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Finite-time control of impulsive hybrid dynamical systems in pest management

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This paper deals with the dynamics of a class of hybrid dynamical systems, which are subject to time-dependent impulsive perturbations within a finite-time interval and describe control strategies for integrated pest management. By using suitably defined Lyapunov functionals, sufficient conditions for the finite-time contractive stability of the null solution are found by means of monotonicity arguments. Finally, a numerical simulation illustrates the theoretical analysis. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

With the development of science and technology, we may now choose from many different approaches as we plan our pest control strategies. In this regard, the traditional approach to pest control relies on the repeated use (or abuse) of chemical controls, that is, on periodic pesticide spraying. Although this is usually the quickest way to contain a pest outbreak, the long-term pesticide usage has been shown to lead, in many cases, to the selection of the pest varieties, which are resistant to pesticides. It has been observed that over 500 arthropod species developed resistance to at least one pesticide [1, 2], whereas certain common pests such as the diamondback moth *Plutella xylostella* and the tobacco budworm *Heliothis virescens* have developed resistance to a variety of pesticides within 1–3 years [3–5]. This leads in turn to the occurrence of the so-called treadmill effect: pests swiftly evolve to withstand chemicals, which encourages even greater pesticide usage, with little useful effects, but causing yet more pest resistance, pest resurgence, and secondary pest outbreaks.

Because of these considerations, integrated pest management (IPM) has emerged in recent years as a cost-effective, multifaceted alternative to pesticide usage, which minimizes the side effects on non-target organisms. One of the core components of IPM strategies is the use of biocontrol agents, which may be predators, parasites, or pathogens of the given pest, that either remove the pest from the environment or interfere with its biological processes [6–11].

It is to be noted that the ultimate goal of IPM strategies is usually not to eradicate pests, as this may not be cost-effective, may affect the ecosystem, or may be plainly impossible. Instead, the focus is on stabilizing the density of the pest population under the economic injury level (EIL), defined in [12] as the amount of pest injury, which justifies the cost of using controls, or the lowest pest density, which causes economic damage. Another important threshold for the pest control strategy is the economic threshold (ET) (lower than the EIL), defined in [13] as the density at which control measures should be used to prevent an increasing pest population from reaching EIL. See also [14] for a comprehensive survey of concepts pertaining to the economics of pest management strategies.

Impulsive dynamical systems are natural choices for the modeling of pest control strategies, because the pest population experiences abrupt changes in size after pesticide usage or predator release (provided that this is the chosen biocontrol agent) and also because reproduction is not a continuous activity, being subject to certain patterns and cycles. If the impulsive perturbations occur at prescribed times, which are independent of the system state, they are termed as time-dependent perturbations, whereas if the impulsive perturbations occur when the trajectories of the system reach a certain manifold in the space of states (for instance, if the pest population reaches a certain density), then the perturbations are termed as state-dependent perturbations.

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In many papers on impulsive pest management strategies, the success of the control strategy is measured in terms of the asymptotic stability properties of the pest-eradication solution (or susceptible pest-eradication solution, if the chosen method of biocontrol is the release of infective pest individuals). Concerning this, the global stability of the aforementioned solution is associated to the global success of the control strategy, whereas its local stability is associated to the conditional success of the control strategy. However, this characterization in terms of Lyapunov stability accounts for the long term behavior of the system (apart from not accounting for either EIL or ET), whereas in concrete situations, positive control results are usually requested within a specified time frame. For instance, pest outbreaks need to be contained at the very least before the crop is harvested. This situation motivates the adaptation of Lyapunov stability concepts to deal with stabilization under maximal bounds within prescribed time intervals.

The concept of finite-time stability was first introduced in the control literature in the early 1950s [15, 16], getting further exposure in [17–19]. See also [20] for a comprehensive overview of this subject. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold within a specified time interval. It should be noted that finite-time stability and Lyapunov stability are independent concepts and, although Lyapunov stability is essentially a qualitative concept, which characterizes the behavior of a system within a infinite time interval, finite-time stability is a quantitative concept, which accounts for the behavior of the system within a finite (and possibly short) time interval. Also, unlike Lyapunov stability, finite-time stability is not an absolute concept, being essentially tied to the bounds imposed on the initial condition and of the state of the system, and a Lyapunov unstable system can be finite-time stable when considered on a suitably small time interval and with respect to appropriate bounds.

Sufficient conditions for the finite-time stability and finite-time stabilization of certain systems have recently been obtained in the control literature [21–23]. Furthermore, a number of authors have considered these concepts in the context of impulsive and hybrid systems [24–28]. However, only a few papers considered Lyapunov-based approaches for discussing the finite-time stability of impulsive systems [29, 30].

Motivated by the aforementioned content, we shall model in this paper the growth of a pest population and its associated pest control strategy by means of using a finite-time impulsive dynamical system, which is subject to time-dependent impulsive perturbations, which occur as a result of birth pulses and of pesticide spraying.

The remaining part of this paper is structured as follows. The next section introduces certain preliminary notions and notations on a class of finite-time impulsive dynamical systems, together with several auxiliary results, which are employed throughout this paper. These conditions are then used in Section 3 to obtain finite-time stability results for the model presented in Section 2. The biological relevance of these results is illustrated in Section 4 through a numerical example, followed by a few concluding remarks.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}_+$ denote the set of nonnegative real numbers, and let $\mathbb{R}^n$ denote the real n-dimensional space endowed with the Euclidean norm $\|\cdot\|$. Let also $\mathbb{Z}$ denote the set of all integers, For any interval $J \subseteq \mathbb{R}_+$ and set $\mathbf{S} \subseteq \mathbb{R}^n$, we shall denote by $\mathbf{C}(J, \mathbf{S})$ the set of functions $\varphi, J \to \mathbf{S}$, which are continuous on J, and by $\mathbf{PC}(J, \mathbf{S})$ be the set of functions $\varphi, J \to \mathbf{S}$, which are piecewise continuous on J and have a finite number points of discontinuity $\tau_k \in J$, $k \in \mathbb{Z}$, where they are continuous from the left. For $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, let us denote $\mathbf{B}_r(\mathbf{x_0}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x_0}\| < r\}$, that is, the open ball of center $\mathbf{x}$ and radius r.

We consider the perturbed differential system in which impulses occur at prescribed times given by

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [T_0, T_0 + T)\backslash\mathcal{T}, \\ \Delta x(t)|_{t=t_k} = I_k(t_k, x(t_k)), & 1 \le k \le N, \\ x(T_0^+) = x_0, \end{cases} \tag{1}$$

where $T_0 \ge 0$, $0 < T < \infty$, and the time interval $[T_0, T_0 + T)$ includes a finite set of resetting points $\mathcal{T} = \{t_1, t_2, \cdots, t_N\}$, $T_0 < t_1 < t_2 < \ldots < t_N < T_0 + T$. Here, $\Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_k)$ for $1 \le k \le N$, f is continuous on $([T_0, T_0 + T)\backslash\mathcal{T}) \times \mathbf{B}_{\boldsymbol{\rho}}(0)$, $\boldsymbol{\rho} > 0$ and $I_k \in \mathbf{C}([T_0, T_0 + T) \times \mathbb{R}^n, \mathbb{R}^n)$. Also, we suppose that $f(t, 0) \equiv 0$ and $I_k(t, 0) \equiv 0$ for all $1 \le k \le N$, so that the system (1) admits the null solution, whose stability we shall discuss in what follows.

To introduce a theoretical framework for our considerations, we also need the following classes of functions.

Definition 1 ([31])

(i) A function a is said to belong to class **K** if $a \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $a(0) = 0$, and $a(u)$ is strictly increasing in u.
(ii) A function a is said to belong to class $\mathbf{K_1}$ if $a \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $a(0) = 0$, and $a(u)$ is nondecreasing in u.

Definition 2 ([31])
Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. Then V is said to belong to class $\mathbf{V}_0$ if

(i) For all $1 \le k \le n$, V is continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$, and for each $x \in \mathbb{R}^n$, the limit

$$\lim_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$$

exists and is finite;
(ii) V is locally Lipschitzian in x.

Definition 3 ([31])
For $(t,x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$, define the upper Dini derivative of V at (t,x) by

$$D^+V(t,x) = \limsup_{h \to 0^+} \frac{1}{h}\left[V(t+h, x+hf(t,x)) - V(t,x)\right].$$

Definition 4 ([15, 19])

(i) Given an initial time T_0 and three positive scalars (T, α, and γ), $\alpha \le \gamma$, the null solution of (1), is said to be finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$, $\alpha \le \gamma$, if for every trajectory $x(t)$, $x(T_0) \in \mathbf{B}_{\boldsymbol{\alpha}}(\mathbf{0})$ implies $x(t) \in \mathbf{B}_{\boldsymbol{\gamma}}(\mathbf{0})$ for all $t \in [T_0, T_0 + T)$.
(ii) The null solution of (1) is said to be finite-time quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$, $\beta < \alpha$, if for every trajectory $x(t)$ with $x(T_0) \in \mathbf{B}_{\boldsymbol{\alpha}}(\mathbf{0})$, there exists a $T_1 \in (T_0, T_0 + T)$ such that $x(t) \in \mathbf{B}_{\boldsymbol{\beta}}(\mathbf{0})$ for all $t \in (T_1, T_0 + T)$.
(iii) The null solution of (1) is said to be finite-time contractively stable with respect to $(\alpha, \beta, \gamma, T_0, T, \|\cdot\|)$, if it is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$, and it is quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$.

Let us observe again that neither finite-time stability nor finite-time contractive stability are absolute concepts, both depending upon the length T of the interval on which they are considered and on the bounds α, β, γ of the initial condition and of the solution, respectively. However, because containing a pest outbreak or obtaining another similar control result is usually requested within a specified time frame, these concepts are better suited for practical purposes than classical Lyapunov stability. In concrete problems, one may think of the significance of α, β, and γ in (at least) two different ways. The first one, more ambitious, is to consider α as a bound on the initial data, β as the ET and γ as the EIL. Obviously, this is the most desirable one, in which the density of the pest population never becomes as high as to cause economic damage and after the allocated time interval of length T in which control measures are employed passes, the density of the pest population is left under the ET. The second one, less ambitious, is to think of α as being a bound on the initial data, β as the EIL and γ as the maximal support capacity of the environment for the given pest. If this course of action is taken, then after the allocated time time interval of length T in which control measures are employed passes, the density of the pest population is left under the EIL, and irreparable damages are avoided, although economic losses may be incurred.

A stronger notion of finite-time stability, which requires the convergence of trajectories to the null solution in finite time, has been discussed in [22, 26, 30] in the absence of the impulsive perturbations, a necessary and sufficient condition for the finite-time stability of a large class of systems being obtained via the use of Lyapunov functionals. The finite-time stabilization (in the same stronger sense) of a general class of nonlinear state-dependent impulsive dynamical systems has been considered in [26] in order to design finite-time stabilizing controllers, which are robust against model uncertainty. Related notions of finite-time stability, defined using polytopes in the state space, rather than open balls, have been employed in [32] and [25]. Concerning [25], the objective of the pest control strategy proposed therein is that the sizes of the pest populations (susceptible and infective) remain under their respective EILs, this condition being used to define the aforementioned polytope, which should contain all trajectories provided that their initial data obey a certain estimation. Note that [25] focuses on finite-time stability alone, rather than on finite-time contractive stability.

3. Main results

In this section, we shall establish a theoretical result, which provides sufficient conditions for the finite-time contractive stability of the null solution of the hybrid dynamical system (1), our approach relying upon the use of Lyapunov functionals. In our settings,

$$I_k(t_k, x(t_k)) = ((1+\beta_k)(1-\delta_k) - 1)\, x(t_k), \quad 1 \le k \le n, \tag{2}$$

where β_k and δ_k account for the percentile jumps in the pest population size as a result of birth pulses, respectively, of pesticide spraying. Although birth pulses and pesticide spraying do not usually occur simultaneously, one can still use the unifying notation introduced in (2) for mathematical convenience by setting the δ_k or β_k corresponding to the event, which does not occur at $t = t_k$ as being equal to 0. See the example given in Section 4 for further details concerning this matter.

Theorem 1
Assume that there exist functions $W_1, W_2 \in \mathbf{K}$, $g \in \mathbf{K_1}$, $c \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $p \in \mathbf{PC}(\mathbb{R}_+, \mathbb{R}_+)$, and $V \in \mathbf{V_0}$ and constants $T^* > T_0$, $\rho > 0$, $\eta > 0$, $(\beta_k)_{k=1}^N \subset [0, \infty)$, $(\delta_k)_{k=1}^N \subset [0, 1)$, and $M \ge \prod_{1 \le k \le N} \frac{1}{(1-\delta_k)}$ such that the following conditions hold:

(i) For $s > 0$, $s < g(s) \le Ms$, and $\inf_{s>0} \frac{c(s)}{s} > 0$, $\inf_{s \ge 0} p(s) > 0$;
(ii) For $(t, x(t)) \in [T_0, T^*] \times \mathbf{B}_{\boldsymbol{\rho}}(\mathbf{0})$,

$$W_1(\|x(t)\|) \le V(t, x(t)) \le W_2(\|x(t)\|);$$

(iii) If $t \in [T_0, T^*] \backslash \mathcal{T}$ and

$$g(V(t,x(t)) \exp(\eta(t - T_0))) > \frac{V(s, x(s))}{M} \textit{ for all } s \in [T_0, t],$$

then

$$D^+V(t, x(t)) \le -p(t)c(V(t, x(t)));$$

(iv) For all $1 \le k \le N$,

$$V(t_k^+, x(t_k) + I_k(t_k, x(t_k))) \le (1+\beta_k)(1-\delta_k)V(t_k, x(t_k));$$

(v)

$$\inf_{t \ge 0} \int_t^{t+\lambda} p(u)du > \sup_{t>0} \int_t^{\tau M t} \frac{du}{c(u)},$$

where $\lambda = \min_{2 \le k \le N}\{t_k - t_{k-1}\}$ and $\tau = \max_{2 \le k \le N}\{\exp(\tilde{m}(t_k - t_{k-1}))\}$, with

$$\tilde{m} \doteq \min\left\{\rho, \eta, \inf_{s>0} \frac{c(s)}{s} \cdot \inf_{s \ge 0} p(s)\right\};$$

(vi) There are $\gamma \in (0, \rho)$ and $H \in (0, \tilde{m})$ with the property that there exist $T \in (0, T^* - T_0)$, $T_1 \in (T_0, T_0 + T)$, $\alpha \in \left(0, W_2^{-1}\left(\frac{W_1(\gamma)}{MM^*}\right)\right)$, and $\beta \in (0, \alpha)$ such that

$$T_1 > \frac{1}{H}\left(\ln \frac{W_2(\alpha)MM^*}{W_1(\beta)} + \eta T_0\right),$$

where

$$M^* \doteq \max\left(1, \prod_{1 \le j \le k} (1+\beta_j)(1-\delta_j), 1 \le k \le N\right).$$

Then the null solution of (1) is finite-time contractively stable with respect to $(\alpha, \beta, \gamma, T_0, T, \|\cdot\|)$.

Proof

Let us observe that because $\alpha < W_2^{-1}\left(\frac{W_1(\gamma)}{MM^*}\right)$ and $MM^* > 1$, it follows that

$$W_2(\alpha) < \frac{W_1(\gamma)}{MM^*} < W_1(\gamma) \le W_2(\gamma),$$

which implies that $\alpha < \gamma$.

We first establish that the null solution of (1) is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$. To this purpose, let us fix $x(T_0) \in \mathbf{B}_{\boldsymbol{\alpha}}(\mathbf{0})$ and prove that $x(t) \in \mathbf{B}_{\boldsymbol{\gamma}}(\mathbf{0})$ for all $t \in [T_0, T_0 + T)$.

Choose $\epsilon \in (H, \tilde{m})$, denote $V(t) = V(t, x(t))$ and define

$$\Phi(t) = V(t)\exp(\epsilon(t - T_0)), \quad t \in [T_0, T_0 + T). \tag{3}$$

Define also $t_0 = T_0$, $t_{N+1} = T_0 + T$, $\beta_0 = 0$, $\delta_0 = 0$, and

$$\tilde{M}_0^* = 1, \ \tilde{M}_k^* = \prod_{1 \le j \le k} (1+\beta_j)(1-\delta_j), \quad 1 \le k \le N, \quad M_k^* = \max_{0 \le l \le k} \tilde{M}_l^*, \quad 0 \le k \le N.$$

Note that, by this definition, $(M_k^*)_{k=0}^N$ is an increasing sequence. We shall prove that

$$\Phi(t) \le MM_k^* V(T_0) \quad \text{for } t \in (t_k, t_{k+1}] \text{ and } 0 \le k \le N, \tag{4}$$

inequality, which is of great importance for our argument, because the growth of the Lyapunov function V will actually be evaluated through obtaining estimations for Φ. We start by proving that, for $l = 0$,

$$\Phi(t) \le MM_l^* V(T_0) = MV(T_0), \quad t \in [T_0, t_1]. \tag{5}$$

Suppose that (5) does not hold, and consequently, there exists $t \in [T_0, t_1)$ such that $\Phi(t) > MV(T_0)$. Let us note that Φ is continuous on $[T_0, t_1]$ and define

$$t^* = \inf\{t \in [T_0, t_1) | \Phi(t) \ge MV(T_0)\}.$$

Because $\Phi(T_0) = V(T_0) < MV(T_0)$, it is seen that $t^* \in (T_0, t_1)$. Also, because of the definition of t^*, it follows that

$$\Phi(t) < MV(T_0) \quad \text{for } t \in [T_0, t^*), \quad \Phi(t^*) = MV(T_0).$$

Using (i), one sees that

$$g(\Phi(t^*)) > \Phi(t^*) = MV(T_0).$$

Let us then define

$$t^{**} = \sup\{t \in [T_0, t^*] | g(\Phi(t)) \le MV(T_0)\},$$

and note that $t^{**} \in (T_0, t^*)$, together with

$$g(\Phi(t)) > MV(T_0) \quad \text{for } t \in (t^{**}, t^*], \quad g(\Phi(t^{**})) = MV(T_0). \tag{6}$$

Also, by (3) and (6)

$$g(V(t, x(t)) \exp(\eta(t - T_0))) \ge g(\Phi(t)) > MV(T_0) \quad \text{for } t \in (t^{**}, t^*],$$

and consequently, because $\Phi(s) \le MV(T_0)$ for $s \in [T_0, t^*]$,

$$g(V(t, x(t)) \exp(\eta(t - T_0))) > \Phi(s) \ge V(s) > \frac{V(s)}{M} \quad \text{for } t \in (t^{**}, t^*] \text{ and } s \in [T_0, t].$$

Because we have verified the hypothesis of (iii) to be true for $(t^{**}, t^*]$, we are now ready to establish the monotonicity of Φ on $(t^{**}, t^*]$. Using condition (iii), it follows that, for $t \in [t^{**}, t^*]$,

$$\begin{aligned} D^+\Phi(t) &= D^+V(t) \exp(\epsilon(t - T_0)) + \epsilon V(t) \exp(\epsilon(t - T_0)) \\ &\le -p(t)c(V(t)) \exp(\epsilon(t - T_0)) + \epsilon V(t) \exp(\epsilon(t - T_0)) \\ &= -V(t) \exp(\epsilon(t - T_0)) \left(p(t)\frac{c(V(t))}{V(t)} - \epsilon \right) \\ &\le 0. \end{aligned} \tag{7}$$

Consequently, Φ is nonincreasing on $[t^{**}, t^*]$, and therefore, $\Phi(t^{**}) \ge \Phi(t^*)$. Because

$$\Phi(t^{**}) < g(\Phi(t^{**})) = MV(T_0) = \Phi(t^*),$$

this is a contradiction, and consequently, the inequality (5) holds true. Let us now suppose that (4) holds for $0 \le k \le l - 1$ and then show that it is also valid for $k = l$. In this regard, note first that

$$\begin{aligned} V(t_l^+) \le (1 + \beta_l)(1 - \delta_l)V(t_l) &= (1 + \beta_l)(1 - \delta_l)V(t_l^-) \\ &\le (1 + \beta_l)(1 - \delta_l)MM_{l-1}^* V(T_0) \exp(-\epsilon(t_l - T_0)) \end{aligned}$$

and consequently

$$\Phi(t_l^+) \le MM_l^* V(T_0).$$

We now prove that

$$\Phi(t_l^-) \le M_{l-1}^* V(T_0). \tag{8}$$

Suppose that (8) does not hold and consequently $\Phi(t_l^-) > M_{l-1}^* V(T_0)$. This implies that either $\Phi(t) > M_{l-1}^* V(T_0)$ for all $t \in (t_{l-1}, t_l]$ or there exists $t \in (t_{l-1}, t_l)$ such that $\Phi(t) \le M_{l-1}^* V(T_0)$.

In the first case, it is seen that

$$\Phi(t) > M_{l-1}^* V(T_0) \ge \frac{\Phi(s)}{M}, \; t \in (t_{l-1}, t_l], \; s \in [T_0, t]. \tag{9}$$

Actually, the max-definition of M_k^*, which ensures that $(M_k^*)_{k=0}^N$ is an increasing sequence, was specifically given for this inequality to hold. The reason is that if $s \in (t_{j-1}, t_j]$, $j < l$ (that is, if s belongs to a 'previous' interval, rather than to the 'current' interval $(t_{l-1}, t_l]$), only the estimation $\Phi(s) \le MM_{j-1}^* V(T_0)$ is available, but this one should still imply the required estimation $\Phi(s) \le MM_{l-1}^* V(T_0)$. Through an iteration of this argument for all l, one obtains that $(M_k^*)_{k=0}^N$ needs to be an increasing sequence.

From (9), one deduces that

$$\Phi(t_l) = \Phi(t_l^-) \ge \frac{\Phi(t_{l-1}^+)}{M}.$$

Then, by (3),

$$MV(t_l^-) \exp(\epsilon(t_l - T_0)) \ge V(t_{l-1}^+) \exp(\epsilon(t_{l-1} - T_0)),$$

that is,

$$MV(t_l^-) \exp(\epsilon(t_l - t_{l-1})) > V\left(t_{l-1}^+\right). \tag{10}$$

Let us define

$$\tau_1 = \max_{l=2,3,\cdots,N}\{\exp(\epsilon(t_l - t_{l-1}))\}.$$

Then

$$\tau MV(t_l^-) > \tau_1 MV(t_l^-) > V\left(t_{l-1}^+\right). \tag{11}$$

By (i), (3), and (9), it is seen that, for $t \in (t_{l-1}, t_l]$ and $s \in [T_0, t]$,

$$g(V(t,x(t))\exp(\eta(t-T_0))) \geq g(\Phi(t)) > \Phi(t) > \frac{\Phi(s)}{M} \geq \frac{V(s)}{M},$$

and by condition (iii), the inequality $D^+V(t) \leq -p(t)c(V(t))$ holds for all $t \in (t_{l-1}, t_l]$. Consequently, from (11), we obtain that

$$\int_{V(t_l^-)}^{V(t_{l-1}^+)} \frac{du}{c(u)} < \int_{V(t_l^-)}^{\tau MV(t_l^-)} \frac{du}{c(u)} \leq \sup_{s>0} \int_s^{\tau Ms} \frac{du}{c(u)}. \tag{12}$$

However, from conditions (iii) and (v), one notes that

$$\int_{V(t_l^-)}^{V(t_{l-1}^+)} \frac{du}{c(u)} \geq \int_{t_{l-1}}^{t_l} p(u)du \geq \int_{t_{l-1}}^{t_{l-1}+\lambda} p(u)du \geq \inf_{s\geq 0} \int_s^{s+\lambda} p(u)du, \tag{13}$$

which leads to a contradiction.

Next, we consider the second case. Let us define

$$t' = \sup\{t \in (t_{l-1}, t_l) | \Phi(t) \leq M_{l-1}^* V(T_0)\}.$$

Then $t' \in (t_{l-1}, t_l)$ and

$$\Phi(t') = M_{l-1}^* V(T_0), \quad \Phi(t) > M_{l-1}^* V(T_0) \text{ for } t \in (t', t_l],$$

which implies that, for $t \in [t', t_l]$ and $s \in [T_0, t]$,

$$g(V(t,x(t))\exp(\eta(t-T_0))) \geq g(\Phi(t)) > \Phi(t) > M_{l-1}^* V(T_0) \geq \frac{\Phi(s)}{M} \geq \frac{V(s)}{M}.$$

By applying the same argument as in the proof of (7), we then obtain that $\Phi(t)$ is nonincreasing in t for $t \in [t', t_l]$. In particular,

$$\Phi(t') \geq \Phi(t_l) = \Phi(t_l^-).$$

However, this contradicts the fact that

$$\Phi(t_l^-) > V(T_0) = \Phi(t').$$

As a result, we can claim that (8) holds. In the following, we shall prove that (4) holds for $k = l$.

Suppose that this assertion is not true. There then exists $t \in (t_l, t_{l+1}]$ such that

$$\Phi(t) > MM_l^* V(T_0).$$

Let us define

$$t'' = \inf\{t \in (t_l, t_{l+1}] | \Phi(t) \geq MM_l^* V(T_0)\}.$$

Then $t'' \in (t_l, t_{l+1})$ and

$$\Phi(t'') = MM_l^* V(T_0), \quad \Phi(t) < MM_l^* V(T_0) \text{ for } t \in (t_l, t'').$$

It is easy to see that

$$g(\Phi(t'')) = g(MM_l^* V(T_0)) > MM_l^* V(T_0)$$

and, by (i) and (8),

$$g\left(\Phi\left(t_l^+\right)\right) \leq MM_l^* V(T_0).$$

Consequently, we may define

$$\widetilde{t} = \sup\{t \in (t_l, t''] | g(\Phi(t)) \le MM_l^* V(T_0)\}.$$

Then $\widetilde{t} \in (t_l, t'']$ and

$$g(\Phi(\widetilde{t})) = MM_l^* V(T_0), \quad g(\Phi(t)) > MM_l^* V(T_0) \text{ for } t \in (\widetilde{t}, t''].$$

Thus, we also have

$$g(V(t,x)\exp(\eta(t-T_0))) \ge g(\Phi(t)) > MM_l^* V(T_0) \ge \Phi(s) \ge \frac{V(s)}{M} \text{ for } t \in (\widetilde{t}, t''] \text{ and } s \in [T_0, t].$$

As performed earlier, we can obtain that $\Phi(t)$ is nonincreasing in t for $t \in [\widetilde{t}, t'']$. Thus, one notes that

$$\Phi(\widetilde{t}) \ge \Phi(t''),$$

which contradicts the fact that

$$\Phi(t'') = MM_l^* V(T_0) = g(\Phi(\widetilde{t})) > \Phi(\widetilde{t}).$$

By the aforementioned discussion , it is seen that (4) holds. Consequently, the following global growth estimation holds

$$\Phi(t) \le MM^* V(T_0) \quad \text{for } t \in [T_0, T_0 + T)$$

and, by (ii),

$$W_1(\|x(t)\|) \le V(t) \le MM^* V(T_0)\exp(-\epsilon(t - T_0)) \quad \text{for } t \in [T_0, T_0 + T).$$

Because, from (ii),

$$V(T_0) \le W_2(\|x(T_0)\|) < W_2(\alpha),$$

one sees using (4) that

$$W_1(\|x(t)\|) \le MM^* W_2(\alpha)\exp(-\epsilon(t - T_0)) < W_1(\gamma) \quad \text{for } t \in [T_0, T_0 + T)$$

and consequently

$$x(t) \in \mathbf{B}_{\boldsymbol{\gamma}}(\mathbf{0}) \quad \text{for } t \in [T_0, T_0 + T).$$

It now follows that the null solution of the system (1) is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$.
Similarly, using again (ii) and (4),

$$W_1(\|x(t)\|) \le MM^* W_2(\alpha)\exp(-\epsilon(T_1 - T_0)), \quad \text{for } t \in (T_1, T_0 + T),$$

which implies, by the choices of ϵ and H,

$$W_1(\|x(t)\|) \le W_1(\beta)\exp(T_1(H - \epsilon) + T_0(\epsilon - \eta)) < W_1(\beta), \quad \text{for } t \in (T_1, T_0 + T),$$

and consequently

$$x(t) \in \mathbf{B}_{\boldsymbol{\beta}}(\mathbf{0}) \quad \text{for } t \in (T_1, T_0 + T).$$

It now follows that the null solution of the system (1) is finite-time quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$, which finishes the proof. ☐

Remark 1

Let us comment on the significance of the hypothesis employed in Theorem 1. In the aforementioned discussion, the meaning of (iii) is that if the value of a function of V is larger than all previous values of V (that is, if V has grown too large), then the derivative of V should satisfy a certain estimation, which in particular ensures its negative sign, in order to ensure that V does not grow even larger. The usage of a general function g (as opposed to using just a multiple of the identity function) allows for some flexibility in the formulation of the growth condition, further softening of this condition being realized through the usage of an exponential in the left-hand side.

The estimation employed in (iv) quantifies the effect of the impulsive perturbations upon the growth of the solution. A typical example of V is the square of the norm (but some other choices are possible too), possibly perturbed, as long as the perturbation obeys some limitations. In this case, W_1 and W_2 from (ii) are also typified by multiples of the square of the norm.

From the aforementioned result, by particularizing $g(t) = Mt$, $p(t) \equiv p > 0$, and $c(u) = u$, one may obtain the following practical consequence.

Corollary 1
Assume that there exist several functions $W_1, W_2 \in \mathbf{K}$, and $V \in \mathbf{V_0}$ and several constants $p > 0$, $T^* > 0$, $\rho > 0$, $\eta > 0$, $\eta > 0$, $(\beta_k)_{k=1}^N \subset [0, \infty)$, $(\delta_k)_{k=1}^N \subset [0, 1)$, and $M \geq \prod_{1 \leq k \leq N} \frac{1}{(1-\delta_k)}$ such that the following conditions hold:

(i) For $(t, x(t)) \in [0, T^*] \times \mathbf{B_\rho}(\mathbf{0})$,

$$W_1(\|x(t)\|) \leq V(t, x(t)) \leq W_2(\|x(t)\|);$$

(ii) If $t \in [T_0, T^*] \backslash \mathcal{T}$ and

$$\frac{V(t, x(t))}{V(s, x(s))} > \frac{\exp(-\eta(t - T_0))}{M^2} \quad \text{for all } s \in [T_0, t],$$

then

$$D^+ V(t, x(t)) \leq -pV(t, x(t)).$$

(iii) For all $1 \leq k \leq N$,

$$V(t_k^+, x(t_k) + I_k(t_k, x(t_k))) \leq (1 + \beta_k)(1 - \delta_k)V(t_k, x(t_k));$$

(iv)

$$\lambda > \frac{\ln(M\tau)}{p},$$

where

$$\lambda = \min_{2 \leq k \leq N}\{t_k - t_{k-1}\}, \quad \tau = \max_{2 \leq k \leq N}\{\exp(\tilde{m}(t_k - t_{k-1}))\}, \quad \tilde{m} = \min\{\rho, \eta, p\};$$

(v) There are $\gamma \in (0, \rho)$ and $H \in (0, \tilde{m})$ with the property that there exist $T \in (0, T^* - T_0)$, $T_1 \in (T_0, T_0 + T)$, $\alpha \in \left(0, W_2^{-1}\left(\frac{W_1(\gamma)}{MM^*}\right)\right)$, and $\beta \in (0, \alpha)$ such that

$$T_1 > \frac{1}{H}(\ln \frac{W_2(\alpha)MM^*}{W_1(\beta)} + \eta T_0),$$

with

$$M^* \doteq \max\left(1, \prod_{1 \leq j \leq k}(1 + \beta_j)(1 - \delta_j), 1 \leq k \leq N\right).$$

Then the null solution of (1) is finite-time contractively stable with respect to $(\alpha, \beta, \gamma, T_0, T, \| \cdot \|)$.

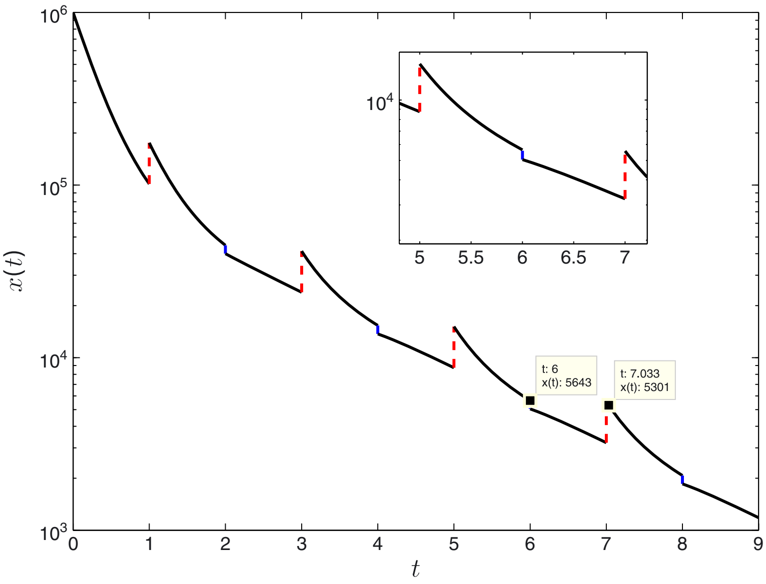

Figure 1. Trajectories $x(t)$ of the system against time.

Remark 2

From the conclusion of Theorem 1, one notes that our result allows significant increases in V as a result of birth pulses, as long as the decrease of V between pulses compensates them properly. That is, adequate policies of continuous control can always keep pests in check, even those with strong natality outbursts.

4. Biological relevance and concluding remarks

As it has been seen in the previous sections, our approach is essentially based on the Lyapunov–Razumikhin method, making use of estimations in terms of guiding Lyapunov functionals, in combination with inf-sup arguments, which establish the impossibility of a first breakdown in various instances. This is performed in order to obtain the finite-time contractive stability of the null solution. Because Lyapunov–Razumikhin method has already been proved its usefulness when dealing with classical Lyapunov stability problems for delay differential equations, the scope of our results may probably be extended to encompass impulsive functional differential systems with delay, after a suitable modification of the dissipativity hypothesis (iii) of Theorem 1 in order to deal with the behavior of V on the 'history' interval $[t-\tau, t]$, τ being the length of the delay. Also, it would be desirable to make our approach resilient to possible errors in the evaluation of V (that is, to allow for further terms in the right-hand side of (iv) in Theorem 1, which would provide an estimation of the errors). Results in this direction (stated as global stability results with convergence rate but easily adaptable to our framework) have already been obtained in [33], but for a less general form of assumptions (ii) and (iii). This line of thought is subject of further research.

Related controllability results have been obtained in [32] for a less general class of differential systems, but with a more general resetting condition, possibly state-dependent. In [32], the focus is on the finite-time stabilization of the hybrid system under consideration, contractivity not being of concern, a similar viewpoint being also followed in [24]. A more restrictive concept of stability, the so-called finite-time contractive stability with fixed settling time, has been recently introduced in [34]. This concept differs from the concept of finite-time contractive stability used in this paper in that the settling time T_1 in (ii) of Definition 4 is specified beforehand. Our approach can also be adapted to yield fixed settling time results similar to Theorems 4 and 5 of [34] at the cost of further strengthening of hypothesis (vi) of Theorem 1.

We now attempt to illustrate the applicability of our abstract controllability results. To this purpose, we introduce a model, which describes the dynamics of a single pest species, which is subject to control measures consisting in periodic pesticide spraying and whose growth occurs in birth pulses. Consider the hybrid dynamical system given by

$$\begin{cases} x'(t) = \int_0^t \exp(-s - \frac{t}{2}) x(s) ds - 3x(t), & t \neq 1, 2, \cdots, 9, \\ \triangle x(t) = (\sqrt{1+2} - 1) x(t), & t = 1, 3, 5, 7, \\ \triangle x(t) = (\sqrt{1-0.2} - 1) x(t), & t = 2, 4, 6, 8, \end{cases} \tag{14}$$

in which $x(t)$ represents the density of the pest population at time t. In the aforementioned model, the integral term $\int_0^t \exp(-s - \frac{t}{2}) x(s) ds$ describes the pest immigration for collective attack and defense, the resetting conditions $x((2k-1)^+) = x((2k-1))\sqrt{1+2}$, $k = 1, 2, 3, 4$, describe the birth pulses, and the other resetting conditions $x((2k)^+) = x(2k)\sqrt{1-0.2}$, $k = 1, 2, 3, 4$, describe the proportional decrease of the pest population size, which occurs as a direct result of pesticide spraying. Let $T_0 = 0, M = 2, \eta = 1, c(s) = s, g(s) = 2s, N = 8, T = 9, T^* = 9, W_1(\|x\|) = \frac{1}{2}\|x\|^2, W_2(\|x\|) = 2\|x\|^2, V(t, x(t)) = x^2(t), H = 0.975, \tau = e$, and $\rho = 10^6$. One then notes that

$\beta_1 = 2$	$\delta_1 = 0$
$\beta_2 = 0$	$\delta_2 = 0.2$
$\beta_3 = 2$	$\delta_3 = 0$
$\beta_4 = 0$	$\delta_4 = 0.2$
$\beta_5 = 2$	$\delta_5 = 0$
$\beta_6 = 0$	$\delta_6 = 0.2$
$\beta_7 = 2$	$\delta_7 = 0$
$\beta_8 = 0$	$\delta_8 = 0.2$

and

$$M^* = \max\left\{1, \prod_{1 \le k \le 8} (1+\beta_k)(1-\delta_k)\right\} = 41.472.$$

Also, if $\frac{x^2(t)}{x^2(s)} \geq \frac{e^{-t}}{4}$, $s \in [0, t]$, that is, $2x(t)\exp(\frac{t}{2}) \geq x(s)$, $s \in [0, t]$, one sees that $x'(t) \leq -x(t)$ for $t > 0$, and consequently

$$D^+ V(t) = 2x(t)x'(t) \leq -2V(t),$$

which implies that (ii) is satisfied with $p(t) \equiv 2$. After analyzing the aforementioned data and using Corollary 1, we conclude that the system (1) is said to be finite-time contractively stable with respect to $(9000, 5301, 10^6, 0, 9, \|\cdot\|)$(Figure 1).

Remark 3

As previously mentioned, the most desirable way to interpret α, β and γ is as a bound on the initial data, the ET and the EIL, respectively, which would provide guidelines to avoid economic losses and would assure the effectiveness of the IPM strategy. Although our example is more of a showcase for the previously constructed abstract framework rather than accurately representing a concrete pest management strategy, being specifically created to fit the hypotheses of Corollary 1, it is still suitable to represent dynamics of species, which are dependent of their entire histories, as specified in the integral term.

Remark 4

Using the aforementioned numerical data, one may estimate T_1, as shown in Figure 1, as being approximately equal to 7.033, which represents the approximate time after which the size of the pest population is reduced under ET, that is, the minimal time for the IPM strategy to reach the desired goal.

Although IPM strategies are a valid alternative to the classical, pesticide-reliant approach, they are also not to be used without precautions, as it has been observed in [35] that because of the commercialization of genetically engineered cotton, which produces an insecticidal protein derived from the biocontrol agent, *Bacillus thuringiensis*, insect pests such as *H. virescens* are adapting to this class of toxins. Our abstract results are then intended to contribute to the design of better, more efficient, and economical IPM strategies, which should be able to achieve their goals within a finite, prescribed time.

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