_r, while having no effect on the average level of $z^*_{d_2}$. This is certainly conceivable, since if y tends to extinction, then the resource x and the top predator z are essentially independent, as the top predator z does not feed upon the resource x. Also, as seen from (5.1) and (6.1), an increase in the carrying capacity of the environment may not necessarily destabilize the intermediate consumer-free periodic solution $(x^*_r(t), 0, z^*_{d_2}(t))$, at least when the functional response g_1 of the intermediate consumer is a Holling type II functional, since $\int_0^T g_1(x^*_r(s))\mathrm{d}s$ is bounded from above as a function of a, but it certainly reduces the chances of having a stable intermediate consumer-free periodic solution, since $\int_0^T g_1(x^*_r(s))\mathrm{d}s$ is decreasing as a function of a. It is then seen that we obtain a paradox of enrichment for our food chain model, albeit in a weaker form. Also, noting that all terms in (5.1) are negative except for $k_1\int_0^T g_1(x^*_r(s))\mathrm{d}s$, we observe that periodic solutions $(x^*_r(t), 0, z^*_{d_2}(t))$ with low x^*_r's are inherently stable rather than unstable, so the paradox of biological control is not present in our model.

To show that our pest management strategy does not over-rely on the use of pesticides, although this, in some sense, has already been observed above, we briefly study below the case in which no pesticides are sprayed (that is, $\delta_1 = \delta_2 = \delta_3 = 0$) and outline the success conditions.

It is seen that in this situation $\ln(1-\delta_1) + rT = rT > 0$ and $\int_0^T z^*_{d_2}(s)\mathrm{d}s = \mu/d_2$. Also, this time $x^*_r(t) = r/a$ for $t \geq 0$ (see (4.3) and (4.4)). We consequently obtain with the help of Theorems 5.1 and 6.1 the following result.

Theorem 7.1. *Suppose that $\delta_1 = \delta_2 = \delta_3 = 0$. Then the following statements hold.*

1. *The resource and intermediate consumer-free periodic solution $(0, 0, z^*_{d_2}(t))$ is unstable.*
2. *The intermediate consumer-free periodic solution $(r/a, 0, z^*_{d_2})$ is stable provided that*

$$(k_1 g_1(r/a) - d_1)\, T < g_2'(0)\mu/d_2,$$

respectively globally asymptotically stable provided that

$$(k_1 g_1(r/a) - d_1)\, T < c_{g_2}\mu/d_2.$$

3. *The intermediate consumer-free periodic solution $(r/a, 0, z^*_{d_2})$ is unstable provided that*

$$(k_1 g_1(r/a) - d_1)\, T \geq g_2'(0)\mu/d_2.$$

It is now easy to see that a voracious top predator can always stabilize the system, driving the intermediate consumer to extinction and the prey to the carrying capacity of the environment. Also, for μ large enough or T small enough, the global stability condition is always satisfied. Note that, for a significant part of the parameter space, the dynamical outcome does not depend upon the initial population sizes, which is perhaps not surprising, having in view that we study a model with predator-dependent functional responses, as opposed to a model with ratio-dependent functional responses.

We may further particularize $g_i(x) = (a_i x)/(1 + b_i x)$, $i \in \{1, 2\}$, and obtain that $(r/a, 0, z^*_{d_2})$ is stable provided that $T < (a_2\mu(a + b_1 r)) \,/\, (d_2(k_1 a_1 r - d_1 a - d_1 b_1 r))$ and unstable provided that the reverse inequality holds, that is, a result similar to Theorem 3.1 in Zhang and Chen [21].

In the situations in which the intermediate consumer-free equilibrium is globally asymptotically stable, or at least the intermediate consumer population stabilizes below the economic injury level, it would be interesting from a practical point of view to give a general estimate of the time required for the intermediate consumer population to drop below the economic injury level. Unfortunately, we were not able to address this issue in this work.

8. Numerical simulations

We are now concerned with the numerical investigation of some situations not covered by our Theorems 5.1 and 6.1 which may lead to a chaotic behavior of the system. Following Klebanoff and Hastings [11] and Kuznetsov and Rinaldi [16], we rescale the variables using the formulas

$$x_1 = \frac{ax}{r}, \qquad x_2 = \frac{ay}{rk_1}, \qquad x_3 = \frac{az}{rk_1k_2}, \qquad s = rt$$

and obtain the following scaled system

$$\text{(SC)}\quad \begin{cases} x_1'(s) = x_1(s)[1-x_1(s)] - \dfrac{m_1x_1(s)}{1+n_1x_1(s)}x_2(s), & s \neq (n+l-1)T_1,\ s \neq nT_1;\\ x_2'(s) = \dfrac{m_1x_1(s)}{1+n_1x_1(t)}x_2(s) - \dfrac{m_2x_2(s)}{1+n_2x_2(s)}x_3(s) - D_1x_2(s), & s \neq (n+l-1)T_1,\ s \neq nT_1;\\ x_3'(s) = \dfrac{m_2x_2(s)}{1+n_2x_2(t)}x_3(s) - D_2x_3(s), & s \neq (n+l-1)T_1,\ s \neq nT_1;\\ \Delta x_1(s) = -\delta_1x_1(s), & s = (n+l-1)T_1;\\ \Delta x_2(s) = -\delta_2x_2(s), & s = (n+l-1)T_1;\\ \Delta x_3(s) = -\delta_3x_3(s), & s = (n+l-1)T_1;\\ \Delta x_1(s) = 0, & s = nT_1;\\ \Delta x_2(s) = 0, & s = nT_1;\\ \Delta x_3(s) = \mu_1, & s = nT_1, \end{cases}$$

where

$$m_1 = \frac{a_1k_1}{a},\qquad n_1 = \frac{b_1r}{a},\qquad n_2 = \frac{rb_2k_1}{a},\qquad D_1 = \frac{d_1}{r},\qquad D_2 = \frac{d_2}{r},\qquad T_1 = rT,$$
$$\mu_1 = \frac{a\mu}{rk_1k_2}.$$

It is easy to see that the corresponding unperturbed system

$$\text{(RSC)}\quad \begin{cases} x_1'(s) = x_1(s)[1-x_1(s)] - \dfrac{m_1x_1(s)}{1+n_1x_1(s)}x_2(s), & s \neq (n+l-1)T_1,\ s \neq nT_1;\\ x_2'(s) = \dfrac{m_1x_1(s)}{1+n_1x_1(t)}x_2(s) - \dfrac{m_2x_2(s)}{1+n_2x_2(s)}x_3(s) - D_1x_2(s), & s \neq (n+l-1)T_1,\ s \neq nT_1;\\ x_3'(s) = \dfrac{m_2x_2(s)}{1+n_2x_2(t)}x_3(s) - D_2x_3(s), & s \neq (n+l-1)T_1,\ s \neq nT_1; \end{cases}$$

has at most five equilibria, namely:

1. The trivial equilibrium $O = (0, 0, 0)$.
2. The intermediate consumer and top predator-free equilibrium $R = (1, 0, 0)$.
3. The top predator-free equilibrium

$$RC = (D_1/(m_1 - n_1D_1), (m_1 - n_1D_1 - D_1)/(m_1 - n_1D_1)^2, 0).$$

4. The positive equilibria

$$P_1 = (x_1^{P1}, D_2/(m_2 - n_2D_2), x_3^{P1}),\quad P_2 = (x_1^{P2}, D_2/(m_2 - n_2D_2), x_3^{P2}),$$

where

$$x_1^{Pi} = \frac{n_1-1}{2n_1} + (-1)^i\frac{\sqrt{(n_1+1)^2 - 4\frac{m_1n_1D_2}{m_2-n_2D_2}}}{2n_1}$$
$$x_3^{Pi} = \frac{1}{m_2-n_2D_2}\left(\frac{m_1x_1^{Pi}}{1+n_1x_1^{Pi}} - D_1\right),\quad i \in \{1,2\}.$$

Note that the first two equilibria exist irrespective of the values of the parameters which characterize the system, while several conditions need to be satisfied for the existence of the last three equilibria.

The dynamics of the unperturbed system (RSC) have been studied in detail by Klebanoff and Hastings in [11] and by Kuznetsov and Rinaldi in [16]. However, the behavior of the perturbed system (SC) is severely affected by our periodic forcing and the qualitative picture bears little resemblance, at least for significant forcing, to that of the unperturbed system.

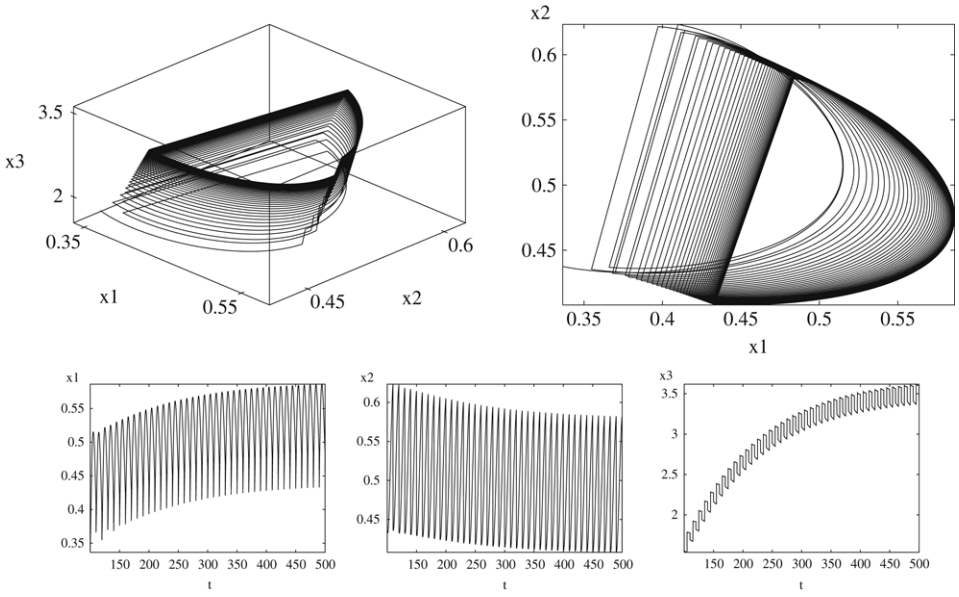

Fig. 8.1. $m_1 = 2.4$, $n_1 = 3$, $m_2 = 0.02$, $n_2 = 0.4$, $D_1 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 10$, $l = 0.5$. The trajectory approaches a periodic orbit of period T_1. The unperturbed system has a top predator-free equilibrium, but no positive equilibria.

From Theorem 5.1, it is easy to see that the intermediate consumer-free periodic solution is unstable provided that $m_2 < m_{2s}$, where

$$m_{2s} = \frac{\ln(1-\delta_2) + \int_0^{T_1} \frac{m_1(x_1)^*(s)}{1+n_1(x_1)^*(s)}\mathrm{d}s - D_1 T_1}{\int_0^{T_1} (x_3)^*_{D_2}(s)\mathrm{d}s}$$

and locally stable provided that the reverse inequality is satisfied.

For $m_1 = 2.4$, $n_1 = 3$, $m_2 = 0.02$, $n_2 = 0.4$, $D_1 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 10$, $l = 0.5$ (part of the values are close to the ones used by McCann and Yodzis in [15]) and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$, it is seen that the intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 0.098$. The unperturbed system has a top predator-free equilibrium, but no positive equilibria.

It is then seen that in this case the trajectory of the perturbed system tends to a periodic orbit of period T_1. Apart from deciding the stability or instability of the intermediate consumer-free periodic solution, the parameter m_2 does not seem to otherwise influence the qualitative properties or the shape of the limiting set. The behavior of the trajectory is depicted in Fig. 8.1.

A related behavior is captured in Fig. 8.2 for $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 11$, $l = 0.5$ and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.329$. The unperturbed system has a top predator-free equilibrium and a positive equilibrium.

In this case the trajectory of the perturbed system tends to a periodic orbit of period $3T_1$.

A typical example of chaotic behavior (strange attractor) is captured in Fig. 8.3 for $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 30$, $l = 0.5$ and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.244$. Again, the unperturbed system has a top predator-free equilibrium and a positive equilibrium. The two-dimensional plot x_2 vs. x_1 and the time series for x_1, x_2, x_3 also indicate that the trajectory has a chaotic behavior. A slight increase in m_2 ($m_2 = 0.109$) "stabilizes" the behavior of the system, and the trajectory tends again to a periodic solution of period T_1.

A somewhat similar situation is captured in Fig. 8.4 for $m_1 = 10$, $n_1 = 2$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 10$, $l = 0.5$ and $x_1(0) = 0.75$, $x_2(0) = 0.49$,

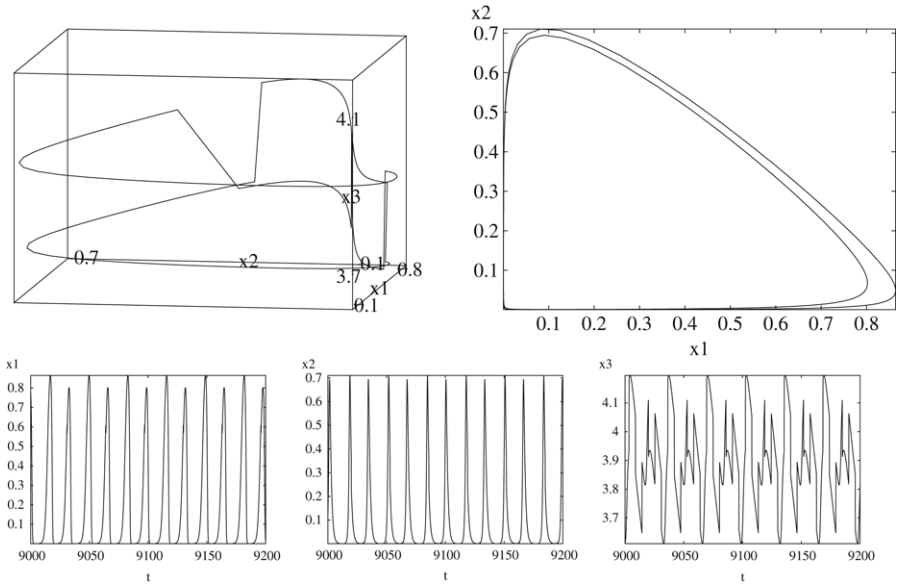

Fig. 8.2. $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 11$, $l = 0.5$. The trajectory approaches a periodic orbit of period $3T_1$. The unperturbed system admits a top predator-free equilibrium and a positive equilibrium.

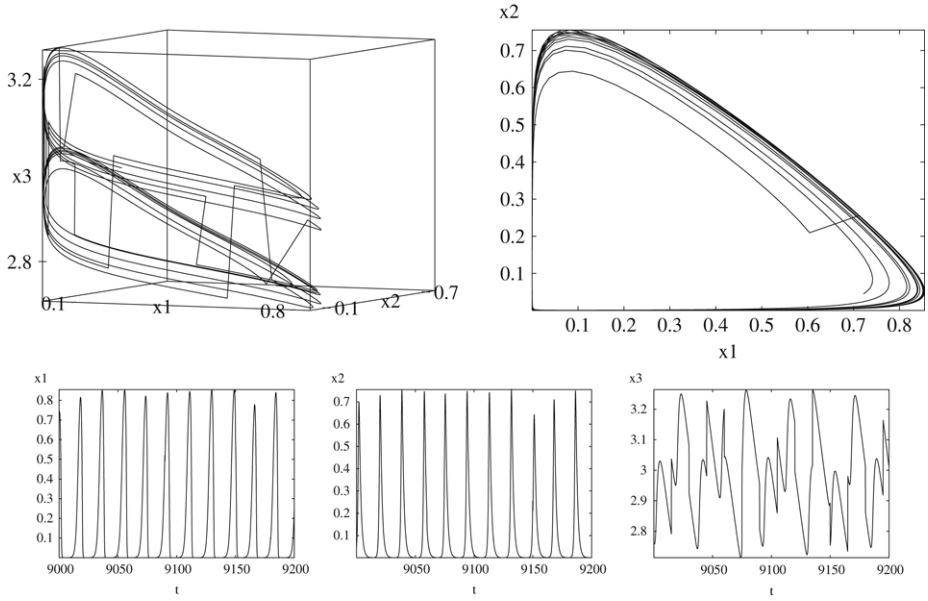

Fig. 8.3. $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 30$, $l = 0.5$. The trajectory is chaotic (bistability-like scenario). The unperturbed system admits a top predator-free equilibrium and a positive equilibrium.

$x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.745$. The unperturbed system has a top predator-free equilibrium and two positive equilibria. A slight increase in m_2 ($m_2 = 0.1119$) "stabilizes" the behavior of the system, and the trajectory tends again to a periodic solution of period T_1. That is, m_2 not only has the potential to stabilize the intermediate consumer-free periodic solution, but also the potential to mitigate the chaotic behavior of a trajectory for certain values significantly smaller than the stabilizing critical value, an increase of m_2 over these values ensuring that the trajectories of the system tend to certain periodic solutions.

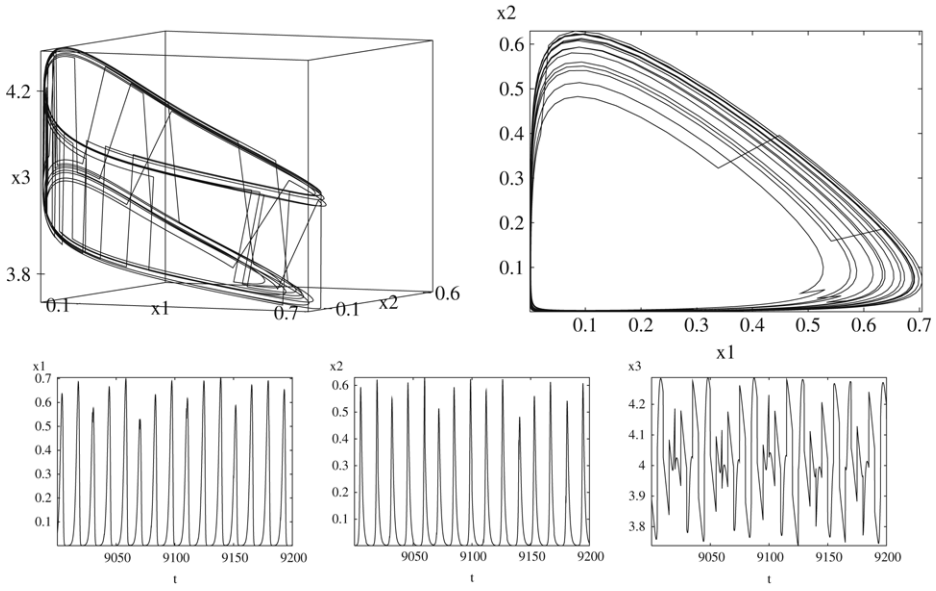

Fig. 8.4. $m_1 = 10$, $n_1 = 2$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 10$, $l = 0.5$. The trajectory is chaotic (bistability-like scenario). The unperturbed system admits a top predator-free equilibrium and two positive equilibria.

9. Concluding remarks

In this paper, an integrated pest management model described through an impulsively perturbed tri-trophic simple food chain system is proposed and investigated. To control the behavior of the system, biological controls, in the form of periodic release of top predators in a fixed number and chemical controls, in the form of periodic pesticide spraying, are employed. It is assumed that as a result of pesticide spraying fixed proportions of resource biomass, intermediate consumer biomass and top predator biomass are degraded each time. Nonlinear general smooth functions are used to model the functional response of the intermediate consumer and of the top predator and a general prey-dependent model is consequently obtained.

By means of the Floquet theory of impulsively perturbed systems of ordinary differential equations, it is seen that the local stability of the intermediate consumer-free periodic solution is governed by a threshold-like inequality, provided that a certain condition on the productivity of the resource is satisfied. If the reverse of the productivity condition is satisfied, then the resource and intermediate consumer-free periodic solution is globally asymptotically stable.

A sufficient condition for the global stability of the intermediate consumer-free periodic solution, corresponding to the ultimate success of our pest management strategy, is established, while it is observed that, biologically speaking, the integrated pest management strategy can be considered successful when the intermediate consumer population stabilizes under a certain economic injury level, not necessarily when it is completely eradicated. Formally, both the local and global stability condition display a significant dependence on the functional response of the top predator.

It is observed that, theoretically speaking, the control strategy can be always made to succeed by the use of proper pesticides, while as far as the biological control is concerned, its global effectiveness can also be reached provided that the top predator is voracious enough, or the number μ of top predators released each time is large enough or the period T is small enough. Any of these features alone can ensure the global success of our control strategy, although in concrete situations these may or may not be biologically feasible or may require a large amount of resources.

Finally, a numerical analysis of some situations leading to a chaotic behavior of the system is also provided.

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