An extended class of nonlinear groups and its applications to the generalized Kortweg-deVries equations

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Abstract. The initial value problem for the generalized Kortweg-deVries equation

$$u_t + (f(u))_x + u_{xxx} = 0, \qquad t, x \in \mathbb{R}$$

is treated in terms of a recent theory of nonlinear operator semigroups associated with semilinear evolution equations in Banach spaces. Two operators A and B are introduced to represent the linear and nonlinear differential operators in the equation and convert the initial-value problem to a semilinear problem

$$\text{(SP)} \qquad u'(t) = (A + B)\, u(t), \quad t > 0; \qquad u(0) = v$$

in the Sobolev space $H^2(\mathbb{R})$. Five energy functionals are then employed to restrict basic properties of $A + B$ as well as the growth of mild solutions to (SP). The solution operators to (SP) are obtained by applying a generation theorem for locally Lipschitzian groups. Here the main point of our argument is to make a precise investigation of the resolvents of $A + B$ and construct a group of locally Lipschitzian operators $G(t)$ on $H^2(\mathbb{R})$ which provides mild solutions to the problem. Also, regularized equations of the form

$$u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t, x \in \mathbb{R},$$

μ being a positive parameter, are studied by means of the same approach and the convergence of the associated groups $G_\mu(t)$ to the group $G(t)$ is discussed.

1 Introduction

This paper is concerned with the initial value problem for the generalized Kortweg-deVries equations

$$\begin{aligned} &u_t + (f(u))_x + u_{xxx} = 0, \qquad &t, x \in \mathbb{R}; \\ &u(0, x) = v(x), \qquad &x \in \mathbb{R}, \end{aligned}$$

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where f is a nonlinear function of class $C^3(\mathbb{R})$ which satisfies the equality $f(0) = 0$ and the growth condition $\overline{\lim}_{|u|\to\infty} f'(u)/|u|^p < \infty$ for some $p \in [0,4)$, and v is a given initial function in $H^2(\mathbb{R})$.

In the case $f(u) = u + u^2/2$, the above equation is known as the Kortweg-deVries equation (usually, abbreviated to a K-dV equation), which is understood to be a general model for the unidirectional propagation of long waves of small amplitude. In fact, u determines the height of a wave at position x and time t with respect to the standard level. The K-dV equation is also formulated to describe physical phenomena such as magnetohydrodynamical waves and interaction of solitons.

In this paper, we convert the initial-value problem for the generalized K-dV equation to a semilinear evolution problem of the form

$$\text{(SP)} \qquad u'(t) = (A+B)\,u(t), \qquad t \in \mathbb{R}; \qquad u(0) = v,$$

in order to treat the Cauchy problem in an operator theoretic fashion. Here A represents the third-order differential operator $-\partial_x^3$ and B stands for the nonlinear first-order differential operator $-\partial_x \circ f$. We then apply a recent theory for semilinear problems involving quasidissipative operators developped in [7], [14], [17], [18] to this semilinear operator $A+B$ and construct a group $G = \{G(t); t \in R\}$ of nonlinear operators on $H^2(\mathbb{R})$ which provides mild solutions to (SP) in the sense that

$$G(t)v = G_A(t)v + \int_0^t G_A(t-s)BG(s)v\,ds$$

for $t \in \mathbb{R}$ and $v \in H^2(\mathbb{R})$, where G_A is the unitary group generated in $L^2(\mathbb{R})$ by A. One of the main features of our argument is that we make use of five energy functionals φ_k, $k = 0,1,2,3,4$ and investigate the growth of the mild solutions and their qualitative properties. More precisely, we show that the group G enjoys exponential type growth conditions with respect to the functionals φ_k, and that the regularity of the mild solutions $u(t) \equiv G(t)v$ is obtained by means of φ_k. In order to apply a recent theory for groups of locally Lipschitzian operators, we necessitate investigating the ranges of $I - \lambda(A+B)$, $|\lambda| < \lambda_0$ and their resolvemts $(I - \lambda(A+B))^{-1}$ in $H^3(\mathbb{R})$ through a fixed point argument. The aimed group $G(t)$ is constructed through the exponential formula

$$G(t)v = H^2\text{-}\lim_{\lambda\downarrow 0}(I - \lambda(A+B))^{-[t/\lambda]}v, \quad t \geq 0\ ,\ v \in H^2(\mathbb{R}).$$

We then consider the initial value problem for the pseudoparabolic regularization of the generalized Kortweg-deVries equations

$$\begin{aligned} &u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t, x \in \mathbb{R} \\ &u_0(0,x) = u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

where μ is a positive parameter. Due to the regularizing effect of the term $-\mu u_{txx}$, a nonlinear group of Fréchet differentiable operators which provides mild solutions to the regularized problem is constructed on $H^1(\mathbb{R})$. In this case $\mu > 0$, f is a nonlinear function of class $C^1(\mathbb{R})$ satisfying $f(0) = 0$ and u_0 is an initial function given in H^1. We also discuss the convergence of the groups G_μ to G as $\mu \to 0$.

2 A generation theorem for nonlinear groups

It is now generally accepted that the theory of semilinear evolution equations proved to be an important tool for the study of many important problems arising in various fields. In this section, folllowing the lines of [7], an attempt is made to provide a generation theorem for nonlinear groups of locally Lipschitz operators associated with a certain class of semilinear evolution problems. For related Hille-Yosida theorems in this case, we refer the reader to [14], [17], [18]. See also [12], [13], [21] for generation theorems for nonlinear semigroups under more general assumptions.

Let $(X, |\cdot|)$ be a real Banach space, D a subset of X and $\varphi : X \to [0, \infty]$ a l.s.c. functional such that $D \subset D(\varphi) = \{v \in X, \varphi(v) < \infty\}$. We denote by X^* the dual of X, and given $v \in X$ and $v^* \in X^*$, the value of v^* at v is written $\langle v, v^* \rangle$. We also denote by $D_\alpha = \{v \in D; \varphi(v) \leq \alpha\}$ a level set of D with respect to φ. The duality mapping of X is the function $F : X \to 2^{X^*}$ defined by

$$Fv = \left\{v^* \in X^*; \langle v, v^* \rangle = |v|^2 = |v^*|^2\right\}.$$

We then define the inner products $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_s$ on $X \times X$ by

$$\langle w, v \rangle_i = \inf \{\langle w, v^* \rangle, v^* \in Fv\},$$

respectively

$$\langle w, v \rangle_s = \sup \{\langle w, v^* \rangle, v^* \in Fv\}.$$

A nonlinear operator $B : D \subset X \to X$ is said to be locally quasidissipative (respectively strongly locally quasidissipative) on $D(B)$ with respect to φ if for each $\alpha \geq 0$ there exist $\omega_\alpha \in \mathbb{R}$ such that

$$\langle Bv - Bw, v - w \rangle_i \leq \omega_\alpha |v - w|^2 \quad \text{for } v, w \in D_\alpha,$$

respectively

$$\langle Bv - Bw, v - w \rangle_s \leq \omega_\alpha |v - w|^2 \quad \text{for } v, w \in D_\alpha.$$

For further properties of the duality mapping and those of quasidissipative operators, see [6] or [20].

By a locally Lipschitzian group on D with respect to φ, we mean a one-parameter family $\mathcal{G} = \{G(t); t \in \mathbb{R}\}$ of (possibly nonlinear) operators from D into itself satisfying the following three conditions below:

(G1) For $v \in D$ and $s, t \in \mathbb{R}$, $\quad G(t) G(s) v = G(t + s) v, \quad G(0) v = v.$

(G2) For $v \in D$, $G(\cdot) v \in C(\mathbb{R}; X)$.

(G3) For each $\alpha > 0$ and each $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that

$$|G(t) v - G(t) w| \leq e^{\omega |t|} |v - w|$$

for $v, w \in D_\alpha = \{v \in D; \varphi(v) \leq \alpha\}$ and $t \in [0, \tau]$.

We consider the semilinear problem

$$\text{(SP)} \qquad u'(t) = (A + B) u(t), \qquad t \in \mathbb{R}; \qquad u(0) = v \in D$$

and we assume the following hypotheses on A, B and D:

(A) $A : D(A) \subset X \to X$ generates a (C_0)-group $G_A = \{G_A(t); t \in \mathbb{R}\}$ such that $|G_A(t)v| \le e^{\omega t}|v|$ for $v \in X$, $t \in \mathbb{R}$ and some $\omega \in \mathbb{R}$.

(B) The level set D_α is closed for each $\alpha \ge 0$ and $B : D \subset X \to X$ is continuous on each D_α.

Since the semilinear problem (SP) does not necessarily admit strong solutions, the variation of constants formula is employed to obtain solutions in a generalized sense. It is then said that a function $u(\cdot) \in C([0,\infty); X)$ is a mild solution to (SP) if $u(t) \in D$ for $t \ge 0$, $Bu(\cdot) \in C([0,\infty); X)$ and the integral equation

$$u(t) = T(t)v + \int_0^t T(t-s)Bu(s)ds$$

is satisfied for each $t \ge 0$. We also say that a semigroup S is associated with (SP), if it provides mild solutions to (SP) in the sense that for each $v \in D$ the function $u(\cdot) = S(\cdot)v$ is a mild solution to (SP).

Under the above hypotheses one can obtain a semilinear Hille-Yosida theorem for locally Lipschitzian groups of nonlinear operators associated with (SP) as follows.

Theorem 2.1. *Let $a, b \ge 0$, A a linear operator in X such that A satisfies condition* (A) *and let B be a nonlinear operator on D such that B satisfies condition* (B) *with respect to a l.s.c. functional φ on X with $D \subset D(\varphi)$. Then the following statements are equivalent:*

(I) *There is a nonlinear group $\mathcal{G} = \{G(t); t \in \mathbb{R}\}$ of locally Lipschitz operators on D satisfying the properties given below:*

(I.1) $G(t)v = G_A(t)v + \int_0^t G_A(t-s)BG(s)v ds$ *for $t \in \mathbb{R}$ and $v \in D$.*

(I.2) *For each $\alpha > 0$ and $\tau > 0$ there is $\omega_1 = \omega_1(\alpha,\tau) \in \mathbb{R}$ such that*

$$|G(t)v - G(t)w| \le e^{\omega_1(\alpha,\tau)|t|}|v-w|$$

for each $v, w \in D_\alpha$.

(I.3) $\varphi(G(t)v) \le e^{a|t|}(\varphi(v) + b|t|)$ *for $t \in \mathbb{R}$ and $v \in D$.*

(II) *The following subtangential condition and semilinear stability condition are satisfied:*

(II.1) *For each $v \in D$ and $\varepsilon > 0$ there are $(h_1, v_{h_1}) \in (0,\varepsilon] \times D$ and $(h_2, v_{h_2}) \in [-\varepsilon, 0) \times D$ such that*

$$(1/h_i)|G_A(h_i)v + h_iBv - v_{h_i}| \le \varepsilon, \quad \varphi(v_{h_i}) \le e^{a|h_i|}(\varphi(v) + (b+\varepsilon)h_i) \quad i = 1, 2.$$

(II.2) *For each $\alpha > 0$ there is $\omega_\alpha \in \mathbb{R}$ such that*

$$\varliminf_{h\to 0}(1/|h|)[|G_A(h)(v-w) + h(Bv - Bw)| - |v-w|] \le \omega_\alpha|v-w|$$

for each $v, w \in D_\alpha$.

Moreover, if D and φ are convex, then the above statements are equivalent to:

(III) *The following denseness condition, quasidissipativity condition and range condition are satisfied:*

(III.1) *$D(A) \cap D$ is dense in D.*

(III.2) *For each $\alpha > 0$ there is $\omega_\alpha \in \mathbb{R}$ such that*

$$\langle (A+B)v - (A+B)w, v-w \rangle_i \leq \omega_\alpha |v-w|^2,$$
$$\langle (A+B)v - (A+B)w, v-w \rangle_s \geq -\omega_\alpha |v-w|^2.$$

(III.3) *To $\alpha > 0$ and $\varepsilon > 0$ there corresponds $\lambda_0 = \lambda_0(\alpha) > 0$ and for $v \in D_\alpha$ and $\lambda \in \mathbb{R}$ with $|\lambda| < \lambda_0(\alpha)$ there exist $v_\lambda \in D(A) \cap D$ and $z_\lambda \in X$ such that $|z_\lambda| < \varepsilon$,*

$$v_\lambda - \lambda (A+B) v_\lambda = v + \lambda z_\lambda, \quad \varphi(v_\lambda) \leq (1 - |\lambda| a)^{-1} (\varphi(v) + (b+\varepsilon)|\lambda|).$$

It should be noted here that the implication from (III) to (I) does not require any convexity on D or φ. Also, if X is a Hilbert space and $(Av, v) = 0$, as is the case of K-dV equation, only the inequality $|(Bv - Bw, v-w)| \leq \omega_0(\alpha)|v-w|^2$ should be verified in place of (III.2). Moreover, if B is a locally Lipschitz operator, then the denseness assumption (III.1) is unnecessary for the implication from (III) to (I).

3 Semilinear evolution problems for the generalized Kortweg-deVries equations

In this section we construct a nonlinear group which provides mild solutions to the initial value problem for the generalized K-dV equation

$$u_t + (f(u))_x + u_{xxx} = 0 \qquad t, x \in \mathbb{R} \tag{3.1}$$
$$u(0,x) = v(x) \qquad x \in \mathbb{R}. \tag{3.2}$$

Here $\mathbb{R}=(-\infty, \infty)$, f in (3.1) is a nonlinear function of class $C^3(\mathbb{R})$ normalized so that $f(0) = 0$ and v in (3.2) is a given initial function in $H^2(\mathbb{R})$. We also assume that f satisfies the growth condition

$$\overline{\lim_{|u|\to\infty}} f'(u) / |u|^p < \infty \tag{3.3}$$

for some real number $p \in [0, 4)$, where f' denotes the derivative of f.

For a study of K-dV equation or its generalized form using compactness methods, we refer the reader to, for instance, Kametaka [11], Tsutsumi and Mukasa [24], Bona and Smith [5]. See also [8] for a discussion on K-dV equation using related methods. In this paper we establish the existence and uniqueness of nonlinear groups of locally Lipschitz operators on $H^2(\mathbb{R})$ which provides mild solutions to the initial value problem (3.1)-(3.2) using the generation theorem stated in the previous section.

In what follows, H^k denotes the Sobolev space $H^k(\mathbb{R})$ for each nonnegative integer k. The inner product and the norm of H^k are denoted by $(\cdot,\cdot)_k$ and $|\cdot|_k$ respectively. In particular, H^0 means just the ordinary Lebesgue space $L^2 = L^2(\mathbb{R})$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. By $C(\mathbb{R}, H^k)$ we mean the space of H^k–valued continuous functions on $\mathbb{R}$. For each integer $m \geq 1$ we write $C^m(\mathbb{R}; H^k)$ for the space of H^k–valued functions which are m times continuously differentiable on $\mathbb{R}$.

Let ∇ be the differential operator d/dx acting from H^1 into L^2. It is obvious that

$$(\nabla v, w) = -(w, \nabla v) \text{ and } (\nabla v, v) = 0 \text{ for } v, w \in H^1. \tag{3.4}$$

The following inequality is well known (see [22]).

Lemma 3.1. *Let $2 \leq q \leq \infty$. If $v \in H^1$, then the inequality*

$$|v|_{L^q} \leq 2^r |\nabla v|^r |v|^{1-r} \tag{3.5}$$

is valid, where $r = (q-2)/2q$ and $|\cdot|_{L^q}$ denotes the norm in the space $L^q(\mathbb{R})$.

Since we aim to rewrite equation (3.1) as an abstract semilinear evolution equation in L^2, of the form

$$(d/dt)\, u(t) = (A + B)\, u(t) \qquad t \in \mathbb{R}, \tag{3.6}$$

we introduce a densely defined, closed linear operator from H^3 into L^2 by

$$Av = -\nabla^3 v \qquad \text{for } v \in H^3. \tag{3.7}$$

It is then seen that A is the infinitesimal generator of a group $G_A = \{G_A(t); t \geq 0\}$ of linear isometries on L^2. More precisely, each of $G_A(t)$ maps H^k into itself and satisfies the relation

$$|G_A(t)\, v|_k = |v|_k \qquad \text{for } v \in H^k \text{ and } k \geq 0. \tag{3.8}$$

Further, we define a nonlinear operator B from H^1 into L^2 by

$$Bv = -\nabla f(v) = -f'(v)\, \nabla v \qquad \text{for } v \in H^1. \tag{3.9}$$

The idea which motivates this approach is that B, as a lower order differential operator, may be regarded as a continuous perturbation of A via a suitable restriction of the domain. The same thing is also valid for quasidissipativity, as seen in the following result.

Proposition 3.1. *The following assertions hold:*

(i) *Let $v \in H^1$ and let $\{v_n\}_{n\geq 1}$ be a sequence in H^1 such that $\sup_{n\geq 1} |v_n|_1 < \infty$. If $v_n \rightharpoonup v$ in L^2, then $Bv_n \rightharpoonup Bv$ in L^2.*

(ii) *Let $v \in H^2$ and let $\{v_n\}_{n\geq 1}$ be a sequence in H^2 such that $\sup_{n\geq 1} |v_n|_2 < \infty$. If $v_n \to v$ in L^2, then $Bv_n \to Bv$ in L^2.*

(iii) *For each $\alpha \geq 0$, there is a number $\omega_0(\alpha) \geq 0$ such that*

$$|(Bv - Bw, v - w)| \leq \omega_0(\alpha)\, |v - w|^2 \tag{3.10}$$

for $v, w \in H^2$ with $|v|_2 \leq \alpha, |w|_2 \leq \alpha$.

(iv) *For each $\alpha \geq 0$, there is a number $\omega_1(\alpha) \geq 0$ such that*

$$|(Bv - Bw, v - w)|_1 \leq \omega_1(\alpha)\, |v - w|_1^2 \tag{3.11}$$

for $v, w \in H^3$ with $|v|_3 \leq \alpha, |w|_3 \leq \alpha$.

Proof. First, we see that for each $u \in H^1$

$$u^2(x) = \int_{-\infty}^{x} u(s)\, u'(s)\, ds - \int_{x}^{\infty} u(s)\, u'(s)\, ds \quad \text{for a.e. } x \in \mathbb{R}.$$

We therefore infer that

$$\left|u^2(x)\right| \leq \int_{-\infty}^{\infty} |u(s)|\, |u'(s)|\, ds \leq (|u|\, |u'|) \text{ for a.e. } x \in \mathbb{R},$$

which implies $|u|_{L^\infty} \leq \left(1/\sqrt{2}\right) |u|_1$. However, for the sake of simplicity, in what follows we will use the inequality

$$|u|_{L^\infty} \leq |u|_1 \text{ for each } u \in H^1. \tag{3.12}$$

Let $v \in H^1$ and let $\{v_n\}_{n\geq 1}$ be a sequence in H^1 such that $\sup\limits_{n\geq 1} |v_n|_1 < \infty$ and $v_n \rightharpoonup v$ in L^2. Denoting $\sup\limits_{n\geq 1} |v_n|_1$ by M_1, an easy computation implies

$$|Bv_n|^2 = \int_{\mathbb{R}} \left|f'(v_n) \nabla v_n\right|^2 dx \leq \left(\sup_{|x|\leq M_1} |f'(s)| \right)^2 \int_{\mathbb{R}} |\nabla v_n|^2\, dx,$$

from which we infer that $\sup\limits_{n\geq 1} |Bv_n| < \infty$ and $v_n \to v$ in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Let $\varphi \in L^2$. Since $C_{\mathrm{c}}^\infty(\mathbb{R})$ is dense in L^2, one can construct a sequence $\{\varphi_m\}_{m\geq 1}$ such that $\varphi_m \in C_{\mathrm{c}}^\infty(\mathbb{R})$ and $\varphi_m \to \varphi$ in L^2 as $m \to \infty$. In view of this and of (3.4), we see that

$$\begin{aligned}
\langle Bv_n - Bv, \varphi_m \rangle &= \langle f(v_n) - f(v), \nabla\varphi_m \rangle \\
&= \int_{C_m} (f(v_n) - f(v))\, \nabla\varphi_m dx \\
&\leq |f(v_n) - f(v)|_{L^2(C_m)} |\nabla\varphi_m|_{L^2(C_m)} \\
&\leq \mathscr{C}(f)\, |v - v_n|_{L^2(C_m)} |\nabla\varphi_m|_{L^2(C_m)},
\end{aligned}$$

where m is an arbitrary nonnegative integer, C_m denotes the (compact) support of φ_m and $\mathscr{C}(f)$ denotes a constant which depends on f, but not on n. Thus it follows that

$$\langle Bv_n - Bv, \varphi_m \rangle \to \infty \text{ as } n \to \infty. \tag{3.13}$$

Since

$$\begin{aligned}
\langle Bv_n - Bv, \varphi \rangle &= \langle Bv_n - Bv, \varphi - \varphi_m \rangle + \langle Bv_n - Bv, \varphi_m \rangle \\
&\leq \left(\sup_{n\geq 1} |Bv_n| + |Bv| \right) |\varphi - \varphi_m| + \langle Bv_n - Bv, \varphi_m \rangle,
\end{aligned}$$

the desired result now follows from (3.13).

(ii) Let $v \in H^2$ and let $\{v_n\}_{n\geq 1}$ be a sequence such that $\sup_{n\geq 1} |v_n|_2 < \infty$ and $v_n \to v$ in L^2 as $n \to \infty$. Since $|f(v_n) - f(v)| \leq \mathscr{C}(f) |v_n - v|$, for $n \geq 1$, we infer that $f(v_n) \to f(v)$ in L^2. In view of the estimate

$$|Bv_n - Bv|^2 \leq |f(v) - f(v_n)| \left|\nabla^2 f(v_n) - \nabla^2 f(v)\right|, \tag{3.14}$$

to derive the desired convergence result it remains to show that $\sup_{n\geq 1} |\nabla^2 f(v_n) - \nabla^2 f(v)| < \infty$. To this end, we observe that

$$\begin{aligned}\left|\nabla^2 f(v_n)\right| &= \left|f''(v_n)(\nabla v_n)^2 + f'(v_n)\nabla^2 v_n\right| \\ &\leq \left|f''(v_n)(\nabla v_n)^2\right| + \left|f'(v_n)\nabla^2 v_n\right| \\ &\leq \sup_{|\xi|\leq M_2} |f''(\xi)| \, |\nabla v_n|_{L^4}^2 + \sup_{|\xi|\leq M_2} |f'(\xi)| \left|\nabla^2 v_n\right|,\end{aligned}$$

where M_2 denotes $\sup_{n\geq 1} |v_n|_2$. It now follows from Lemma 3.1 that

$$\left|\nabla^2 f(v_n)\right| \leq \mathscr{C}_1(f)\left(\left|\nabla^2 v_n\right|^{1/2} |\nabla v_n|^{3/2} + \left|\nabla^2 v_n\right|\right),$$

and since $\{v_n\}_{n\geq 1}$ is convergent in H^2, this yields

$$\sup_{n\geq 1} \left|\nabla^2 f(v_n) - \nabla^2 f(v)\right| < \infty. \tag{3.15}$$

Combining (3.14) and (3.15) we now obtain the desired result.

(iii) Let $\alpha \geq 0$, and define $\omega_0(\alpha)$ by

$$\omega_0(\alpha) = (1/2)\sup\left\{|f''(v)\nabla v|_{L^\infty(\mathbb{R})}; v \in H^2, |v|_2 \leq \alpha\right\}. \tag{3.16}$$

For $\theta \in [0,1]$, let us denote $z_\theta(\cdot) = \theta v(\cdot) + (1-\theta) w(\cdot)$. Suppose that $v, w \in H^2$, $|v|_2 \leq \alpha$, $|w|_2 \leq \alpha$, and for $\theta \in [0,1]$, let us denote $z_\theta(\cdot) = \theta v(\cdot) + (1-\theta) w(\cdot)$. Therefore, by (3.4) it follows that

$$\begin{aligned}(Bv - Bw, v - w) &= (f(v) - f(w), \nabla(v - w)) \\ &= \left(\int_0^1 d/d\theta\,(f(z_\theta))\, d\theta, \nabla(v - w)\right) \\ &= \left(\int_0^1 f'(z_\theta)\, d\theta, \nabla\left(1/2\,(v - w)^2\right)\right) \\ &= (-1/2)\left(\int_0^1 f''(z_\theta)(\nabla z_\theta)\, d\theta, (v - w)^2\right).\end{aligned}$$

From this relation we deduce that $|(Bv - Bw, v - w)| \leq \omega_0(\alpha) |v - w|^2$, which proves the third statement.

(iv) Let $\alpha \geq 0$, and define $\omega_1(\alpha)$ by

(3.17)
$$\omega_1(\alpha) = \max\left\{\omega_0(\alpha), (\sup|f'''(v)|_{L^\infty})(\sup|\nabla v|_{L^\infty})^2 + (\sup|f''(v)|_{L^\infty})\left(\sup\left|\nabla^2 v\right|_{L^\infty}\right)\right.$$

$$+ (3/2) (\sup |f'' (v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty})\},$$

each of the supremums being considered over the set $\{v \in H^3, |v|_3 \le \alpha\}$. As a first step, we see that

One may see that

$$\begin{aligned}
(\nabla (Bv - Bw), \nabla (v-w)) &= \left(f'(v) \nabla v - f'(w) \nabla w, \nabla^2 (v-w)\right) \\
&= \left(\int_0^1 (d/d\theta) (f'(z_\theta) \nabla z_\theta) \, d\theta, \nabla^2 (v-w)\right) \\
&= \left(\int_0^1 f''(z_\theta) (v-w) \nabla z_\theta d\theta, \nabla^2 (v-w)\right) \\
&\quad + \left(\int_0^1 f'(z_\theta) (\nabla v - \nabla w) \, d\theta, \nabla^2 (v-w)\right).
\end{aligned}$$

We denote by T_1 and T_2, respectively, the first and the second term of the right-hand side of the above inequality. By (3.4), it follows that

$$\begin{aligned}
|T_1| &= \left|\left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta\right) (v-w), \nabla^2 (v-w)\right)\right| \\
&= \left|\left(\nabla \left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta\right) (v-w)\right), \nabla (v-w)\right)\right| \\
&\le \left|\left(\left(\int_0^1 f'''(z_\theta) (\nabla z_\theta)^2 \, d\theta\right) (v-w), \nabla (v-w)\right)\right| \\
&\quad + \left|\left(\left(\int_0^1 f''(z_\theta) \nabla^2 z_\theta d\theta\right) (v-w), \nabla (v-w)\right)\right| \\
&\quad + \left|\left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta\right) \nabla(v-w), \nabla(v-w)\right)\right| \\
&\le \left[(\sup |f'''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty})^2 + (\sup |f''(v)|_{L^\infty}) \left(\sup \left|\nabla^2 v\right|_{L^\infty}\right)\right] \\
&\quad \cdot (|v-w|, |\nabla (v-w)|) + (\sup |f''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty}) |\nabla (v-w)|^2
\end{aligned}$$

and

$$\begin{aligned}
|T_2| &= \left|\left(\left(\int_0^1 f'(z_\theta) \, d\theta\right) \nabla (v-w), \nabla^2 (v-w)\right)\right| \\
&= \left|\left(\int_0^1 f'(z_\theta) \, d\theta, (1/2) \nabla \left((\nabla (v-w))^2\right)\right)\right| \\
&= (1/2) \left|\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta, (\nabla (v-w))^2\right)\right| \\
&\le (1/2) (\sup |f''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty}) |\nabla (v-w)|^2.
\end{aligned}$$

Since $(|v-w|, |\nabla (v-w)|) \le |w-v| |\nabla (v-w)| \le |v-w|_1^2$, we obtain the desired result. □

By this proposition, it is seen that the nonlinear differential operator B is continuous and quasidissipative on each subset $\{v \in H^2; |v|_2 \le \alpha\}$, $\alpha \ge 0$. In view of this fact, we can employ the classical notion of mild solution.

Definition 3.1. Let $v \in H^2$. A H^2-valued function $u(\cdot)$ on $\mathbb{R}$ is said to be a mild solution of (3.6) (or of (3.1)) with $u(0) = v$ if $u(\cdot) \in C(\mathbb{R}; H^2)$ and satisfies

$$u(t) = G_A(t)\, v + \int_0^t G_A(t-s)\, Bu(s)\, ds \qquad \text{for } t \in \mathbb{R}. \tag{3.18}$$

Let $v \in H^2$ and $u(\cdot) \in C(\mathbb{R}; H^2)$. As is easily seen, $u(\cdot)$ is a mild solution of (3.6) with $u(0) = v$ if and only if satisfies

$$(u(t), w) = (v, w) + \int_0^t \left\{ \left(\nabla^2 u(s), \nabla w\right) - \left(\nabla f(u(s)), w\right) \right\} ds \tag{3.19}$$

for $t \in \mathbb{R}$ and $w \in H^1$. By Theorem 2.1 and Proposition 3.1, we have the following result which guarantees the uniqueness of mild solutions.

Proposition 3.2. *Let $u(\cdot), \hat{u}(\cdot)$ be mild solutions of* (3.6) *with initial data $u(0) = v$ and $\hat{u}(0) = v$. Then for each $\tau > 0$ we have*

$$|u(t) - \hat{u}(t)| \le e^{\omega_0(\alpha)|t|}\, |v - \hat{v}| \qquad \text{for } t \in [-\tau, \tau]. \tag{3.20}$$

where α is chosen such that $|u(t)|_2 \le \alpha$ and $|\hat{u}(t)|_2 \le \alpha$ for each $t \in [-\tau, \tau]$ and $\omega_0(\alpha)$ is the constant provided for α by Lemma 3.1.

We next discuss the existence of nonlinear groups of locally Lipschitz operators on H^2 which provide mild solutions to (3.6). To this purpose, we introduce five l.s.c. functionals $\varphi_k : H^k \to \mathbb{R}$, $k = 0, 1, 2, 3, 4$ which will also be employed to establish the regularity properties of the solution. We define

$$\begin{aligned}
&\varphi_0(v) = |v|, && v \in L^2;\\
&\varphi_1(v) = (1/2)\,|\nabla v|^2 - \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx, && v \in H^1;\\
&\varphi_2(v) = (1/2)\,\left|\nabla^2 v\right|^2 + (5/6)\left(f(v), \nabla^2 v\right), && v \in H^2;\\
&\varphi_3(v) = \left|\nabla^3 v + \nabla f(v)\right|, && v \in H^3;\\
&\varphi_4(v) = \left|\nabla^3 f(v) + \nabla f(v)\right|_1, && v \in H^4.
\end{aligned} \tag{3.21}$$

Since we have established the continuity and quasidissipativity of B on level sets of H^2 with respects to the usual Sobolev space norm, it then becomes necessary to show that φ_0, φ_1, φ_2 are equivalent to $|\cdot|_0$, $|\cdot|_1$, $|\cdot|_2$, in the sense that the level sets induced are equivalent. However, the functionals φ_k appear to be more intimately related to the physical structure of the model since, as will be seen in Theorem 4.2, φ_1 and φ_2 are actually invariants for the generalized Kortweg-deVries equation (3.1). It is also important to see that $\varphi_3(v) = |(A+B)v|_3$ for $v \in H^3$ and $\varphi_4(v) = |(A+B)v|_4$ for $v \in H^4$.

Lemma 3.2. *The following affirmations hold:*

(i) *For each α_0, $\alpha_1 \geq 0$ there is $\beta_1 = \beta_1(\alpha_0, \alpha_1) \geq 0$ such that if $v \in H^1$, $\varphi_0(v) \leq \alpha_0$ and $\varphi_1(v) \leq \alpha_1$, then $|\nabla v| \leq \beta_1$.*

(ii) *For each α_0, α_1, $\alpha_2 \geq 0$ there is $\beta_2 = \beta_2(\alpha_0, \alpha_1, \alpha_2) \geq 0$ such that if $v \in H^2$, $\varphi_0(v) \leq \alpha_0$ and $\varphi_1(v) \leq \alpha_1$, then $|\nabla v| \leq \beta_2$.*

Proof. (i) In view of the growth condition (3.3), one finds $\overline{C}_1$ and $\overline{C}_2 \in \mathbb{R}$ such that $f'(s) \leq \overline{C}_1 + \overline{C}_2 |s|^p$ for each $s \in \mathbb{R}$, and by integration we find that $\int_0^s f(\xi)\, d\xi \leq C_1 |s|^2 + C_2 |s|^{p+2}$ for each $s \in \mathbb{R}$ and some C_1, C_2. The use of Lemma 3.1 leads us to the estimate

$$\begin{aligned}(3.22)\qquad \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx &\leq C_1 |v|^2 + C_2 |v|_{L^{p+2}}^{p+2} \\ &\leq C_1 |v|^2 + C_2 2^{p/2} |\nabla v|^{p/2} |v|^{(p+4)/2}\end{aligned}$$

for each $v \in H^1$. Invoking Young's inequality, one obtains

$$C_2 2^{p/2} |\nabla v|^{p/2} |v|^{(p+4)/2} \leq (1/4) |\nabla v|^2 + (1/4)(4-p) \left(C_2 (4p)^{p/4} |v|^{(p+4)/2}\right)^{4/4-p}.$$

Hence we get the estimate

$$(3.23)\qquad |\nabla v|^2 \leq 4(\alpha_1 + C_1 \alpha_0)^2 + (4-p)\left(C_2 (4p)^{p/4} \alpha_0^{(p+4)/2}\right)^{4/4-p},$$

from which the desired result follows.

(ii) By (3.4) and (3.21) it is seen that

$$\left|\nabla^2 v\right|^2 = 2\varphi_2(v) - (5/3)\left(f'(v), (\nabla v)^2\right),$$

and therefore

$$(3.24)\qquad \left|\nabla^2 v\right|^2 \leq 2\alpha_2 + (5/3)\mathscr{C}(f, \alpha_0, \beta_1)\beta_1^2$$

Here, by $\mathscr{C}(f, \alpha_0, \beta_1)$ is denoted a positive constant which depends only on f, α_0, β_1. □

In view of Theorem 2.1, it is necessary to show that the range condition (III.3) is verified with respect to $\varphi = \varphi_2$. Therefore, to proceed any further, we need the following technical lemma, which gives an estimate for intermediary terms arising from the computation of the values of φ_2.

Lemma 3.3. *For each $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$ there are $a = a(\alpha_0, \alpha_1) \geq 0$ and $b = b(\alpha_0, \alpha_1) \geq 0$ such that*

$$(3.25)\qquad \left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \leq a\varphi_2(v) + b$$

for $v, w \in H^2$ with $\varphi_0(v) \leq \alpha_0$, $\varphi_1(v) \leq \alpha_1$, $\varphi_0(w) \leq \alpha_0$, $\varphi_1(w) \leq \alpha_1$.

Proof. Let α_0, α_1 be positive numbers and let v, $w \in H^2$ so that $\varphi_0(v) \leq \alpha_0$, $\varphi_0(w) \leq \alpha_0$, $\varphi_1(v) \leq \alpha_1$, $\varphi_1(w) \leq \alpha_1$. We may apply Lemma 3.2 to find $\beta_1 = \beta_1(\alpha_0, \alpha_1) \geq 0$, and inequality (3.13) now implies that there is $\gamma_1 = \gamma_1(\alpha_0, \alpha_1)$ so that $|v|_{L^\infty} \leq \gamma_1$, $|w|_{L^\infty} \leq \gamma_1$. Denote $\sup\limits_{|\xi|\leq\gamma_1} |f'''(\xi)|$ by $\mathscr{C}(f, \alpha_0, \alpha_1)$. It is then easily seen that

$$\begin{aligned}
&\left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \\
&\quad \leq (1/6)\left|f'''(v)(\nabla v)^3\right|\left|\nabla^2 v\right| + (5/6)\left|f'(v)\nabla^2 v\right|\left|f'(w)\nabla w\right|
\end{aligned}$$

We denote by T_1 and T_2 respectively the first and the second term of the right hand side of the above inequality, and we observe that

$$\begin{aligned}
T_1 = (1/6)\left|f'''(v)(\nabla v)^3\right|\left|\nabla^2 v\right| &\leq (1/6)\,\mathscr{C}(f, \alpha_0, \alpha_1)\,|\nabla v|_{L^6}^3\left|\nabla^2 v\right| \\
&\leq (1/3)\,\mathscr{C}(f, \alpha_0, \alpha_1)\left|\nabla^2 v\right|^2|\nabla v|^2 \\
&\leq (1/3)\,\mathscr{C}(f, \alpha_0, \alpha_1)\,\beta_1^2\left|\nabla^2 v\right|^2
\end{aligned}$$

and

$$\begin{aligned}
T_2 = (5/6)\left|f'(v)\nabla^2 v\right|\left|f'(w)\nabla w\right| &\leq (5/6)\,\overline{\mathscr{C}}^2(f, \alpha_0, \alpha_1)\left|\nabla^2 v\right||\nabla w| \\
&\leq (5/12)\,\overline{\mathscr{C}}^2(f, \alpha_0, \alpha_1)\,\beta_1\left(\left|\nabla^2 v\right|^2 + 1\right),
\end{aligned}$$

where $\overline{\mathscr{C}}(f, \alpha_0, \alpha_1) = \sup\limits_{|\xi|\leq\gamma_1}|f'(\xi)|$. Using (3.23), one then obtains

$$\begin{aligned}
&\left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \\
&\quad \leq (2/3)\left(\mathscr{C}\beta_1^2 + (5/4)\,\overline{\mathscr{C}}^2\beta_1\right)\left(\varphi_2(v) + (5/6)\,\mathscr{C}\beta_1^2\right) + (5/12)\,\overline{\mathscr{C}}^2\beta_1 \\
&= a(\alpha_0, \alpha_1)\,\varphi_2(v) + b(\alpha_0, \alpha_1).
\end{aligned}$$

□

4 Resolvents of the semilinear operator $A + B$

The next result shows that a generalized form of the range condition is fulfilled.

Theorem 4.1. *Let $v \in H^3$, $\varepsilon > 0$ and suppose that $\alpha_0, \alpha_1, \alpha_2 > 0$ are chosen such that $\varphi_0(v) + \varepsilon < \alpha_0$, $\varphi_1(v) + \varepsilon < \alpha_1$, and $e^{2a}(|\varphi_2(v)| + (b + \varepsilon)) < \alpha_2$, where $a = a(\alpha_0, \alpha_1)$ and $b = b(\alpha_0, \alpha_1)$ are numbers as in Lemma 3.3. Then there is a number $\lambda_0 = \lambda_0(\alpha_1, \alpha_2, \alpha_3, \varepsilon)$, $0 < \lambda_0 < \min\{1, 1/2a, 1/\omega_0\}$ such that for each $\lambda \in (-\lambda_0, \lambda_0)$ there is an unique element $v_\lambda \in H^3$ which satisfies*

(4.1) $$v_\lambda - \lambda(A + B)v_\lambda = v,$$

and

(4.2) $$\varphi_0(v_\lambda) \leq \varphi_0(v) + |\lambda|\,\varepsilon,$$

$$\varphi_1(v_\lambda) \leq \varphi_1(v) + |\lambda|\varepsilon,$$
$$\varphi_2(v_\lambda) \leq (1 - |\lambda| a)^{-1} (\varphi_2(v) + |\lambda|(b + \varepsilon)),$$
$$\varphi_3(v_\lambda) \leq (1 - |\lambda|\omega_0)^{-1} \varphi_3(v).$$

Furthermore, if $v \in H^4$, then $v_\lambda \in H^4$ and satisfies

$$\varphi_4(v_\lambda) \leq (1 - |\lambda|\omega_1)^{-1} \varphi_4(v).$$

Proof. Let $v \in H^3$, $\varepsilon > 0$ and suppose that α_0, α_1, α_2 are numbers as indicated above. By virtue of Lemma 3.2 we can choose β_0, β_1, $\beta_2 > 0$ such that

$$\left\{w \in H^2, \varphi_k(w) \leq \alpha_k, k = 0,1,2\right\} \subset \left\{w \in H^2, \left|\nabla^k w\right| \leq \beta_k, k = 0,1,2\right\}. \tag{4.3}$$

We denote

$$L_0 = \sup\left\{|Bw|\,; w \in H^1, |w| \leq \beta_0, |\nabla w| \leq \beta_1\right\}; \tag{4.4}$$
$$L_1 = \sup\left\{|\nabla Bw|\,; w \in H^2, \left|\nabla^k w\right| \leq \beta_k, k = 0,1,2\right\}. \tag{4.5}$$

Further, choose $\beta_3 > 0$ such that $|\nabla^3 v| + 2L_0 \leq \beta_3$ and set

$$L_2 = \sup\left\{\left|\nabla^2 Bw\right|; w \in H^3, \left|\nabla^k w\right| \leq \beta_k, k = 0,1,2,3\right\}. \tag{4.6}$$

By (4.3) and Proposition 3.1, there is a number $\delta = \delta(|v|_3, \varepsilon) > 0$ such that if $w \in H^3$, $|w - v| < \delta$ and $\left|\nabla^k w\right| \leq \max\left\{\beta_k, \left|\nabla^k v\right| + L_k\right\}$, $k = 0,1,2$, then

$$\begin{aligned} &|Bw - Bv| \leq \varepsilon/2, \\ &|f(w) - f(v)|\,\beta_3 \leq \varepsilon/2, \\ &|\nabla Bw - \nabla Bv|\,\beta_3 \leq \varepsilon/5, \\ &|f'(w) - f'(v)|\,\beta_3 \left(\left|\nabla^2 v\right| + L_2\right) \leq \varepsilon/5. \end{aligned} \tag{4.7}$$

We now try to obtain (4.1) and (4.2) using a fixed point argument. Set

$$\lambda_0 = \min\{1,\ \delta/\beta_3,\ \varepsilon/(2\beta_3),\ 1/(2a),\ 1/\omega_0\} \tag{4.8}$$

and let $\lambda \in (-\lambda_0, \lambda_0)$, $\lambda \neq 0$ be fixed. Let K be a subset of H^3 defined by

$$K = \left\{w \in H^3;\ |w - v| \leq |\lambda|\beta_3,\ \left|\nabla^k w\right| \leq \beta_k,\ k = 0,1,2,3\right\}. \tag{4.9}$$

It is easily seen that K is convex, bounded and closed in L^2. Since L^2 is reflexive, K is weakly compact in L^2. We define a mapping $\Gamma : K \to H^3$ by

$$\Gamma w = (I - \lambda A)^{-1}(v + \lambda Bw) \qquad \text{for } w \in K. \tag{4.10}$$

Since the resolvent function $(I - \lambda A)^{-1}$ is strongly strongly continuous in L^2, it is also weakly weakly continuous, and Proposition 3.1 implies that Γ is weakly weakly continuous on L^2. We now show that Γ maps K into itself. To prove this, let $w \in K$ and note $z = \Gamma w$. It is seen that

$$|z - v|^2 = (z - v, \lambda Az) + (z - v, \lambda Bw)$$

$$
\begin{aligned}
&= (-v, \lambda Az) + (z - v, \lambda Bw) \\
&= \lambda (Av, z) + \lambda (z - v, Bw) \\
&\le |\lambda| \, |z - v| \left(\left|\nabla^3 v\right| + |Bw| \right),
\end{aligned}
$$

and thus

$$
|z - v| \le |\lambda| \left(\left|\nabla^3 v\right| + L_0 \right) \le |\lambda| \, \beta_3. \tag{4.11}
$$

Also, since (4.10) implies

$$
\lambda Az = z - v - \lambda Bw, \tag{4.12}
$$

it follows from (4.4) that

$$
\begin{aligned}
\left|\nabla^3 z\right| &\le (1/\left|\lambda\right|) \, |z - v| + |Bw| \\
&\le \left|\nabla^3 v\right| + 2L_0 \le \beta_3.
\end{aligned} \tag{4.13}
$$

We next prove that $\left|\nabla^k z\right| \le \left|\nabla^k v\right| + L_k$, $k = 0, 1, 2$, which will enable us to use the estimates in (4.7). Using again (3.4), we obtain

$$
\begin{aligned}
|z|^2 &= (z, v + \lambda Az + \lambda Bw) \\
&= (z, v) + \lambda (z, Bw) \\
&\le |z| \left(|v| + |\lambda| \, |Bw| \right).
\end{aligned}
$$

Together with (4.4) and (4.8), this implies $|z| \le |v| + |\lambda| \, L_0$. Repeating the same argument as above, it is seen that

$$
\begin{aligned}
|\nabla z|^2 &= \left(-\nabla^2 z, v + \lambda Az + \lambda Bw\right) \\
&= -\left(\nabla^2 z, v + \lambda Bw\right) = (\nabla z, \nabla v + \lambda \nabla Bw) \\
&\le |\nabla z| \left(|\nabla v| + |\lambda| \, |\nabla Bw| \right),
\end{aligned}
$$

and thus one obtains from (4.5) and (4.8) that $|\nabla z| \le |\nabla v| + |\lambda| \, L_1$.

Let now $z_n \to z$ in H^2, $z_n \in C_c^\infty(\mathbb{R})$. By (3.4) and (4.12), we have

$$
\begin{aligned}
\left(\nabla^2 z, \nabla^2 z\right) &= \left(\nabla^2 z, \nabla^2 z - \nabla^2 z_n\right) + \left(\nabla^2 z, \nabla^2 z_n\right) \\
&\le \left|\nabla^2 z\right| \left|\nabla^2 z - \nabla^2 z_n\right| + \left(v + \lambda Az + \lambda Bw, \nabla^4 z_n\right).
\end{aligned} \tag{4.14}
$$

Since (4.12) implies that $Az \in H^2$, using (3.4) we obtain that

$$
\left(v + \lambda Az + \lambda Bw, \nabla^4 z_n\right) \to \left(\nabla^2 v, \nabla^2 z\right) + \lambda \left(\nabla^2 Az, \nabla^2 z\right) + \lambda \left(\nabla^2 Bw, \nabla^2 z\right)
$$

as $n \to \infty$. Combining this with (4.14), one may obtain that $|\nabla^2 z| \le |\nabla^2 v| + L_2$. We next prove that $\varphi_k(z) \le \alpha_k$ for $k = 0, 1, 2$, and so (4.3) will imply that $\left|\nabla^k z\right| \le \beta_k$, $k = 0, 1, 2$. From (3.4) and (4.12) it is clear that

$$
\begin{aligned}
|z|^2 &= (z, v + \lambda Az + \lambda Bw) \\
&= (z, v) + \lambda (z, Bw - Bz)
\end{aligned}
$$

$$\leq |z| \left(|v| + |\lambda| \, |Bw - Bz|\right),$$

from which one may get

$$|z| \leq |v| + |\lambda| \left(|Bw - Bv| + |Bz - Bv|\right).$$

Therefore, by (4.7), z satisfies the inequality

$$|z| \leq |v| + |\lambda| \, \varepsilon < \alpha_0. \tag{4.15}$$

Note that in the above estimates we have also used the fact that $(Bz, z) = 0$ for each $z \in H^1$.

We now try to estimate $\varphi_1(z)$. By (3.21) we have

$$\begin{aligned} \varphi_1(z) - \varphi_1(v) &= (1/2) \left|\nabla^2 z\right|^2 - (1/2) \left|\nabla^2 v\right|^2 - \int_{-\infty}^{\infty} \int_{v(x)}^{z(x)} f(\xi) \, d\xi dx \\ &= (1/2) \left|\nabla^2 z\right|^2 - (1/2) \left|\nabla^2 v\right|^2 \\ &\quad - \int_{-\infty}^{\infty} f\left(\theta(x) z(x) + (1 - \theta(x)) v(x)\right) \left(z(x) - v(x)\right) dx. \end{aligned} \tag{4.16}$$

Put $w_1(\cdot) = \theta(\cdot) z(\cdot) + (1 - \theta(\cdot)) v(\cdot)$. From (4.8) we infer that

$$|w_1 - v| = |\theta(z - v)| \leq |\lambda| \, \beta_3 < \delta. \tag{4.17}$$

Since $|w - v| < \delta$, relation (4.7) leads us to the estimate

$$|(f(w_1) - f(w), z - v)| \leq \varepsilon / \beta_3 \, |z - v| \leq |\lambda| \, \varepsilon. \tag{4.18}$$

Therefore, by (4.16) and (4.18) it follows that

$$\varphi_1(z) - \varphi_1(v) \leq (1/2) |\nabla z|^2 - (1/2) |\nabla v|^2 - (f(w), z - v) + |\lambda| \, \varepsilon.$$

Since $(f(w), z - w) = \lambda (f(w), Az + Bw)$ and $(f(w), Bw) = 0$, it is seen that

$$\varphi_1(z) - \varphi_1(v) \leq (1/2) |\nabla z|^2 - (1/2) |\nabla v|^2 - \lambda (\nabla Bw, \nabla z) + |\lambda| \, \varepsilon \tag{4.19}$$

and this implies that $\varphi_1(z) \leq \varphi_1(v) + |\lambda| \, \varepsilon < \alpha_1$.

To estimate $\varphi_2(z)$ we first observe that, by virtue of (3.4),

(4.20)

$$\begin{aligned} \left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) &= \left(f(z) - f(v), \nabla^2 z\right) + \left(f(v), \nabla^2 z - \nabla^2 v\right) \\ &= \left(f'(w_1)(z - v), \nabla^2 z\right) - (\nabla Bv, z - v) \\ &= \left((f'(w_1) - f'(z))(z - v), \nabla^2 z\right) + \left(f'(z)(z - v), \nabla^2 z\right) \\ &\quad - (\nabla Bv - \nabla Bw, z - v) - (\nabla Bw, z - v) \\ &= \left((f'(w_1) - f'(z)), \nabla^2 z (z - v)\right) + \left(f'(z) \nabla^2 z, z - v\right) \\ &\quad - (\nabla Bv - \nabla Bw, z - v) - (\nabla Bw, z - v), \end{aligned}$$

where w_1 is defined as above.

Since

$$\left|\left(\left(f'\left(w_1\right)-f'\left(z\right)\right),\left(\nabla^2 z\right)\left(z-v\right)\right)\right| \leq \left(\left|f'\left(w_1\right)-f'\left(v\right)\right|+\left|f'\left(v\right)-f'\left(z\right)\right|\right)\left|\nabla^2 z\right|\left|z-v\right|,$$

the estimates in (4.17) imply

$$\left|f'\left(w_1\right)-f'\left(v\right)\right| \leq \varepsilon/\left(5\beta_3\left(\left|\nabla^2 v\right|+L_2\right)\right),$$

and also

$$\left|f'\left(z\right)-f'\left(v\right)\right| \leq \varepsilon/\left(5\beta_3\left(\left|\nabla^2 v\right|+L_2\right)\right).$$

Hence $\left|\left(\left(f'\left(w_1\right)-f'\left(z\right)\right),\left(\nabla^2 z\right)\left(z-v\right)\right)\right| \leq 2\varepsilon\left|\lambda\right|\left|\nabla^2 z\right|/\left(\left(\left|\nabla^2 v\right|+L_2\right)\right)$, and, since $\left|\nabla^2 z\right| \leq \left|\nabla^2 v\right|+L_2$,

$$\left|\left(\left(f'\left(w_1\right)-f'\left(z\right)\right),\left(\nabla^2 z\right)\left(z-v\right)\right)\right| \leq 2\varepsilon\left|\lambda\right|/5. \tag{4.21}$$

From (4.7) and (4.11), it is easy to see that

$$\left|\left(\nabla Bv-\nabla Bw,z-v\right)\right| \leq \left|\nabla Bv-\nabla Bw\right|\left|z-v\right| \leq \varepsilon\left|\lambda\right|/5. \tag{4.22}$$

We now estimate the term $\left(f'\left(z\right)\nabla^2 z,z-v\right)-\left(\nabla Bw,z-v\right)$ from the right-hand side of (4.20). From (3.4) and (4.12) we have

$$\begin{aligned}\left(f'\left(z\right)\nabla^2 z,z-v\right)-\left(\nabla Bw,z-v\right) &= \left(f'\left(z\right)\nabla^2 z,\lambda\left(Az+Bw\right)\right)-\lambda\left(\nabla Bw,Az+Bw\right)\\ &= \lambda\left(f'\left(z\right)\nabla^2 z,\lambda\left(Az+Bw\right)\right)-\lambda\left(\nabla Bw,Az\right)\\ &= -\lambda\left(f'\left(z\right)\nabla^2 z,\nabla^3 z+f'\left(w\right)\nabla w\right)-\lambda\left(\nabla Bw,Az\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\nabla^2 z,\nabla^2 z\right) &= \left(\nabla^2 z,\nabla^2 v+\lambda\nabla^2 Az+\lambda\nabla^2 Bw\right)\\ &= \left(\nabla^2 z,\nabla^2 v\right)+\lambda\left(\nabla^2 z,\nabla^2 Bw\right)\\ &= \left(\nabla^2 z,\nabla^2 v\right)+\lambda\left(Az,\nabla Bw\right),\end{aligned}$$

from which we infer that

$$\begin{aligned}\left(f'\left(z\right)\nabla^2 z,z-v\right)+\left(\nabla Bw,z-v\right) = &-\lambda\left(f'\left(z\right)\nabla^2 z,\nabla^3 z+f'\left(w\right)\nabla w\right)\\ &+\left(\nabla^2 z,\nabla^2 v\right)-\left(\nabla^2 z,\nabla^2 z\right).\end{aligned}$$

In order to proceed any further, we need the important "integration by parts" formula

$$5\left(f'\left(z\right)\nabla^2 z,\nabla^3 z\right)=\left(\nabla Bz,Az\right)+\left(f'''\left(z\right)\left(\nabla z\right)^3,\nabla^2 z\right). \tag{4.23}$$

To prove this, we first see that

$$\begin{aligned}\left(f'\left(z\right)\nabla^2 z,\nabla^3 z\right) &= -\left(\nabla\left(f'\left(z\right)\nabla^2 z\right),\nabla^2 z\right)\\ &= -\left(f''\left(z\right)\nabla z\nabla^2 z,\nabla^2 z\right)-\left(f'\left(z\right)\nabla^3 z,\nabla^2 z\right)\\ &= -\left(f''\left(z\right)\nabla z\nabla^2 z,\nabla^2 z\right)-\left(f'\left(z\right)\nabla^2 z,\nabla^3 z\right).\end{aligned} \tag{4.24}$$

This implies

$$
\begin{aligned}
2\left(f'(z)\nabla^2 z, \nabla^3 z\right) &= -\left(f''(z)\nabla z \nabla^2 z, \nabla^2 z\right) \\
&= -(1/2)\left(\nabla\left(f''(z)(\nabla z)^2\right) - f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= (1/2)\left(f''(z)(\nabla z)^2, \nabla^3 z\right) + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= (1/2)\left(\nabla(f'(z)\nabla z) - f'(z)\nabla^2 z, \nabla^3 z\right) \\
&\quad + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= -(1/2)\left(\nabla Bz, \nabla^3 z\right) - (1/2)\left(f'(z)\nabla^2 z, \nabla^3 z\right) \\
&\quad + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right),
\end{aligned}
$$

from which the desired equality follows. Note that (4.23) justifies also the choice of the functional φ_2 in (3.21).

Thus it follows from (4.20) through (4.23) that

$$
\begin{aligned}
&\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) \\
&\qquad \le 3|\lambda|\varepsilon/5 - \lambda\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + \left(\nabla^2 v, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 z\right) \\
&\qquad\quad - \lambda\left((1/5)(\nabla Bz, Az) + \left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right) \\
&\qquad \le 3|\lambda|\varepsilon/5 - (\lambda/5)(\nabla Bz, Az) + \left(\nabla^2 v, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 z\right) \\
&\qquad\quad - (\lambda/5)\left[\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + 5\left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right].
\end{aligned}
$$

Since $\lambda(\nabla Bw, Az) = (\nabla^2 z, \nabla^2 z) - (\nabla^2 v, \nabla^2 z)$, we see that

(4.25)
$$
\begin{aligned}
&\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) \\
&\qquad \le 3|\lambda|\varepsilon/5 - (\lambda/5)\left[\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + 5\left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right] \\
&\qquad\quad - (\lambda/5)(\nabla Bz - \nabla Bw, Az) - (6/5)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right)\right).
\end{aligned}
$$

Using (4.7) one gets

(4.26)
$$
\begin{aligned}
|(\nabla Bz - \nabla Bw, Az)| &\le (|\nabla Bz - \nabla Bv| + |\nabla Bv - \nabla Bw|)|Az| \\
&\le 2\varepsilon/5.
\end{aligned}
$$

We therefore infer from (4.25), (4.26) and Lemma 3.3 that

$$
\begin{aligned}
\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) + (6/5)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right)\right) \\
\le \varepsilon|\lambda| + |\lambda|(a\varphi_2(z) + b)/5.
\end{aligned}
$$

Since

$$
\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right) \ge (1/2)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 v\right)\right),
$$

we see that

$$
\varphi_2(z) - \varphi_2(v) \le (5/6)\varepsilon|\lambda| + |\lambda|(a\varphi_2(z) + b).
$$

From this inequality we deduce

$$\begin{aligned}\varphi_2(z) &\le (1-|\lambda|a)^{-1}(\varphi_2(v)+(5/6)\varepsilon|\lambda|+|\lambda|b)\\ &< (1-|\lambda|a)^{-1}(\varphi_2(v)+|\lambda|(b+\varepsilon)).\end{aligned}$$

Noting that (4.8) implies $(1-|\lambda|a)^{-1}\le 1+2|\lambda|a<e^{2a}$, one gets

$$\varphi_2(z)\le e^{2a}(|\varphi_2(v)|+|\lambda|(b+\varepsilon))<\alpha_2, \tag{4.27}$$

and so we obtain that $z\in K$. Applying Tihonov's Fixed Point Theorem, we get the existence of a fixed point v_λ satisfying (4.1).

As seen in Proposition 3.1, $A+B-\omega_0(\alpha_2)I$ is dissipative on $\{v\in H^2; |v|_2\le\alpha_2\}$, and this implies the uniqueness of v_λ. Also, since $A+B$, $-A-B$ are quasidissipative, we get from (4.1) that

$$|\lambda|\varphi_3(v_\lambda)\le(1-|\lambda|\omega_0)^{-1}|\lambda(A+B)v|=|\lambda|(1-|\lambda|\omega_0)^{-1}\varphi_3(v),$$

and so $\varphi_3(v_\lambda)\le(1-|\lambda|\omega_0)^{-1}\varphi_3(v)$. It is easily seen that if $v\in H^4$, then $v_\lambda\in H^4$ and, by the same reasoning, $\varphi_4(v_\lambda)\le(1-|\lambda|\omega_1)^{-1}\varphi_4(v)$, which finishes the proof. □

We can now employ the generation result stated in Section 2 to obtain the existence of a nonlinear group of locally Lipschitz operators on H^2 which provides mild solutions to the initial value problem for the generalized Kortweg-deVries equation (3.1).

Theorem 4.2. *There exists a nonlinear group $G=\{G(t);t\in\mathbb{R}\}$ on H^2 such that the following properties are satisfied:*

(i) *For each $v\in H^2$, $G(\cdot)v\in C(\mathbb{R};H^2)$ and $G(\cdot)v$ satisfies*

$$G(t)v=G_A(t)v+\int_0^t G_A(t-s)BG(s)v\,ds \qquad \text{for } t\in\mathbb{R}. \tag{4.28}$$

(ii) *$\varphi_0(G(t)v)=\varphi_0(v)$ and $\varphi_1(G(t)v)=\varphi_1(v)$ for $t\in\mathbb{R}$ and $v\in H^2$.*

(iii) *For each $\alpha_0,\alpha_1\ge 0$ there exist positive numbers $a=a(\alpha_0,\alpha_1)$ and $b=b(\alpha_0,\alpha_1)$ such that*

$$\varphi_2(G(t)v)\le e^{a|t|}(\varphi_2(v)+b|t|) \tag{4.29}$$

for $v\in H^2$ with $\varphi_0(v)\le\alpha_0$, $\varphi_1(v)\le\alpha_1$ and $t\in\mathbb{R}$.

(iv) *Each of $G(t)$ maps H^3 into itself and H^4 into itself.*

(v) *For each $\alpha_k\ge 0$, $k=0,1,2$ and each $\tau>0$, there exists a positive number $\omega_0=\omega_0(\alpha_0,\alpha_1,\alpha_2,\tau)$ such that*

$$\varphi_3(G(t)v)\le e^{\omega_0|t|}\varphi_3(v) \tag{4.30}$$

for $t\in[-\tau,\tau]$ and for $v\in H^3$ with $\varphi_k(v)\le\alpha_k$, $k=0,1,2$. Consequently, if $v\in H^3$, then $G(\cdot)v\in C(\mathbb{R};H^3)\cap C^1(\mathbb{R};L^2)$.

(vi) *For each* $\alpha_k \geq 0$, $k = 0,1,2,3$ *and each* $\tau > 0$ *there exists a positive number* $\omega_1 = \omega_1(\alpha_0, \alpha_1, \alpha_2, \alpha_{3,}\tau)$ *such that*

$$\varphi_4(G(t)v) \leq e^{\omega_1|t|}\varphi_4(v) \tag{4.31}$$

for $t \in [-\tau, \tau]$ *and for* $v \in H^4$ *with* $\varphi_k(v) \leq \alpha_k$, $k = 0,1,2,3$. *Consequently, if* $v \in H^4$, *then* $G(\cdot)v \in C(\mathbb{R}; H^4) \cap C^1(\mathbb{R}; H^1)$. *In particular, if* $v \in H^4$, *then* $u(t,x) = [G(t)v]x$, $(t,x) \in \mathbb{R} \times \mathbb{R}$ *satisfies the equation* (3.1) *pointwise on* $\mathbb{R} \times \mathbb{R}$.

Proof. Let $v \in H^2$. Since the range condition in Theorem 4.1 is verified for $v \in H^3$, we need to use a density argument. We choose a sequence $\{v_n\}$ in H^3, $v_n \to v$ in H^2 as $n \to \infty$. Let $\varepsilon \in (0,1)$ and let also α_0, α_1, $\alpha_2 > 0$ so that

$$\sup_{n\geq 1}\varphi_k(v_n) + \varepsilon < \alpha_k,\ k = 0,1 \text{ and } e^{2a}\left(\sup_{n\geq 1}|\varphi_2(v_n)| + (b+\varepsilon)\right) < \alpha_2,$$

where $a = a(\alpha_0, \alpha_1)$ and $b = b(\alpha_0, \alpha_1)$ are numbers as in Lemma 3.3. Applying Theorem 4.1, one gets $\lambda_0 > 0$ such that to each $\lambda \in (-\lambda_0, \lambda_0)$ and $n \geq 1$ there corresponds an unique $v_{\lambda,n} \in K$ which satisfies

$$v_{\lambda,n} - \lambda(A+B)v_{\lambda,n} = v_n; \tag{4.32}$$

and

$$\begin{aligned}
\varphi_0(v_{\lambda,n}) &\leq \varphi_0(v_n) + |\lambda|\varepsilon;\\
\varphi_1(v_{\lambda,n}) &\leq \varphi_1(v_n) + |\lambda|\varepsilon;\\
\varphi_2(v_{\lambda,n}) &\leq (1-|\lambda|a)^{-1}(\varphi_2(v_n) + |\lambda|(b+\varepsilon));\\
\varphi_3(v_{\lambda,n}) &\leq (1-|\lambda|\omega_0)^{-1}\varphi_3(v_n).
\end{aligned} \tag{4.33}$$

Since, as seen in Proposition 3.1, B is quasidissipative on level sets of H^2, (4.32) implies $|v_{\lambda,n} - v_{\lambda,m}| \leq (1-|\lambda|\omega_0)^{-1}|v_n - v_m|$, and so $v_{\lambda,n}$ converges in L^2 to some v_λ. Since K is weakly compact in H^3, one can extract a subsequence v_{λ,n_k} such that $v_{\lambda,n_k} \rightharpoonup v_\lambda$ in H^3. We now prove that this implies $v_{\lambda,n_k} \to v_\lambda$ in H^2. By (3.4), one obtains

$$(\nabla(v_{\lambda,n_k} - v_\lambda), \nabla(v_{\lambda,n_k} - v_\lambda)) = -\left(v_{\lambda,n_k} - v_\lambda, \nabla^2(v_{\lambda,n_k} - v_\lambda)\right),$$

which implies that $|\nabla(v_{\lambda,n_k} - v_\lambda)|^2 \to 0$, and therefore $v_{\lambda,n_k} \to v_\lambda$ in H^1. Also,

$$\left(\nabla^2(v_{\lambda,n_k} - v_\lambda), \nabla^2(v_{\lambda,n_k} - v_\lambda)\right) = -\left(\nabla(v_{\lambda,n_k} - v_\lambda), \nabla^3(v_{\lambda,n_k} - v_\lambda)\right),$$

from which we get the required H^2-convergence of v_{λ,n_k} to v_λ.

From (ii) of Proposition 3.1, we see that $Av_{\lambda,n_k} \rightharpoonup Av_\lambda$ and $Bv_{\lambda,n_k} \to Bv_\lambda$, and since

$$v_{\lambda,n_k} - \lambda(A+B)v_{\lambda,n_k} = v_{n_k},$$

passing to limit as $k \to \infty$ we get

$$v_\lambda - \lambda(A+B)v_\lambda = v.$$

Moreover, since $\varphi_0(v_{\lambda,n_k}) \le \varphi_0(v_{n_k}) + |\lambda|\varepsilon$ and $v_{n_k} \to v$ in H^2, it is seen that $\varphi_0(v_\lambda) \le \varphi_0(v) + |\lambda|\varepsilon$. From (4.33) we also infer that

$$\begin{aligned}(4.34) \qquad (1/2)\left|\nabla v_{\lambda,n_k}\right|^2 &- \int_{-\infty}^{\infty}\int_0^{v_{\lambda,n_k}(x)} f(\xi)\,d\xi dx \\ &\le (1/2)\left|\nabla v_{n_k}\right|^2 - \int_{-\infty}^{\infty}\int_0^{v_{n_k}(x)} f(\xi)\,d\xi dx + |\lambda|\varepsilon.\end{aligned}$$

Now, since

$$\begin{aligned}(4.35) \qquad &\left|\int_{-\infty}^{\infty}\int_{v_{\lambda,n_k}(x)}^{v_\lambda(x)} f(\xi)\,d\xi dx\right| \\ &= \left|\int_{-\infty}^{\infty}\left(\int_0^1 f\left(\theta v_\lambda(x) + (1-\theta)v_{\lambda,n_k}(x)\right)d\theta\right)\left(v_\lambda(x) - v_{\lambda,n_k}(x)\right)dx\right| \\ &\le \left|\int_0^1 f\left(\theta v_\lambda(\cdot) + (1-\theta)v_{\lambda,n_k}(\cdot)\right)d\theta\right|\left|v_\lambda - v_{\lambda,n_k}\right|,\end{aligned}$$

and the first factor in the right-hand side is uniformly bounded with respect to k, from (4.34) we find that

$$\begin{aligned}(4.36) \qquad (1/2)\left|\nabla v_\lambda\right|^2 &- \int_{-\infty}^{\infty}\int_0^{v_\lambda(x)} f(\xi)\,d\xi dx \\ &\le (1/2)\left|\nabla v\right|^2 - \int_{-\infty}^{\infty}\int_0^{v(x)} f(\xi)\,d\xi dx + |\lambda|\varepsilon.\end{aligned}$$

Thus, it follows that $\varphi_1(v_\lambda) \le \varphi_1(v) + |\lambda|\varepsilon$. We now try to prove that $\varphi_2(v_\lambda) \le (1-|\lambda|a)^{-1}(\varphi_2(v) + |\lambda|(b+\varepsilon))$. From (3.4), it is seen that $(5/6)\left(f(v_{\lambda,n_k}), \nabla^2 v_{\lambda,n_k}\right) = (5/6)\left(Bv_{\lambda,n_k}, \nabla v_{\lambda,n_k}\right)$, and using the continuity of B on level sets we easily obtain that

$$(4.37) \qquad (5/6)\left(f(v_{\lambda,n_k}), \nabla^2 v_{\lambda,n_k}\right) \to (5/6)\left(f(v_\lambda), \nabla^2 v_\lambda\right) \qquad \text{as } k \to \infty.$$

Combining (4.37) and (4.33) we deduce that $\varphi_2(v_\lambda) \le (1-|\lambda|a)^{-1}\left(\varphi_2(v) + |\lambda|(b+\varepsilon)\right)$ as required. Applying now Theorem 2.1, we conclude that there exists a nonlinear group $G = \{G(t); t \in \mathbb{R}\}$ of locally Lipschitz operators on H^2 such that $G(\cdot)v \in C(\mathbb{R}; L^2)$ for each $v \in H^2$ and (4.28) is satisfied together with

$$(4.38) \qquad \varphi_0(G(t)v) \le \varphi_0(v) \text{ and } \varphi_1(G(t)v) \le \varphi_1(v)$$

for $v \in H^2$ and $t \in \mathbb{R}$, and

$$(4.39) \qquad \varphi_2(G(t)v) \le e^{a|t|}\left(\varphi_2(v) + b|t|\right)$$

for $v \in H^2$ with $\varphi_0(v) \le \alpha_0$, $\varphi_1(v) \le \alpha_1$ and $t \in \mathbb{R}$.

We will now prove that actually $G(\cdot)v$ belongs to $C(\mathbb{R}; H^2)$ for each $v \in H^2$.

Let $v \in H^2(\mathbb{R})$, $t \in \mathbb{R}$ and let $\{t_n\}$ be a sequence such that $t_n \to t$ as $n \to \infty$. From (ii) of Lemma 3.2, (4.38) and (4.39), there is $\beta_2 = \beta_2(\{t_n\}, t)$ such that $|\nabla^2 G(t_n) v| \leq \beta_2$ for $n \geq 1$. Since

$$\begin{aligned}|\nabla G(t_n) v - \nabla G(t) v|^2 &= \left|\left(\nabla^2 G(t_n) v - \nabla^2 G(t) v, G(t_n) v - G(t) v\right)\right| \\ &\leq M\,|G(t_n) v - G(t) v|,\end{aligned}$$

we get that $G(\cdot) v \in C(\mathbb{R}, H^1)$. To show that $G(\cdot)v$ belongs to $C(\mathbb{R}; H^2)$, we first prove its continuity with respect to the weak topology of H^2. Then we use the exponential growth condition (4.39) to show the continuity of $|\nabla^2 G(\cdot)|$. The conclusion will then follow from a strong convergence criterion for uniformly convex spaces. We first see that

$$\left(\nabla^2 G(t_n) v, \psi\right) = -\left(\nabla G(t_n) v, \nabla\psi\right) \to -\left(\nabla G(t) v, \nabla\psi\right)$$

as $n \to \infty$, for each $\psi \in C_c^\infty(\mathbb{R})$. Since $(\nabla G(t) v, \nabla\psi) = -(\nabla^2 G(t) v, \psi)$ one obtains that $G(t_n) v \rightharpoonup G(t) v$ in H^2 as $n \to \infty$. Moreover, from (4.39) and the group property of G we obtain

$$\varphi_2(G(t_n) v) = \varphi_2(G(t_n - t) G(t) v) \leq e^{a|t_n - t|}\left(\varphi_2(G(t) v) + b\,|t_n - t|\right).$$

We therefore infer that

$$\overline{\lim_{n\to\infty}}\, \varphi_2(G(t_n) v) \leq \varphi_2(G(t) v). \tag{4.40}$$

Since $G(t_n) v \rightharpoonup G(t) v$ in H^2, $G(t_n) v \to G(t) v$ in H^1 and

$$(5/6)\left(f(G(t_n) v), \nabla^2 G(t_n) v\right) = (5/6)\left(BG(t_n) v, \nabla G(t_n) v\right),$$

we also have

$$\underline{\lim}_{n\to\infty}\, \varphi_2(G(t_n) v) \geq \varphi_2(G(t) v). \tag{4.41}$$

From (4.40) and (4.41) one obtains that

$$\lim_{n\to\infty} \varphi_2(G(t_n) v) = \varphi_2(G(t) v).$$

Noting that

$$\begin{aligned}\varphi_2(G(t_n) v) &= (1/2)\left|\nabla^2 G(t_n) v\right| + (5/6)\left(f\left(G(t_n) v, \nabla^2 G(t_n) v\right)\right) \\ &= (1/2)\left|\nabla^2 G(t_n) v\right| + (5/6)\left(BG(t_n) v, \nabla G(t_n) v\right),\end{aligned}$$

since $G(t_n) v \to G(t) v$ in H^1 as $n \to \infty$, we get

$$\left|\nabla^2 G(t_n) v\right| \to \left|\nabla^2 G(t) v\right| \qquad \text{as } n \to \infty. \tag{4.42}$$

Since $G(t_n) v \rightharpoonup G(t) v$ in H^2, (4.42) implies via a strong convergence criterion for uniformly convex spaces that $\nabla^2 G(t_n) v \to \nabla^2 G(t) v$ in L^2, and so $G(t_n) v \to G(t) v$ in H^2 as requested. Also, the inequalities in (4.38) imply

$$\varphi_0(G(t) v) \leq \varphi_0(v) = \varphi_0(G(-t) G(t) v) \leq \varphi_0(G(t) v)$$

and

$$\varphi_1(G(t)v) \le \varphi_1(v) = \varphi_1(G(-t)G(t)v) \le \varphi_1(G(t)v),$$

hence $\varphi_0(G(t)v) = \varphi_0(v)$ and $\varphi_1(G(t)v) = \varphi_1(v)$ for each $v \in H^2$.

By Theorems 2.1 and 4.2, each of $G(t)$ maps H^3 into itself and (4.32) easily implies (4.36). Next, let us suppose that $v \in H^3$. It is seen that

$$e^{-\omega_0|t_n - t|}\varphi_3(G(t_n)v) \le \varphi_3(G(t)v) \le e^{\omega_0|t_n-t|}\varphi_3(G(t_n)v),$$

which implies in turn

$$\lim_{n\to\infty}\varphi_3(G(t_n)v) = \varphi_3(G(t)v). \tag{4.43}$$

Since

$$\left(\nabla^3 G(t_n)v + \nabla f(G(t_n)v), \psi\right) = -\left(\nabla^2 G(t_n)v + f(G(t_n)v), \nabla\psi\right)$$

for each $\psi \in C_c^\infty(\mathbb{R})$, using the H^2-continuity of $G(\cdot)v$ one obtains that

$$\nabla^3 G(t_n)v + \nabla f(G(t_n)v) \rightharpoonup \nabla^3 G(t)v + \nabla f(G(t)v) \qquad \text{as } n\to\infty$$

and this, together with (4.43), implies

$$\nabla^3 G(t_n)v + \nabla f(G(t_n)v) \to \nabla^3 G(t)v + \nabla f(G(t)v) \qquad \text{as } n\to\infty. \tag{4.44}$$

But $\nabla f(G(t_n)v) = -BG(t_n)v \to -BG(t)v = \nabla f(G(t)v)$, so

$$\nabla^3 G(t_n)v \to \nabla^3 G(t)v \qquad \text{as } n\to\infty,$$

and therefore $G \in C(\mathbb{R}; H^3) \cap C^1(\mathbb{R}; L^2)$. In the same way one can get that, for $v \in H^4$, $G \in C(\mathbb{R}; H^4) \cap C^1(\mathbb{R}; H^1)$, and in this further case $u(t,x)$ satisfies the equation (3.1) pointwise on $\mathbb{R}\times\mathbb{R}$. □

Remark 4.1. It is easily seen that differentiating a solution of (3.1) with respect to t one reduces its regularity from $C(\mathbb{R}; H^3)$ to $C(\mathbb{R}; L^2)$, that is, with three x-derivatives. With regard to this, it should be mentioned that if the function f in (3.1) is of class $C^\infty(\mathbb{R})$ and satisfies (3.3), then $G(\cdot)v \in \bigcap_{m=0}^{[k/3]} C^m\left(\mathbb{R}; H^{k-3m}\right)$ for each $v \in H^k$ and for any integer $k \ge 3$. It is also interesting to note that the regularity of the group $G = \{G(t); t\in\mathbb{R}\}$ with respect to t is established with the aid of the l.s.c. functionals φ_2, φ_3 and φ_4.

5 Regularized dispersive equations

In this section we establish the existence of nonlinear groups $G_\mu = \{G_\mu(t); t\in\mathbb{R}\}$ of Fréchet differentiable operators on H^1 which provide mild solutions to the initial value problem for nonlinear dispersive equations of the form

$$u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t, x \in \mathbb{R} \tag{5.1}$$

$$(5.2) \qquad u(0,x) = u_0(x), \qquad x \in \mathbb{R}.$$

Here $\mu > 0$, f is merely a nonlinear function of class $C^1(\mathbb{R})$ such that $f(0) = 0$ and u_0 is an initial function given in H^1. The convergence of $G_\mu = \{G_\mu(t)\,; t \in \mathbb{R}\}$ to the group $G = \{G(t)\,; t \in \mathbb{R}\}$ obtained in Theorem 4.2 will be also discussed in the next section.

Equation (5.1) is regarded as a pseudoparabolic regularization of the generalized K-dV equation (3.1). An equation related to (5.1) is the long wave equation

$$(5.3) \qquad u_t + u_x + uu_x - u_{xxt} = 0,$$

which was proposed by Benjamin, Bona and Mahony in [2] as a substitute for K-dV equation. A derivation of the equation (5.3) was also given in Benjamin [3]. Since then, many works have been devoted to the study of equations of type (5.3) (we refer the reader to, for instance Iwamiya, Oharu and Takahashi [10], Medeiros and Menzala [15], Medeiros and Miranda [16]). See also Avrin and Goldstein [1], Goldstein, Kajikya and Oharu [9] for a discussion on the equation (5.3) in several space variables, Tsutsumi and Mukasa [24] for other parabolic regularizations of (3.1), Bona and Chen [4], Bona and Smith [5], Takahashi [23] for problems related to (5.1). Also, initial-boundary value problems for a class of equations which significantly generalize (5.3) were treated by Oharu and Takahashi in [19] using nonlinear operator theory.

In order to derive a semilinear evolution equation in L^1 which is equivalent to (5.1), we begin by defining some necessary operators and stating their properties.

Let Δ be the one-dimensional Laplace operator defined by $\Delta v = \nabla^2 v$ for $v \in H^2$. It is easy to see that $I - \mu\Delta$ has a bounded inverse $(I - \mu\Delta)^{-1}$ on L^2 which satisfies the relation

$$(5.4) \qquad \left((I-\mu\Delta)^{-1} v, w\right) = \left(v, (I-\mu\Delta)^{-1} w\right), \qquad \text{for } v, w \in L^2.$$

Moreover, since (3.4) implies

$$\left((I-\mu\Delta)^{-1} \nabla v, v\right) = -\left((I-\mu\Delta)^{-1} v, \nabla v\right) \qquad \text{for } v \in H^1,$$

from (5.4) one may deduce that

$$(5.5) \qquad \left((I-\mu\Delta)^{-1} \nabla v, v\right) = 0 \qquad \text{for } v \in H^1.$$

We introduce a densely defined and closed linear operator A_μ in L^2 by

$$(5.6) \qquad A_\mu v = (1/\mu)\left(\nabla v - (I-\mu\Delta)^{-1} \nabla v\right) \qquad \text{for } v \in H^1$$

and a nonlinear operator B_μ from H^1 into H^2 by

$$(5.7) \qquad B_\mu v = -(I-\mu\Delta)^{-1} \nabla f(v) \qquad \text{for } v \in H^1.$$

Further, we define new scalar products on H^1 and H^2 by

$$(u,v)_{1,\mu} = (u,v) + \mu(\nabla u, \nabla v) \qquad \text{for } u, v \in H^1,$$

respectively

$$(u,v)_{2,\mu} = (u,v) + 2\mu(\nabla u, \nabla v) + \mu^2(\nabla^2 u, \nabla^2 v) \qquad \text{for } u, v \in H^2,$$

and we denote by $|\cdot|_{1,\mu}$ and $|\cdot|_{2,\mu}$ the norms induced on H^1, respectively on H^2, by these scalar products.

Let now k be an arbitrary positive integer. It is easy to see that A_μ maps H^{k+1} into H^k, and

$$\left(\nabla^k A_\mu u, \nabla^k u\right) = (1/\mu)\left(\nabla^{k+1} u, \nabla^k u\right) - (1/\mu)\left((I-\mu\Delta)^{-1}\nabla^{k+1} u, \nabla^k u\right) \tag{5.8}$$

for $u \in H^{k+1}$. Therefore, from (5.5) and (5.8) it follows that $\left(\nabla^k A_\mu u, \nabla^k u\right) = 0$ for $u \in H^{k+1}$. Hence $(A_\mu u, u)_k = 0$ for each $v \in H^{k+1}$, and so A_μ is the generator of a (C_0)-group $T_\mu = \{T_\mu(t); t \in \mathbb{R}\}$ such that each of $T_\mu(t)$ maps H^k into itself and satisfies the relation

$$|T_\mu(t)\, v|_k = |v|_k \qquad \text{for each } t \in \mathbb{R} \text{ and } v \in H^k.$$

Also, one may see that

$$\left|(I-\mu\Delta)^{-1} w\right|_{2,\mu} = |w| \qquad \text{for } w \in H^2. \tag{5.9}$$

Now equation (5.1) can be rewritten as a semilinear evolution equation in $\left(H^1, |\cdot|_\mu\right)$ of the form

$$(d/dt)\, u_\mu(t) = (A_\mu + B_\mu)\, u_\mu(t), \qquad t \in \mathbb{R}. \tag{5.10}$$

Our purpose is to construct the solution operator groups for (5.10) on H^1 using the generation theorem stated in Section 2. To this goal, we need to establish further regularity properties of the operators B_μ, as will be done in the next proposition.

Proposition 5.1. *Let μ in $(0,1)$. For the nonlinear operator B_μ, the following statements hold:*

(i) *For each $\alpha \geq 0$ there is a number $\omega = \omega(\alpha, \mu) \geq 0$ such that*

$$|B_\mu v - B_\mu w|_{1,\mu} \leq \omega\, |v - w|$$

for $v, w \in H^1$ with $|v|_{1,\mu} \leq \alpha$ and $|w|_{1,\mu} \leq \alpha$.

(ii) *B_μ is continuously Fréchet differentiable on H^1 and its Fréchet derivative $dB_\mu v$ at $v \in H^1$ is given by*

$$dB_\mu(v)\, w = -\nabla\left((I-\mu\Delta)^{-1}\left(f'(v)\, w\right)\right) \qquad \text{for } w \in H^1.$$

Proof. Let $\alpha \geq 0$ and set

$$\omega = \mu^{-1/2} \sup\left\{|f'(v)|_{L^\infty}; v \in H^1, |v|_{1,\mu} \leq \alpha\right\}.$$

By (3.4) and the definition of B_μ, we see that

$$|B_\mu v - B_\mu w|_\mu^2 = (B_\mu v - B_\mu w, B_\mu v - B_\mu w) - \mu\left(\Delta\left(B_\mu v - B_\mu w\right), B_\mu v - B_\mu w\right)$$

$$= -\left(\nabla f(v) - \nabla f(w), B_\mu v - B_\mu w\right)$$
$$= \left(f(v) - f(w), \nabla\left(B_\mu v - B_\mu w\right)\right).$$

Hence $|B_\mu v - B_\mu w|_{1,\mu}^2 \leq w|v-w|\mu^{1/2}\,|\nabla(B_\mu v - B_\mu w)|$, from which the desired estimate follows. To get (ii), we note that

$$\left|B_\mu(v+w) - B_\mu v + \nabla(I-\mu\Delta)^{-1}\left(f'(v)\,w\right)\right|_{1,\mu}^2$$
$$= -\left(B_\mu(v+w) - B_\mu(v) + \nabla\left((I-\mu\Delta)^{-1}\right)\left(f'(v)\,w\right), \nabla f(v+w) - \nabla f(v)\right)$$
$$\quad - \left(\nabla\left(B_\mu(v+w) - \nabla B_\mu v + \nabla(I-\mu\Delta)^{-1}\left(f'(v)\,w\right)\right), f'(v)\,w\right)$$
$$= \left(\nabla\left(B_\mu(v+w) - B_\mu(v) - \nabla\left((I-\mu\Delta)^{-1}\right)\left(f'(v)\,w\right)\right), f(v+w) - f(v) - f'(v)\,w\right),$$

which implies

$$\left|B_\mu(v+w) - B_\mu v - \nabla(I-\mu\Delta)^{-1}\left(f'(v)\,w\right)\right|_{1,\mu} \leq \mu^{-1/2}\left|f(v+w) - f(v) - f'(v)\,w\right|.$$

Therefore, $\left|B_\mu(v+w) - B_\mu v - \nabla(I-\mu\Delta)^{-1}\left(f'(v)\,w\right)\right|_{1,\mu} = o\left(|w|_{1,\mu}\right)$, and so (ii) is proved.

□

We are now in position to state the main result of this section.

Theorem 5.1. *For each $\mu > 0$ there exists a nonlinear group $G_\mu = \{G_\mu(t)\,; t\in\mathbb{R}\}$ of locally Lipschitz operators on H^1 which satisfies the properties given below:*

(i) *If $v \in H^1$, then $G_\mu(\cdot)\,v \in C(\mathbb{R};H^1)\cap C^1(\mathbb{R};L^2)$ and*

$$G_\mu(t)\,v = T_\mu(t)\,v + \int_0^t T_\mu(t-s)\,B_\mu G_\mu(s)\,v ds,$$

for $t\in\mathbb{R}$ and $v\in H^1$.

(ii) *If $v \in H^2$, then $G_\mu(\cdot)\,v \in C(\mathbb{R};H^2)\cap C^1(\mathbb{R};H^1)\cap C^2(\mathbb{R};L^2)$ and satisfies the equation in $C(\mathbb{R};H^1)$*

$$(d/dt)\,G_\mu(t)\,v = (A_\mu + B_\mu)\,G_\mu(t)\,v \qquad \text{for } t\in\mathbb{R}.$$

(iii) *Each of $G_\mu(t)$ is continuously Fréchet differentiable on H^1.*

(iv) *$\varphi_{0,\mu}(G_\mu(t)\,v) = \varphi_{0,\mu}(v)$ for $t\in\mathbb{R}$ and $v\in H^1$, where the functional $\varphi_{0,\mu}$ is defined by*

$$\varphi_{0,\mu}(v) = |v|_{1,\mu} \text{ for } v\in H^1.$$

(v) *$\varphi_1(G_\mu(t)\,v) = \varphi_1(v)$ for $t\in\mathbb{R}$ and $v\in H^1$, where φ_1 is the functional defined by (3.21).*

Proof. One may show that $A_\mu + B_\mu$ satisfies the following range condition

For each $\alpha \geq 0$ there is a number $\lambda_\mu = \lambda_\mu(\alpha) > 0$ such that for any $v\in H^1$ with $|v|_{1,\mu} \leq \alpha$ and for any $\lambda \in (-\lambda_\mu, \lambda_\mu)$ there is an element $v_\lambda \in H^1$ such that

$$v_\lambda - \lambda\,(A_\mu + B_\mu)\,v_\lambda = v,$$

$$\varphi_{0,\mu}(v_\lambda) \le \varphi_{0,\mu}(v) + |\lambda|\,\varepsilon,$$
$$\varphi_1(v_\lambda) \le \varphi_1(v) + |\lambda|\,\varepsilon.$$

The proof is similar to that of Theorem 6.1, and the desired result follows easily from Theorem 2.1 and Proposition 5.1. □

Remark 5.1. Note that the differentiation of a solution reduces its Sobolev regularity from $C(\mathbb{R}; H^1)$ to $C(\mathbb{R}; L^2)$, that is, with one x-derivative, and so if $f \in C^\infty(\mathbb{R})$, then $G_\mu(\cdot)\, v \in \bigcap_{m=0}^{k} C^m\left(\mathbb{R}; H^{k-m}\right)$ for each $v \in H^k$ and for each integer $k \ge 1$.

6 A convergence theorem for nonlinear groups

In the previous sections we have obtained the existence of the nonlinear groups $G = \{G(t)\,; t \ge 0\}$ and $G_\mu = \{G_\mu(t)\,; t \ge 0\}$ of locally Lipschitz operators on H^2, respectively on H^1, which provides mild solutions to the initial value problems for the generalized K-dV equation (3.1), respectively to its pseudoparabolic regularization (5.1). Here we discuss the convergence of G_μ to G. For this purpose, we assume that the nonlinear function f is of class $C^3(\mathbb{R})$ and satisfies (3.3).

In what follows $\varphi_{k,\mu}$, $k = 0, 1, 2, 3$ denote the functionals

$$\text{(6.1)} \qquad \varphi_{0,\mu}(v) = |v|_{1,\mu} = \left(|v|^2 + \mu\,|\nabla v|^2\right)^{1/2}, \qquad v \in H^1;$$

$$\varphi_{1,\mu}(v) = \varphi_1(v) = (1/2)\,|\nabla v|^2 - \int_{-\infty}^{\infty}\int_0^{v(x)} f(\xi)\, d\xi dx, \qquad v \in H^1;$$

$$\varphi_{2,\mu}(v) = (1/2)\left|\nabla^2 v\right|^2 + (\mu/12)\left|\nabla^3 v\right|^2 + (5/6)\left(f(v), \nabla^2 v\right) - (5\mu/12)\left(f'(v), \left(\nabla^2 v\right)^2\right), \qquad v \in H^3;$$

$$\varphi_{3,\mu}(v) = \varphi_3(v) = \left|\nabla^3 v + \nabla f(v)\right|, \qquad v \in H^3.$$

As done in Section 3, we first establish the relation between $\varphi_{k,\mu}$-boundedness and norm boundedness.

Lemma 6.1. *Let $\mu \in (0, 1)$. For each α_0, α_1, $\alpha_2 \ge 0$, there is $\beta_2 = \beta_2(\alpha_0, \alpha_1, \alpha_2) \ge 0$, independent of μ, such that if $v \in H^3$ and $\varphi_{0,\mu}(v) \le \alpha_0$, $\varphi_{1,\mu}(v) \le \alpha_1$, $\varphi_{2,\mu}(v) \le \alpha_2$, then $|\nabla^2 v| \le \beta_2$ and $\mu^{1/2}\,|\nabla^3 v| \le \beta_2$.*

Proof. Let $\mu \in (0, 1)$, $\alpha_0, \alpha_1, \alpha_2 \ge 0$ and $v \in H^3$ such that $\varphi_{0,\mu}(v) \le \alpha_0$, $\varphi_{1,\mu}(v) \le \alpha_1$, $\varphi_{2,\mu}(v) \le \alpha_2$. Then, in a way similar to that used to establish Lemma 3.2, we see that there are numbers $\beta_0 = \beta_0(\alpha_0) \ge 0$ and $\beta_1 = \beta_1(\alpha_0, \alpha_1) \ge 0$ such that $|v| \le \beta_0$ and $|\nabla v| \le \beta_1$. We further note that

$$\text{(6.2)} \qquad C\left|\nabla^2 v\right|^2 \le (1/2)\left(C^2\,|\nabla v|^2 + \left|\nabla^3 v\right|^2\right) \qquad \text{for each } v \in H^3 \text{ and } C \in \mathbb{R}_+.$$

Since $\varphi_{2,\mu}(v) \le \alpha_2$, (6.1) implies

$$(1/2)\left|\nabla^2 v\right|^2 + (\mu/12)\left|\nabla^3 v\right|^2$$

$$\leq \alpha_2 + (5/6)\left(f'(v), (\nabla v)^2\right) + (5\mu/12)\left(f'(v), \left(\nabla^2 v\right)^2\right).$$

Since $|w|_{L^\infty} \leq |w|_1$ for each $w \in H^1$, it follows that there is $\gamma_1 = \gamma_1(\alpha_0, \alpha_1) \geq 0$ such that $|v|_{L^\infty} \leq \gamma_1$. We then define

$$\mathscr{C} = \mathscr{C}(f', \alpha_0, \alpha_1) = \sup\{|f'(x)|; |x| \leq \gamma_1\}.$$

Therefore, (6.1) and (6.2) lead us to the following estimate

$$\begin{aligned}(1/2)\left|\nabla^2 v\right|^2 + (\mu/12)\left|\nabla^3 v\right|^2 &\leq \alpha_2 + (5/6)\mathscr{C}|\nabla v|^2 + (5\mu/12)\mathscr{C}\left|\nabla^2 v\right|^2\\ &\leq \alpha_2 + (5/6)\mathscr{C}|\nabla v|^2 + (5\mu/72)\left(9\mathscr{C}^2|\nabla v|^2 + \left|\nabla^3 v\right|^2\right).\end{aligned}$$

Hence

$$(1/2)\left|\nabla^2 v\right|^2 + (\mu/72)\left|\nabla^3 v\right|^2 \leq \alpha_2 + (5/24)\mathscr{C}\beta_1^2(4 + 3\mathscr{C})$$

and so Lemma 6.1 is proved. □

Since we aim to apply Theorem 2.1, we now establish the quasidissipativity of the operators B_μ on level sets with respect to φ_k, $k = 0, 1, 2$.

Lemma 6.2. *Let $\mu \in (0,1)$. For each $\alpha_0, \alpha_1, \alpha_2 \geq 0$ there exists a number $\omega_0 = \omega_0(\alpha_0, \alpha_1, \alpha_2)$, independent of μ, such that*

$$\left|(B_\mu v - B_\mu w, v - w)_{1,\mu}\right| \leq \omega_0 |v - w|^2 \tag{6.3}$$

and

$$\left|(B_\mu v - B_\mu w, v - w)_{2,\mu}\right| \leq \omega_0 |v - w|^2 \tag{6.4}$$

for $v, w \in H^3$ with $\varphi_{k,\mu}(v) \leq \alpha_k$ and $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0, 1, 2$.

Proof. Let $\alpha_0, \alpha_1, \alpha_2 \geq 0$ and $v, w \in H^3$ so that $\varphi_{k,\mu}(v) \leq \alpha_k$ and $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0, 1, 2$. It is clear that

$$\begin{aligned}(B_\mu v - B_\mu w, v - w)_{1,\mu} &= ((I - \mu\Delta)(B_\mu v - B_\mu w), v - w)\\ &= (f(v) - f(w), \nabla(v - w))\end{aligned}$$

and

$$\begin{aligned}&(B_\mu v - B_\mu w, v - w)_{2,\mu}\\ &\quad= ((I - \mu\Delta)(B_\mu v - B_\mu w), v - w) - \mu\left((I - \mu\Delta)(B_\mu v - B_\mu w), \nabla^2(v - w)\right)\\ &\quad= (f(v) - f(w), \nabla(v - w)) + \mu\left(\nabla(f(v) - f(w)), \nabla^2(v - w)\right),\end{aligned}$$

from which we easily get the required conclusion. □

We next prove the range condition for $A_\mu + B_\mu$.

Theorem 6.1. *Let $v \in H^3$, $\varepsilon > 0$, and suppose that $\alpha_0, \alpha_1, \alpha_2 \geq 0$ are chosen so that $\varphi_{0,\mu}(v) + \varepsilon < \alpha_0$, $\varphi_{1,\mu}(v) + \varepsilon < \alpha_1$ and $e^{2a}\{|\varphi_{2,\mu}(v)| + 1 + \varepsilon\} < \alpha_2$ for all $\mu \in (0,1)$. Let $a = a(\alpha_0, \alpha_1)$, $b = b(\alpha_0, \alpha_1)$ be positive numbers as in Lemma 3.3, and let $\omega_0 = \omega_0(\alpha_0, \alpha_1, \alpha_2)$ be a positive number as in Lemma 6.2. Then there are numbers $\mu_0 = \mu_0(\alpha_0, \alpha_1, \alpha_2) > 0$ and $\hat{\lambda}_0 = \hat{\lambda}_0(|v|_3, \varepsilon)$, $0 < \hat{\lambda}_0 \leq \min\{1, 1/2a, 1/\omega_0\}$ such that for each $\mu \in (0, \mu_0)$ and each $\lambda \in \left(-\hat{\lambda}_0, \hat{\lambda}_0\right)$ there exists an unique element $v_{\lambda,\mu} \in H^3$ which satisfies*

$$v_{\lambda,\mu} - \lambda(A_\mu + B_\mu) v_{\lambda,\mu} = v, \tag{6.5}$$

and

$$\begin{aligned} \varphi_{0,\mu}(v_{\lambda,\mu}) &\leq \varphi_{0,\mu}(v) + |\lambda|\varepsilon, \\ \varphi_{1,\mu}(v_{\lambda,\mu}) &\leq \varphi_{1,\mu}(v) + |\lambda|\varepsilon, \\ \varphi_{2,\mu}(v_{\lambda,\mu}) &\leq (1 - |\lambda| a)^{-1}\{\varphi_{2,\mu}(v) + |\lambda|(b + 1 + \varepsilon)\}, \\ \varphi_{3,\mu}(v_{\lambda,\mu}) &\leq (1 - |\lambda| a)^{-1}\varphi_{3,\mu}(v). \end{aligned} \tag{6.6}$$

Proof. The proof is similar to that of Theorem 4.1. We first choose $\beta_k > 0$, $k = 0, 1, 2$, so that

$$\left\{w \in H^3; \varphi_{k,\mu}(w) \leq \alpha_k,\ k = 0,1,2\right\} \subset \left\{w \in H^3; \left|\nabla^k w\right| \leq \beta_k,\ k = 0,1,2\right\}$$

for $\mu \in (0,1)$, and $|\nabla^3 v| + 2L_0 \leq \beta_3$, where

$$L_0 = \sup\left\{|Bw|; w \in H^1, |w| \leq \beta_0, |\nabla w| \leq \beta_1\right\}.$$

Denote also L_1, L_2 as in the proof of Theorem 4.1 and

$$M_1 = \sup\{|f'(w)|_{L^\infty}; |w| \leq \beta_0, |\nabla w| \leq \beta_1\}, \tag{6.7}$$

$$M_2 = \sup\{|f''(w)|_{L^\infty}; |w| \leq \beta_0, |\nabla w| \leq \beta_1\}. \tag{6.8}$$

It follows that there exists a positive number $\delta = \delta(|v|_3, \varepsilon)$ such that if $w \in H^3$, $|w - v| < \delta$ and $\left|\nabla^k w\right| \leq \max\left\{\beta_k, \left|\nabla^k v\right| + L_k\right\}$, $k = 0, 1, 2$, then the inequalities in (4.7) are satisfied.

Define

$$\hat{\lambda}_0 = \min\{1, \delta/\beta_3, \varepsilon/(2\beta_3), 1/\omega_0, 1/(2a)\},$$

and let also $\mu_0 = \mu_0(\alpha_0, \alpha_1, \alpha_2)$ be a positive number such that $5\mu_0 M_1/6 \leq 1$ and

$$a\mu_0(4 + 25M_1^2\beta_2^3)/48 + 5M_2\beta_2^2\mu_0^{1/2}(\beta_2 + \mu_0^{1/2}L_0) \leq 1,$$

where ω_0 is a positive number as in Lemma 6.2.

For each $\lambda \in \left(-\hat{\lambda}_0, \hat{\lambda}_0\right)$, $\lambda \neq 0$ and $\mu \in (0, \mu_0)$, we define a subset $K_{\lambda,\mu}$ of H^3 by

$$K_{\lambda,\mu} = \left\{w \in H^3; |v - w|_{2,\mu} \leq |\lambda|\beta_3, \left|\nabla^k w\right| \leq \beta_k,\ k = 0,1,2,3\right\} \tag{6.9}$$

and an operator $\Gamma_{\lambda,\mu} : K_{\lambda,\mu} \to H^3$ by

$$\Gamma_{\lambda,\mu} w = (I - \lambda A_\mu)^{-1}(v + \lambda B_\mu w) \qquad \text{for } w \in K_{\lambda,\mu}. \tag{6.10}$$

Let now $w \in K_{\lambda,\mu}$ and write $z = \Gamma_{\lambda,\mu} w$ for simplicity. We see that

$$z = v + \lambda A_\mu z + \lambda B_\mu w. \tag{6.11}$$

Since

$$|z - v|_{2,\mu} = |(I - \mu\Delta)(z - v)|^2, \tag{6.12}$$

it follows that

$$\begin{aligned} |z - v|_{2,\mu} &= \lambda\left((I - \mu\Delta)(z - v), Az + Bw\right) \\ &= -\lambda\left((I - \mu\Delta) v, Az\right) + \lambda\left((I - \mu\Delta)(z - v), Bw\right). \end{aligned}$$

Hence

$$\begin{aligned} |z - v|_{2,\mu} &= \lambda\left(Av, (I - \mu\Delta) z\right) + \lambda\left((I - \mu\Delta)(z - v), Bw\right) \\ &= \lambda\left(Av, (I - \mu\Delta)(z - v)\right) + \lambda\left((I - \mu\Delta)(z - v), Bw\right) \\ &= \lambda\left((I - \mu\Delta)(z - v), Av + Bw\right), \end{aligned}$$

and by virtue of (6.12), we deduce the estimate

$$\left(|z - v|^2 + 2\mu\left|\nabla(z - v)\right|^2 + \mu^2\left|\nabla^2(z - v)\right|^2\right)^{1/2} \leq |\lambda|\left(\left|\nabla^3 v\right| + L_0\right) \leq |\lambda|\,\beta_3. \tag{6.13}$$

Since $|z - v|_{2,\mu} = |(I - \mu\Delta) A_\mu z|^2$, invoking Minkowski' s inequality we obtain

$$\begin{aligned} \left|\nabla^3 z\right| &= |A_\mu z|_{2,\mu} \\ &\leq |\lambda|^{-1}\left(|z - v|_{2,\mu} + |B_\mu w|_{2,\mu}\right) \\ &\leq \left|\nabla^3 v\right| + 2L_0 \leq \beta_3. \end{aligned} \tag{6.14}$$

Since $|B_\mu w| \leq |Bw|$ for $w \in H^1$, the estimates $\left|\nabla^k z\right| \leq \left|\nabla^k v\right| + \lambda L_k, \quad k = 0, 1, 2$ are obtained as in the proof of Theorem 4.1. By (5.6), (5.7) and (6.11), we see that

$$(z - v, z) + \mu\left(\nabla(z - v), \nabla z\right) = \lambda\left(Az + Bw, z\right).$$

Therefore, since $(Az, z) = (Bz, z) = 0$, it follows that

$$(z, z) + \mu\left(\nabla z, \nabla z\right) = (v, z) + \mu\left(\nabla v, \nabla z\right) + \lambda\left(Bw - Bz, z\right).$$

In view of (4.7), we conclude that

$$\varphi_{0,\mu}(z) \leq \varphi_{0,\mu}(v) + |\lambda|\,\varepsilon. \tag{6.15}$$

We now prove the second inequality in (6.6). Noting that

$$\left|\int_{-\infty}^{\infty}\int_{v(x)}^{z(x)} f(\xi)\, d\xi dx - \int_{-\infty}^{\infty} f(w(x))\,(z(x) - v(x))\, dx\right|$$

$$\leq \left| \int_0^1 f\left(\theta v + (1-\theta) z\right) d\theta - f(w) \right| |z - v|,$$

from (6.13) we obtain the inequality

$$(\nabla z, \nabla z) - \int_{-\infty}^{\infty} \int_0^{z(x)} f(\xi) \, d\xi dx \leq (\nabla v, \nabla z) - \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi) \, d\xi dx + |\lambda| \varepsilon,$$

and hence

$$\varphi_{1,\mu}(z) \leq \varphi_{1,\mu}(v) + |\lambda| \varepsilon. \tag{6.16}$$

We next show that

$$\varphi_{2,\mu}(z) \leq (1 - |\lambda| a)^{-1} \left\{ \varphi_{2,\mu}(v) + |\lambda| (b + 1 + \varepsilon) \right\}. \tag{6.17}$$

The application of the Main Value Theorem implies

$$\begin{aligned}
\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) &= \left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 z\right) + \left(f(v), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) \\
&= \left(f'(w_1) \nabla^2 z, z - v\right) - (\nabla B v, z - v) \\
&= \left(\left(f'(w_1) - f'(z)\right) \nabla^2 z, z - v\right) + \left(f'(z) \nabla^2 z, z - v\right) \\
&\quad + (\nabla B w - \nabla B v, z - v) - (\nabla B w, z - v),
\end{aligned}$$

where $w_1(\cdot) = \theta(\cdot) z(\cdot) + (1 - \theta(\cdot)) v(\cdot)$. In view of (4.23), it is easy to check that

$$\begin{aligned}
\left(f'(z) \nabla^2 z, z - v\right) &- \mu \left(f'(z) \nabla^2 z, \nabla^2 z - \nabla^2 v\right) \\
&= \left(f'(z) \nabla^2 z, \lambda (Az + Bw)\right) \\
&= (-\lambda/5) \left[5 \left(f'(z) \nabla^2 z, \nabla^3 z\right) - \left(f'''(z) (\nabla z)^3, \nabla^2 z\right)\right] \\
&\quad - (\lambda/5) \left[\left(f'''(z) (\nabla z)^3, \nabla^2 z\right) + 5 \left(f'(z) \nabla^2 z, f'(w) \nabla w\right)\right].
\end{aligned}$$

Furthermore, from (3.4) and (6.11) we obtain

$$(\nabla B w, z - v) = \lambda \left(\nabla^2 B_\mu w, \nabla^2 z\right) = \left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 v\right).$$

Since $(\nabla A_\mu z, Az) = 0$, this implies

$$\lambda \mu \left(\nabla^3 B_\mu w, \nabla^3 z\right) = -\left[\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 v\right)\right] + \lambda \left(\nabla B w, \nabla^3 z\right).$$

We are now ready to prove the estimate (6.17). By virtue of (4.23) and Lemma 3.3, we obtain

$$\begin{aligned}
(1 - |\lambda| a) \varphi_{2,\mu}(z) \leq \varphi_{2,\mu}(v) &+ |\lambda| (b + \varepsilon) + (5\mu/12) \left(f'(z), \left(\nabla^2 z - \nabla^2 v\right)^2\right) \\
&- (1/2) \left|\nabla^2 z - \nabla^2 v\right|^2 + (5\mu/12) \left(f'(v) - f'(z), \left(\nabla^2 v\right)^2\right) \\
&- (\mu/6) \left|\nabla^3 z - \nabla^3 v\right|^2 + (\mu/12) |\lambda| a \left[-\left|\nabla^3 z\right|^2 + 5 \left(f'(z), \left(\nabla^2 z\right)^2\right)\right].
\end{aligned}$$

From this inequality we can easily get

(6.18)

$$(1-|\lambda|\,a)\,\varphi_{2,\mu}(z)\le\varphi_{2,\mu}(v)+|\lambda|\,(b+\varepsilon)+(1/12)\,|\lambda|\,a\mu\left[-\left|\nabla^3 z\right|+5\left(f'(z),\left(\nabla^2 z\right)^2\right)\right]$$
$$+(5/12)\lambda\mu\left(\int_0^1 f''\left(\theta v+(1-\theta)\,z\right)d\theta\left(\nabla^2 v\right)^2,A_\mu z+B_\mu w\right).$$

From (6.18) and (6.13) one may see that

$$\begin{aligned}\mu&\left|\left(\int_0^1 f''\left(\theta v+(1-\theta)\,z\right)d\theta\left(\nabla^2 v\right)^2,A_\mu z+B_\mu w\right)\right|\\&\le M_2\beta_2^2\mu\left(\left|\nabla^3 v\right|+L_0\right)\\&\le M_2\beta_2^2\mu^{1/2}\left(\beta_2+\mu^{1/2}L_0\right).\end{aligned}$$

Using (6.18), we obtain

$$\begin{aligned}5\left(f'(z),\left(\nabla^2 z\right)^2\right)&\le 5M_1\left|\nabla^2 z\right|_{L^4}^2\\&\le 5\sqrt{M}_1\left|\nabla^3 z\right|^{1/2}\left|\nabla^2 z\right|^{3/2}\\&\le(1/2)\left(4\left|\nabla^3 z\right|+25/2M_1^2\left|\nabla^2 z\right|^3\right)\\&\le\left|\nabla^3 z\right|^2+1+25/4M_1^2\left|\nabla^2 z\right|^3,\end{aligned}$$

from which we get the required conclusion.

Thus it is shown that (6.13) is verified, and so $\Gamma_{\lambda,\mu}v\in K_{\lambda\mu}$. The conclusion of this theorem is obtained in a manner similar to that of Theorem 4.1, noting that the last inequality in (6.5) follows from Lemma 6.2 and from relation (5.9).

□

By virtue of Theorems 2.1 and 6.1 one obtains a result on the regularity of the groups $G_\mu=\{G_\mu(t)\,;t\in\mathbb{R}\}$.

Theorem 6.2. *Let $\mu\in(0,1)$. Let $G_\mu=\{G_\mu(t)\,;t\in\mathbb{R}\}$ be the nonlinear group of locally Lipschitz operators obtained in Theorem 5.1. In addition to the properties stated in Theorem 5.1, we have the following*

(i) *$G_\mu(\cdot)\,v\in\bigcap_{m=0}^{3}C^m\left(\mathbb{R};H^{3-m}\right)$ for $v\in H^3$ and $G_\mu(\cdot)\,v\in\bigcap_{m=0}^{4}C^m\left(\mathbb{R};H^{4-m}\right)$ for $v\in H^4$. In particular, if $v\in H^4$, then $u(t,x)=[G_\mu(t)\,v](x)$ satisfies equation (5.1) pointwise on $\mathbb{R}\times\mathbb{R}$.*

(ii) *The exponential formula localized with respect to $\{\varphi_{k,\mu}\}_{k=0,1,2}$*

$$G_\mu(t)\,v=H^2\text{-}\lim_{n\to\infty}\left(I-(t/n)\left(A_\mu+B_\mu\right)\right)^{-n}v$$

holds for $t\in\mathbb{R}$ and the convergence is uniform on each bounded subinterval of $\mathbb{R}$.

(iii) *For each* $\alpha_0, \alpha_1, \alpha_2 \geq 0$ *and* $\tau > 0$, *there are numbers* $\hat{a} = \hat{a}(\alpha_0, \alpha_1) > 0$, $\hat{b} = \hat{b}(\alpha_0, \alpha_1) > 0$, $\hat{\omega}_0 = \hat{\omega}_0(\alpha_0, \alpha_1, \alpha_2, \tau)$ *and* $\mu_0 = \mu_0(\alpha_0, \alpha_1, \alpha_2, \tau)$ *such that:*

(iii.1) $\varphi_{2,\mu}(G_\mu(t)v) \leq e^{\hat{a}|t|}\left(\varphi_{2,\mu}(v) + \hat{b}|t|\right)$,

for $\mu \in (0, \mu_0)$, $t \in [-\tau, \tau]$ *and for* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0, 1, 2$.

(iii.2) $\varphi_3(G_\mu(t)v) \leq e^{\hat{\omega}_0|t|}\varphi_3(v)$,

for $\mu \in (0, \mu_0)$, $t \in [-\tau, \tau]$ *and for* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0, 1, 2$

(iii.3) $|G_\mu(t)v - G_\mu(t)w|_\mu \leq e^{\hat{\omega}_0|t|}|v - w|_\mu$,

for $\mu \in (0, \mu_0)$, $t \in [-\tau, \tau]$ *and for* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0, 1, 2$ *and* $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0, 1, 2$.

We are now in position to prove the convergence theorem

Theorem 6.3. *The following statements hold:*

(i) $(I - \lambda(A + B))^{-1}v = H^2\text{-}\lim_{n\to\infty}(I - \lambda(A_\mu + B_\mu))^{-1}v$

for $v \in H^3$ *and for* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \min\left\{\lambda_0(|v|_3, \varepsilon), \hat{\lambda}_0(|v|_3, \varepsilon)\right\}$, *where* $\varepsilon > 0$, $\lambda_0 = \lambda_0(|v|_3, \varepsilon)$ *is the number given in Theorem 2.1 and* $\hat{\lambda}_0 = \hat{\lambda}_0(|v|_3, \varepsilon)$ *is the number given in Theorem 6.1.*

(ii) *If* $v \in H^2$, $v_\mu \in H^3$, $v_\mu \to v$ *in* H^2 *as* $\mu \to 0$ *and* $\mu|\nabla^3 v_\mu|^2 \leq M$ *as* $\mu \to 0$ *for some* $M \geq 0$, *then*

$$G(t)v = H^1\text{-}\lim_{\mu\to 0}G_\mu(t)v_\mu \qquad \textit{for } t \in \mathbb{R}$$

and the convergence is uniform on each bounded subinterval of $\mathbb{R}$.

If in particular $v \in H^3$, *then*

$$G(t)v = H^1\text{-}\lim_{\mu\to 0}G_\mu(t)v \qquad \textit{for } t \in \mathbb{R}$$

and the convergence is uniform on each bounded subinterval of $\mathbb{R}$.

Proof. (i) Let $v \in H^3$, $\varepsilon > 0$ and let $\lambda \in \mathbb{R}$ be such that $|\lambda| < \min\left\{\lambda_0, \hat{\lambda}_0\right\}$. If we write $v_\lambda = (I - \lambda(A + B))^{-1}v$ and $v_{\lambda,\mu} = (I - \lambda(A_\mu + B_\mu))^{-1}v$, then v_λ satisfies (4.1) and $v_{\lambda,\mu}$ makes sense and satisfies (6.5) for $\mu > 0$ sufficiently small. It is obvious that $\varphi_{k,\mu}(v_\lambda) \to \varphi_k(v_\lambda)$ as $\mu \to 0_+$, for $k = 0, 1, 2$. Therefore, we see from Lemma 6.2 that

$$\begin{aligned}(6.19)\qquad &|((A_\mu + B_\mu)v_{\lambda,\mu} - (A_\mu + B_\mu)v_\lambda, v_{\lambda,\mu} - v_\lambda)\\ &\quad + 2\mu(\nabla(A_\mu + B_\mu)v_{\lambda,\mu} - \nabla(A_\mu + B_\mu)v_\lambda, \nabla(v_{\lambda,\mu} - v_\lambda))|\\ &= |(B_\mu v_{\lambda,\mu} - B_\mu v_\lambda, v_{\lambda,\mu} - v_\lambda) + \mu(\nabla B_\mu v_{\lambda,\mu} - \nabla B_\mu v_\lambda, \nabla(v_{\lambda,\mu} - v_\lambda))|\\ &\leq \hat{\omega}_0|v_{\lambda,\mu} - v_\lambda|^2.\end{aligned}$$

An easy computation yields

$$(6.20)\qquad \lambda\left((A_\mu + B_\mu)v_\lambda - (A + B)v_\lambda, v_{\lambda,\mu} - v_\lambda - \mu\nabla^2(v_{\lambda,\mu} - v_\lambda)\right)$$

$$
\begin{aligned}
&= \lambda \left((A_\mu + B_\mu) v_\lambda - (A_\mu + B_\mu) v_{\lambda,\mu}, v_{\lambda,\mu} - v_\lambda \right) \\
&\quad + \mu \left(\nabla (A_\mu + B_\mu) v_\lambda - \nabla (A_\mu + B_\mu) v_{\lambda,\mu}, \nabla (v_{\lambda,\mu} - v_\lambda) \right) \\
&\quad + |v_{\lambda,\mu} - v_\lambda|^2 + \mu \left| \nabla (v_{\lambda,\mu} - v_\lambda) \right|^2,
\end{aligned}
$$

Therefore, from (6.19) and (6.20) we obtain

$$
\begin{aligned}
(6.21) \qquad & (1 - |\lambda| \hat{\omega}_0) \left| v_{\lambda,\mu} - v_\lambda \right|^2 \\
& \qquad \le |\lambda| \left| (A_\mu + B_\mu) v_\lambda - (A + B) v_\lambda \right| \left| v_{\lambda,\mu} - v_\lambda - \mu \nabla^2 (v_{\lambda,\mu} - v_\lambda) \right|.
\end{aligned}
$$

Since $(A_\mu + B_\mu) v_\lambda \to (A + B) v_\lambda$ in L^2 as $\mu \to 0_+$ and $\overline{\lim_{\mu \downarrow 0}} |v_{\lambda,\mu}|_3 < \infty$, it follows that $v_{\lambda,\mu} \to v_\lambda$ in L^2 as $\mu \to 0_+$. Noting that $|\nabla w| \le |w|^{1/2} |\nabla^2 w|^{1/2}$ and $|\nabla^2 w| \le |w|^{1/3} |\nabla^3 w|^{2/3}$ for $w \in H^3$, we conclude that $v_{\lambda,\mu} \to v_\lambda$ in H^2 as $\mu \to 0_+$. Thus assertion (i) is proved.

We now prove (ii). If $v \in H^3$, then the H^2-convergence of $G_\mu(t) v$ to $G(t) v$ follows easily.

If $v \in H^2$, we construct $\{v_\mu\} \subset H^3$, $v_\mu \to v$ as $\mu \to 0$ and $\mu |\nabla^3 v_\mu|^2 \le \overline{M}$ for some $\overline{M} \ge 0$. Let also $\{v_\lambda\} \subset H^3$ such that $v_\lambda \to v$ as $\lambda \to 0$ and $\lambda |\nabla^3 v_\lambda|^2 \le \overline{M}_1$ for some $M_1 \ge 0$. Then

$$
\begin{aligned}
\left| G_\mu(t) v_\mu - G(t) v \right|_\mu &\le \left| G(t) v - G(t) v_\lambda \right|_\mu + \left| G(t) v_\lambda - G_\mu(t) v_\lambda \right|_\mu \\
&\quad + \left| G_\mu(t) v_\lambda - G_\mu(t) v_\mu \right|_\mu \\
&\le \left| G(t) v - G(t) v_\lambda \right|_1 + \left| G(t) v_\lambda - G_\mu(t) v_\lambda \right|_1 \\
&\quad + e^{\hat{\omega}_0 |t|} \left| v_\lambda - v_\mu \right|_\mu
\end{aligned}
$$

if $\varphi_{k,\mu}(v_\lambda) \le \alpha_k$, $\varphi_{k,\mu}(v_\mu) \le \alpha_k$ for each $k = 0, 1, 2$.

From the above relation it is seen that $G_\mu(t) v_\mu \to G(t) v$ in L^2 as $\mu \to 0$. We also see that

$$
\begin{aligned}
\varphi_{0,\mu}(G_\mu(t) v_\mu) &\le \varphi_{0,\mu}(v_\mu) \\
\varphi_{1,\mu}(G_\mu(t) v_\mu) &\le \varphi_{1,\mu}(v_\mu) \\
\varphi_{2,\mu}(G_\mu(t) v_\mu) &\le e^{\hat{a}|t|} \left(\varphi_{2,\mu}(v_\mu) + \hat{b} |t| \right)
\end{aligned}
$$

and so $\varphi_{k,\mu}(G_\mu(t) v_\mu) \le \gamma_k$, for some $\gamma_k \in \mathbb{R}$, $k = 0, 1, 2$ and $t \in [-\tau, \tau]$, $\tau > 0$. In view of this, there exists $\beta > 0$ such that $|\nabla^2 G_\mu(t) v_\mu| \le \beta$ for each $\mu > 0$ and $t \in [-\tau, \tau]$. Since

$$
\left| \nabla (G_\mu(t) v_\mu - G(t) v) \right|^2 \le \left| \nabla^2 G_\mu(t) v_\mu - \nabla^2 G(t) v \right| \left| G_\mu(t) v_\mu - G(t) v \right|
$$

we have that $G_\mu(t) v_\mu \to G(t) v$ in H^1 as $\mu \to 0$, which finishes the proof. □

References

[1] J. Avrin and J. A. Goldstein *Global existence for the Benjamin-Bona-Mahony equation in arbitrary dimensions*, Nonlinear Anal., **9** (1985), 861-865.

[2] T. Benjamin, J. Bona and J. Mahony *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London, Ser. A, **272** (1972), 47-78.

[3] T. Benjamin *Lectures on nonlinear wave motion*, Lectures in Applied Mathematics vol. **15**, 3-47, AMS, Providence, R.I. ,1974.

[4] J. Bona and H. Chen *Comparison of model equations for small-amplitude long waves*, Nonlinear Anal., **38** (1999), 625-647.

[5] J. Bona and S. Smith *The initial-value problem for the Korteweg-deVries equation* Philos. Trans. Roy. Soc. London, Ser. A, **278** (1975), 555-601.

[6] I. Ciorănescu *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Math. and Its Appl., vol. **62**, Kluwer, Dordrecht, 1990.

[7] P. Georgescu and S. Oharu, *Generation and characterization of locally Lipschitzian semigroups associated with semilinear evolution equations*, to appear in Hiroshima Math. J. .

[8] J. A. Goldstein, S. Oharu and T. Takahashi *A semigroup approach to the generalized KdV equation*, preprint.

[9] J. A. Goldstein, R. Kajikya and S. Oharu *On some nonlinear dispersive equations in several space variables*, Differential and Integral Equations, **3** 1990, 617-632.

[10] T. Iwamiya, S. Oharu and T. Takahashi *On the semigroup approach to some nonlinear dispersive equations*, Numerical Analysis of Evolution Equations (Kyoto, 1978), in Lecture Notes in Num. Appl. Anal. , vol. **1**, 95-134, Kinokuniya Book Store Co., Tokyo, Japan, 1979.

[11] Y. Kametaka *Korteweg-deVries equation I-IV* Proc. Japan Acad., **45** (1969), 552-558, 656-665.

[12] Y. Kobayashi *Difference approximations of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups* , J. Math. Soc. Japan, **27** (1975), 640-665.

[13] K. Kobayasi, Y. Kobayashi and S. Oharu *Nonlinear evolution operators in Banach spaces* Osaka J. Math, **21** (1984), 281-310.

[14] T. Matsumoto, S. Oharu and H. R. Thieme *Nonlinear perturbations of a class of integrated semigroups*, Hiroshima Math. J., **26** (1996), 433-473.

[15] L. A. Medeiros and G. Menzala *Existence and uniqueness for periodic solutions of the Benjamin-Bona-Mahony equation*, Siam J. Math. Anal., **8** (1977), 792-799.

[16] L. A. Medeiros and M. Miranda *Weak solutions for a nonlinear dispersive equation*, J. Math. Anal. Appl., **59** (1977), 432-441.

[17] S. Oharu and T. Takahashi *Characterization of nonlinear semigroups associated with semilinear evolution equations*, Trans. Amer. Math. Soc., **311** (1989), 593-619.

[18] S. Oharu and T. Takahashi *On nonlinear evolution operators associated with some nonlinear dispersive equations*, Proc. Amer. Math. Soc., **97** (1986), 139-145.

[19] S. Oharu and T. Takahashi *On some nonlinear dispersive systems and the associated nonlinear evolution operators*, Recent Topics in Nonlinear PDE (Hiroshima, 1983) , in Lecture Notes in Num. Appl. Anal., vol. **6**, 125-142, Kinokuniya Book Store Co. , Tokyo, Japan, 1983.

[20] N. H. Pavel *Differential Equations, Flow Invariance and Applications*, Research Notes in Math., vol. **113**, Pitman, London, 1984.

[21] N. H. Pavel *Nonlinear evolution equations governed by f-quasidissipative operators*, Nonlinear Anal., **5** (1981), 449-468.

[22] S. I. Sobolev *Sur les Équationes aux Deriveés Partielles Hyperboliques Non-Lineaires*, Edizione Cremonese, Roma, 1961.

[23] T. Takahashi *On the pseudo-parabolic regularization for the generalized Kortweg-deVries equation*, Proc. Japan Acad., **55** (1979), 290-292.

[24] M. Tsutsumi and T. Mukasa *Parabolic regularization of the generalized Kortweg-deVries equation*, Funkcialaj Ekvacioj, **14** (1971), 89-110.

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