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Flow invariance for semilinear evolution equations under generalized dissipativity conditions

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Abstract

Let X be a real Banach space, let $A : D(A) \subset X \to X$ be a linear operator which is the infinitesimal generator of a (C_0)-semigroup and let $B : D \subset X \to X$ be a nonlinear perturbation which is continuous on level sets of D with respect to a lower semicontinuous (l.s.c.) functional φ. We discuss the existence of a nonlinear semigroup S providing mild solutions to the semilinear abstract Cauchy problem

$$(\mathrm{SP}; x)\ u'(t) = (A + B)u(t), \quad t > 0; \quad u(0) = x \in D$$

and satisfying a certain Lipschitz-like estimation and an exponential growth condition. Using the discrete schemes approximation, it is proved that the combination of a subtangential condition and a semilinear stability condition in terms of a metric-like functional is necessary and sufficient for the generation of such a semigroup S.

MSC: 47H20; 47H06

Keywords: Semilinear evolution equations; Flow invariance; Generalized dissipativity; Discrete schemes; Subtangential condition; Stability condition

1. Introduction

The aim of this paper is to study the generation of nonlinear semigroups associated with semilinear problems of the form

$$(\mathrm{SP}; x) \begin{cases} u'(t) = (A + B)u(t), & t > 0; \\ u(0) = x \in D, \end{cases}$$

where $A : D(A) \subset X \to X$ is a linear operator which generates a (C_0)-semigroup $T = \{T(t); t \geq 0\}$ on a real Banach space $(X, |\cdot|)$ and $B : D \subset X \to X$ is a nonlinear perturbation which is continuous on level sets D_α of D with respect to a l.s.c. functional φ, in terms of necessary and sufficient conditions. Here $D_\alpha = \{x \in D; \varphi(x) \leq \alpha\}$ for $\alpha > 0$ and it is supposed that the effective domain $D(\varphi) = \{x \in X; \varphi(x) < \infty\}$ of φ is contained in D.

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This framework has been found useful for the treatment of a wide class of nonlinear evolution equations because of its universality, with applications ranging from nonlinear diffusion and convection phenomena to muscle contraction or population dynamics (see, for instance, [10,15,16] or [17]). It is therefore important for both theoretical and practical reasons to treat problems of type (SP; x) through the use of perturbation arguments and obtain semigroup generation results, which in concrete problems amount to validating the model and obtaining regularity properties of the solutions.

As problems of type (SP; x) may not necessarily admit strong or classical solutions, the weaker notion of a mild solution, defined using the variation of constants formula, is employed throughout this paper.

Using the discrete schemes approximation, it is then proved that the generation of a nonlinear semigroup S on D providing mild solutions to (SP; x) and satisfying an exponential growth condition in terms of the l.s.c. functional φ together with a quasi-Lipschitz condition in terms of a metric-like functional V is assured by the coupling of two necessary and sufficient conditions, namely a subtangential condition (ST) and a semilinear stability condition (S):

(ST) For $x \in D$ and $\varepsilon > 0$ there is a pair $(h, x_h) \in (0, \varepsilon] \times D$ such that

$$h^{-1}|T(h)x + hBx - x_h| \le \varepsilon \quad \text{and} \quad \varphi(x_h) \le e^{ah}(\varphi(x) + (b+\varepsilon)h).$$

(S) For $\alpha > 0$ there is an $\omega(\alpha) \in \mathbb{R}$ such that

$$\liminf_{h\downarrow 0} h^{-1}[V(T(h)x + hBx, T(h)y + hBy) - V(x, y)] \le \omega(\alpha)V(x, y)$$

for all $x, y \in D_\alpha$.

Here, $a, b > 0$. The above result is our main theorem, which is fully stated in Section 2. Section 3 is devoted to showing the necessity of our subtangential and semilinear stability conditions and to indicating the relation between condition (S) and some classical dissipativity assumptions. In Section 4, it is seen that the subtangential condition holds uniformly on level sets of D with respect to φ and the construction of ε-approximate solutions is also indicated. The local mild solution to (SP; x) is then obtained as a uniform limit of ε-approximate solutions by using an important coupling estimate given in Section 5. The proof of the limiting result does not employ the classical method of Crandall and Liggett, but rather the construction of a forward difference scheme and of certain auxiliary sequences of discrete approximate solutions together with coupling estimates, and is given in Section 6. This method is specifically tailored to the semilinear character of the problem under consideration. Finally, the global existence of the mild solution is obtained via a classical argument.

We use the approach given in [8,3], papers to which our work is related. However, it is to be noted that the construction of elements (z_k^ε) and $(z_k^{\hat{\varepsilon}})$ in the proof of our Theorem 5.1 differs from those indicated in [8, Proposition 5.1] or [3, Theorem 6.1]. Consequently, the convergence estimations are somewhat simpler and some technicalities may be avoided, particularly the refining argument indicated in the proof of [3, Theorem 6.1].

Stability conditions similar to (S) are used in [8] to treat semilinear Cauchy problems for $V(x, y) = |x - y|$, while subtangential conditions of type (ST) are employed in [18] to treat time-dependent dissipative perturbations of generators of (C_0)-semigroups. In the above-mentioned papers, these conditions are global rather than local, i.e., the l.s.c. functional φ is not employed.

In this regard, the l.s.c. functional φ is usually constructed having in mind the particularities of the system rather than through a general technique, so that the estimations in terms of φ would correspond to a priori estimations, energy estimations, total population size estimations or other types of estimations which insure the global existence of the solutions together with their asymptotic properties. Other hints as to the expression of the functional φ for some particular reaction–diffusion systems may be provided by the shape of the positively invariant regions, if any. See [16] or [10] for nontrivial examples of functionals φ.

Metric-like functionals are used in [11–13] to introduce dissipativity-like conditions for nonlinear and continuous operators. The use of such functionals makes possible the treatment of evolution equations which involve operators that do not satisfy the classical dissipativity (or accretivity) conditions.

Another approach for treating problems of type (SP; x) is imposing compactness assumptions in place of dissipativity-like assumptions (such as the semilinear stability condition (S)) and obtaining the existence of mild solutions via a relative compactness argument instead of a discrete approximation one, but under these hypotheses one may obtain local existence results only. See, for instance [1], where a certain set of contractivity assumptions is required, on operator B, by means of the Hausdorff measure of noncompactness, or [2], where X is assumed to be reflexive and the semigroup generated by A is assumed to be compact. See also [19] for a survey of existence results for initial value problems featuring both dissipativity and compactness approaches, or [6] for related results concerning evolution equations in Hilbert spaces.

2. Main result

Let X be a real Banach space with norm $|\cdot|$ and let D be a subset of X. For $w \in \mathbb{R}$, denote $w^+ = \max\{w, 0\}$. Also, for $x_0 \in X$ and $r > 0$ denote by $B(x_0; r)$ the closed ball with center x_0 and radius r in X, that is, the set $\{x \in X; |x - x_0| \le r\}$. Let $\varphi : X \to [0, \infty]$ be a l.s.c. functional on X such that $D \subset D(\varphi)$, where $D(\varphi)$ is the effective domain of φ. Given $\alpha > 0$, we denote by D_α the level set $\{x \in D; \varphi(x) \le \alpha\}$.

To define a dissipativity-type condition, we shall employ a metric-like functional $V : X \times X \to [0, \infty)$ satisfying the following properties:

(V1) There is an $L > 0$ such that

$$|V(x, y) - V(\hat{x}, \hat{y})| \le L(|x - \hat{x}| + |y - \hat{y}|) \quad \text{for all } x, \hat{x}, y, \hat{y} \in X.$$

(V2) $V(x, x) = 0$ for all $x \in D$.

(V3) For each $\alpha > 0$ there are $c_1(\alpha) > 0$, $c_2(\alpha) > 0$ such that

$$c_1(\alpha)|x - y| \le V(x, y) \le c_2(\alpha)|x - y|$$

for all x, y in D_α.

We consider the semilinear problem (SP; x) mentioned in the introduction, where the operators $A : D(A) \subset X \to X$ and $B : D \to X$ satisfy hypotheses (A) and (B) mentioned below:

(A) A generates a (C_0)-semigroup $T = \{T(t); t \ge 0\}$ on X such that $|T(t)x| \le \mathrm{e}^{wt}|x|$ for all $x \in X$, $t \ge 0$ and some $w \in \mathbb{R}$.

(B) The level sets D_α are closed and $B : D \to X$ is continuous on D_α for all $\alpha > 0$.

It is then said that a function $u \in C([0, \infty); X)$ is a mild solution to (SP; x) if $u(t) \in D$ for $t \ge 0$, $Bu(\cdot) \in C([0, \infty); X)$ and the integral equality

$$u(t) = T(t)x + \int_0^t T(t - s)Bu(s)\mathrm{d}s$$

is satisfied for each $t \ge 0$.

Our main result can now be stated as follows:

Theorem 2.1. *Let $a, b > 0$. Suppose that the operators A and B satisfy hypotheses* (A) *and* (B)*, respectively, and that the metric-like functional V satisfies* (V1)–(V3). *The following statements are then equivalent:*

(I) *There is a nonlinear semigroup $S = \{S(t); t \ge 0\}$ on D such that the following integral equality, quasi-Lipschitz estimation and exponential growth condition are satisfied:*

(M) $S(t)x = T(t)x + \int_0^t T(t - s)BS(s)x\mathrm{d}s$ *for all $x \in D$ and $t \ge 0$.*

(QL) *For all $\alpha > 0$ and $\tau > 0$ there is $w_1 = w_1(\alpha, \tau) \in \mathbb{R}$ such that*

$$V(S(t)x, S(t)y) \le \mathrm{e}^{w_1(\alpha,\tau)t}V(x, y) \quad \text{for all } x, y \in D_\alpha \text{ and } t \in [0, \tau].$$

(GC) $\varphi(S(t)x) \le \mathrm{e}^{at}(\varphi(x) + bt)$ *for all $x \in D$ and $t \ge 0$.*

(II) *The semilinear operator $A + B$ satisfies the following subtangential and semilinear stability condition:*

(ST) *For all $x \in D$ and $\varepsilon > 0$ there is $(h, x_h) \in (0, \varepsilon] \times D$ such that*

$$h^{-1}|T(h)x + hBx - x_h| \le \varepsilon \quad \text{and} \quad \varphi(x_h) \le \mathrm{e}^{ah}(\varphi(x) + (b + \varepsilon)h).$$

(S) *For all $\alpha > 0$ there is $w = w(\alpha) \in \mathbb{R}$ such that*

$$\liminf_{h\downarrow 0} h^{-1}[V(T(h)x + hBx, T(h)y + hBy) - V(x, y)] \le w(\alpha)V(x, y)$$

for all $x, y \in D_\alpha$.

In the first statement, condition (M) states that for all $x \in D$ the function $t \mapsto S(t)x$ is a mild solution to (SP; x) on $[0, \infty)$, while condition (QL), apart from providing a quasi-Lipschitz estimate which characterizes the continuity of the nonlinear semigroup S on level sets of D, assures also the uniqueness of a mild solution to (SP; x). In the second statement, conditions (ST) and (S) replace the more usual coupling of a range condition and a quasidissipativity condition.

In the rest of our paper, T can be any (C_0)-semigroup satisfying condition (A). However, for computational simplicity, we shall assume in the following that T is a (C_0)-contraction semigroup. It is also to be noted that our result does not hold in the case in which T generates an arbitrary (C_0)-semigroup, that is, a semigroup which satisfies $|T(t)x| \le Ce^{wt}|x|$ for all $x \in X$, $t \ge 0$ and some $w \in \mathbb{R}$, $C \ge 1$. This happens because the standard renorming technique which is used in the linear case does not work here, since our stability condition (S) is norm dependent. Also, the iterative estimates employed in the proof of Theorem 6.1 would contain powers of C in the right-hand side, a fact which would make them inoperant.

For $A \equiv O$ and $\varphi \equiv 0$, conditions (ST) and (S) reduce to (ST′) $\liminf_{h\downarrow 0} h^{-1} d(x + hBx, D) = 0$ for all $x \in D$ and to (S′) $\langle Bx - By, x - y\rangle_s \le w|x - y|$ for all $x, y \in D$, respectively. However, in view of the remarks indicated after the statement of Lemma 4.2, condition (ST′) is equivalent to (ST″) $\lim_{h\downarrow 0} h^{-1} d(x + hBx, D) = 0$ for all $x \in D$. Since under these circumstances B is continuous on D and (ST″) holds, condition (S′) is equivalent to (S″) $\langle Bx - By, x - y\rangle_i \le w|x - y|$ for all $x, y \in D$ (see, for instance, [9, Lemma 4.1]). It is then seen that our main result, Theorem 2.1, contains as a particular case a well-known invariance result obtained by Martin [14, Theorem 5].

Also, our Theorem 2.1 may be regarded as a partial extension to the semilinear case of [12, Theorem 2.1], where the continuous case is treated under a more general growth condition in terms of a so-called uniqueness function. Moreover, it should be noted that in some situations it might be significantly easier to verify conditions of type (ST) rather than the usual range conditions, which amount in many concrete problems to solving elliptic equations and proving nontrivial elliptic estimates.

3. The subtangential and semilinear stability conditions: Necessity

In this section we prove the implication from (I) to (II).

Let $S = \{S(t); t \ge 0\}$ be a nonlinear semigroup on D satisfying (M), (QL) and (GC). Let also $x \in D$ and $\varepsilon > 0$. It is easy to see that condition (M) implies that

$$\lim_{h\to 0} h^{-1}|S(h)x - T(h)x - hBx| = 0 \tag{3.1}$$

and consequently there is $\delta_\varepsilon > 0$ such that $h^{-1}|S(h)x - T(h)x - hBx| < \varepsilon$ for all $h \in (0, \delta_\varepsilon]$. Also, $\varphi(S(h)x) \le e^{ah}(\varphi(x) + bh)$, so (ST) is satisfied with $x_h = S(h)x$.

Let $x, y \in D_\alpha$ and $\tau > 0$. Using (V1) and (QL) we remark that

$$\begin{aligned} h^{-1}[V(T(h)x + hBx, T(h)y + hBy) - V(x, y)] &\le h^{-1}(e^{w_1(\alpha,\tau)h} - 1)V(x, y) \\ &\quad + L(h^{-1}|S(h)x - T(h)x - hBx| + h^{-1}|S(h)y - T(h)y - hBy|) \end{aligned}$$

for all $h \in (0, \tau]$. Passing to limit as $h \downarrow 0$ in the above relation and using (3.1) we obtain that

$$\liminf_{h\downarrow 0} h^{-1}(V(T(h)x + hBx, T(h)y + hBy) - V(x, y)) \le \liminf_{h\downarrow 0} h^{-1}(e^{w_1(\alpha,\tau)h} - 1)V(x, y),$$

and so (S) is satisfied with $w(\alpha) = \liminf_{h\downarrow 0} w_1(\alpha, h)$. The implication from (I) to (II) is now completely proved.

We now make some remarks concerning the relation between (S) and some more usual dissipativity conditions. It is easily seen that if $x, y \in D(A) \cap D$, then

$$\liminf_{h\downarrow 0} h^{-1}[V(T(h)x + hBx, T(h)y + hBy) - V(x, y)] = D_+V(x, y)((A + B)x, (A + B)y),$$

where $D_+V(x, y)(\xi, \eta)$ is defined by

$$D_+V(x, y)(\xi, \eta) = \liminf_{h\downarrow 0} h^{-1}(V(x + h\xi, y + h\eta) - V(x, y)).$$

In view of the above equality, (S) may be interpreted as a dissipativity-like condition. Also, it may be seen that (S) is satisfied if the following condition holds:

(VQD) For all $\alpha > 0$ there is $w_2(\alpha) \in \mathbb{R}$ such that

$$D^- V(x, y)(Bx, By) \le w_2(\alpha) V(x, y) \quad \text{for all } x \in D_\alpha$$

(that is, B is "V-quasidissipative" on level sets of D with respect to α), where $D^- V(x, y)(\xi, \eta)$ is defined by

$$D^- V(x, y)(\xi, \eta) = \limsup_{h \uparrow 0} h^{-1}(V(x + h\xi, y + h\eta) - V(x, y)).$$

In the proofs of the above results, a crucial role is played by the estimations indicated in conditions (V3).

4. Construction of approximate solutions

We now prove the implication from (II) to (I), that is, our nonlinear semigroup generation theorem. In this section we discuss the construction of approximate solutions for the Cauchy problem (SP; x). To this end, we employ [3, Theorems 3.1 and 5.1] and [8, Lemmas 5.1 and 5.2], which read as follows.

Lemma 4.1 *([3, Theorem 3.1]). Suppose that condition* (ST) *is satisfied. Let $x \in D$, $\varepsilon \in (0, 1)$, $\beta > \varphi(x)$, $M \ge 0$ and let $r = r(x, \beta, \varepsilon) > 0$ be chosen such that*

$$|Bx - By| \le \varepsilon/4 \quad \text{and} \quad \sup_{s \in [0,r]} |T(s)Bx - Bx| \le \varepsilon/4 \quad \text{for each } y \in D_\beta \cap B(x, r), \tag{4.1}$$

and

$$|By| \le M \quad \text{for each } y \in D_\beta \cap B(x, r). \tag{4.2}$$

We define

$$H(x, \beta, \varepsilon) = \sup \left\{ \tau > 0; \tau(M + 1) + \sup_{s \in [0,\tau]} |T(s)x - x| \le r \text{ and } \mathrm{e}^{a\tau}(\varphi(x) + (b + \varepsilon)\tau) \le \beta \right\}. \tag{4.3}$$

Let $h \in [0, H(x, \beta, \varepsilon))$ and $y \in D$ satisfying

$$|y - T(h)x| \le h(M + 1) \quad \text{and} \quad \varphi(y) \le \mathrm{e}^{ah}(\varphi(x) + (b + \varepsilon)h). \tag{4.4}$$

Then for each $\eta > 0$ with $h + \eta \le H(x, \beta, \varepsilon)$ there is $z \in D_\beta \cap B(x, r)$ satisfying

$$(1/\eta)|z - T(\eta)y - \eta B(y)| \le \varepsilon \quad \text{and} \quad \varphi(z) \le \mathrm{e}^{a\eta}(\varphi(y) + (b + \varepsilon)\eta). \tag{4.5}$$

Lemma 4.2 *([3, Theorem 5.1]). Suppose that condition* (ST) *is satisfied. Let $x \in D$, $R > 0$, $\beta > \varphi(x)$ and let $M > 0$ be such that $|By| \le M$ for $y \in D_\beta \cap B(x, R)$. Let $\tau > 0$ and $\varepsilon_0 \in (0, 1)$ be chosen so that*

$$\tau(M + 1) + \sup_{t \in [0,\tau]} |T(t)x - x| \le R \quad \text{and} \quad \mathrm{e}^{a\tau}(\varphi(x) + (b + \varepsilon_0)\tau) < \beta.$$

Then for each $\varepsilon \in (0, \varepsilon_0]$ there exist a sequence $(t_i^\varepsilon)_{i=0}^{N_\varepsilon}$ and a sequence $(x_i^\varepsilon)_{i=0}^{N_\varepsilon}$ in $D_\beta \cap B(x, R)$ such that

(i) $t_0^\varepsilon = 0$, $x_0^\varepsilon = x$, $t_{N_\varepsilon}^\varepsilon = \tau$;
(ii) $0 < t_{i+1}^\varepsilon - t_i^\varepsilon \le \varepsilon$ *for* $0 \le i \le N_\varepsilon - 1$;
(iii) $|x_i^\varepsilon - T(t_i^\varepsilon)x| \le t_i^\varepsilon(M + 1)$ *and* $\varphi(x_i^\varepsilon) \le \mathrm{e}^{at_i^\varepsilon}(\varphi(x) + (b + \varepsilon)t_i^\varepsilon)$ *for* $0 \le i \le N_\varepsilon$;
(iv) $|x_{i+1}^\varepsilon - T(t_{i+1}^\varepsilon - t_i^\varepsilon)x_i^\varepsilon - (t_{i+1}^\varepsilon - t_i^\varepsilon)Bx_i^\varepsilon| \le (t_{i+1}^\varepsilon - t_i^\varepsilon)\varepsilon$ *and* $\varphi(x_{i+1}^\varepsilon) \le \mathrm{e}^{a(t_{i+1}^\varepsilon - t_i^\varepsilon)}(\varphi(x_i^\varepsilon) + (b + \varepsilon)(t_{i+1}^\varepsilon - t_i^\varepsilon))$ *for* $0 \le i \le N_\varepsilon - 1$;
(v) $x_i^\varepsilon \in D_\beta \cap B(x, R)$ *for* $0 \le i \le N_\varepsilon$;
(vi) *For* $0 \le i \le N_\varepsilon - 1$ *there is* $r_i^\varepsilon \in (0, \varepsilon]$ *such that* $|By - Bx_i^\varepsilon| \le \varepsilon/4$ *for* $y \in B(x_i, r_i^\varepsilon) \cap D_\beta$, $\sup_{t \in [0, r_i^\varepsilon]} |T(t)Bx_i^\varepsilon - Bx_i^\varepsilon| \le \varepsilon/4$ *and* $(t_{i+1}^\varepsilon - t_i^\varepsilon)(M + 1) + \sup_{t \in [0, t_{i+1}^\varepsilon - t_i^\varepsilon]} |T(t)x_i^\varepsilon - x_i^\varepsilon| \le r_i^\varepsilon$.

We note first that if we apply Lemma 4.1 with $h = 0$ and y satisfying (4.4) we obtain that for all $0 < \eta \le H(x, \beta, \varepsilon)$ there is $z \in D_\beta \cap B(x, r)$ satisfying (4.5), that is, the subtangential condition (ST) holds uniformly on level sets of D with respect to φ. This uniformity result plays a very important role in various parts of our proof, including the construction of approximate solutions and the proof of the coupling estimates leading to our limiting result. Actually, for our argument, the uniformity of the subtangential condition is a prerequisite.

Using the above-defined sequences $(t_i^\varepsilon)_{i=0}^{N_\varepsilon}$ and $(x_i^\varepsilon)_{i=0}^{N_\varepsilon}$, one may define an approximate solution $u_\varepsilon : [0, \tau) \to X$ as a piecewise continuous function by

$$u_\varepsilon(t) = T(t - t_i^\varepsilon)x_i^\varepsilon + (t - t_i^\varepsilon)Bx_i^\varepsilon \quad \text{for } t \in [t_i^\varepsilon, t_{i+1}^\varepsilon), 0 \le i \le N_\varepsilon - 1. \tag{4.6}$$

Lemma 4.3 *([7, Lemma 5.2]). Let $(\overline{s}_n)_{n\ge0}$ be a strictly increasing sequence in $[0, \infty)$ and let $(y_n)_{n\ge0}$ be a sequence in D. The following identity holds:*

$$\begin{aligned} y_n &= T(\overline{s}_n - \overline{s}_0)y_0 + \sum_{k=0}^{n-1}(\overline{s}_{k+1} - \overline{s}_k)T(\overline{s}_n - \overline{s}_{k+1})By_k \\ &\quad + \sum_{k=0}^{n-1} T(\overline{s}_n - \overline{s}_{k+1})[y_{k+1} - T(\overline{s}_{k+1} - \overline{s}_k)y_k - (\overline{s}_{k+1} - \overline{s}_k)By_k]. \end{aligned}$$

Lemma 4.4 *([7, Lemma 5.1]). Let $\overline{\varepsilon} > 0$, $M > 0$ and let $(\overline{s}_n)_{n\ge0} \subset [0, \infty)$ be a strictly increasing sequence in $[0, \infty)$, $(y_n)_{n\ge0}$ be a sequence in D such that $|By_n| \le M$ and*

$$|y_{n+1} - T(\overline{s}_{n+1} - \overline{s}_n)y_n - (\overline{s}_{n+1} - \overline{s}_n)By_n| \le (\overline{s}_{n+1} - \overline{s}_n)\varepsilon$$

for $n \ge 0$. If $\overline{s}_n \uparrow s$ as $n \to \infty$, then the sequence $(y_n)_{n\ge0}$ is convergent in X.

5. Coupling estimates

A first step towards the estimation of the difference of two approximate solutions u_ε and $u_{\hat\varepsilon}$ corresponding to different error parameters ε and $\hat\varepsilon$ is represented by the following lemma, which may be interpreted as a natural extension of the subtangential condition, since two base data y and $\hat y$ are involved, instead of a single one.

Lemma 5.1. *Let $x, \hat x \in D$ and assume that $\varepsilon, \hat\varepsilon, \beta, M = M(x, \hat x, \beta, \varepsilon, \hat\varepsilon)$, $r = r(x, \beta, \varepsilon)$, $\hat r = r(\hat x, \beta, \hat\varepsilon)$ are chosen such that*

$$|By - Bx| \le \varepsilon/4 \quad \text{and} \quad |By| \le M \quad \text{for } y \in D_\beta \cap B(x, r), \tag{5.1}$$

$$|By - B\hat x| \le \hat\varepsilon/4 \quad \text{and} \quad |By| \le M \quad \text{for } y \in D_\beta \cap B(\hat x, \hat r), \tag{5.1$'$}$$

$$\varepsilon, \hat\varepsilon \in (0, 1/3), \beta > \max\{\varphi(x), \varphi(\hat x)\} \tag{5.2}$$

$$\sup_{s\in[0,r]} |T(s)Bx - Bx| \le \varepsilon/4, \qquad \sup_{s\in[0,\hat r]} |T(s)Bx - Bx| \le \hat\varepsilon/4. \tag{5.3}$$

Let $h \in [0, H(x, \beta, \varepsilon))$, $\hat h \in [0, H(\hat x, \beta, \hat\varepsilon))$ and $y, \hat y \in D$ be chosen so that

$$|y - T(h)x| \le h(M + 1), \qquad \varphi(y) \le e^{ah}(\varphi(x) + (b + \varepsilon)h) \tag{5.4}$$

and

$$|\hat y - T(\hat h)\hat x| \le \hat h(M + 1), \qquad \varphi(\hat y) \le e^{a\hat h}(\varphi(\hat x) + (b + \hat\varepsilon)\hat h), \tag{5.4$'$}$$

where $H(x, \beta, \varepsilon)$, $H(\hat x, \beta, \hat\varepsilon)$ are defined as in (4.3). Then for each $\delta > 0$ and each $\eta > 0$ satisfying $h+\eta \le h(x, \beta, \varepsilon)$ and $\hat h + \eta \le h(\hat x, \beta, \hat\varepsilon)$ there exist $z \in D_\beta \cap B(x, r)$ and $\hat z \in D_\beta \cap B(\hat x, \hat r)$ such that

$$|z - T(\eta)y - \eta By| < 2\eta\varepsilon; \tag{5.5}$$

$$|\hat z - T(\eta)\hat y - \eta B\hat y| < 2\eta\hat\varepsilon; \tag{5.5$'$}$$

$$\varphi(z) \le e^{a\eta}(\varphi(y) + (b + \varepsilon)\eta); \tag{5.6}$$

$$\varphi(\hat z) \le e^{a\eta}(\varphi(\hat y) + (b + \hat\varepsilon)\eta), \tag{5.6$'$}$$

and the elements $z, \hat{z}$ satisfy

$$V(z,\hat{z}) \le e^{\omega(\beta)^+\eta}[V(y,\hat{y}) + \eta(L(\varepsilon+\hat{\varepsilon})+\delta)]. \tag{5.7}$$

Proof. First of all, we note that $y \in D \cap B(x,r)$ (and similarly $\hat{y} \in D \cap B(\hat{x},\hat{r})$), since

$$|y-x| \le |y-T(h)x| + |T(h)x-x| \le h(M+1) + |T(h)x-x| \le r.$$

We inductively construct sequences $(s_n)_{n\ge 0}$, $(x_n)_{n\ge 0}$, $(\hat{x}_n)_{n\ge 0}$ such that the following properties (i), (ii) and (vi) through (viii)′ are satisfied for all $n \ge 0$, while properties (iii) through (v) are satisfied for $n \ge 1$.

(i) $s_0 = 0$, $x_0 = y$, $\hat{x}_0 = \hat{y}$;
(ii) $0 < s_n < s_{n+1}$ and $\lim_{n\to\infty} s_n = \eta$;
(iii) $|x_n - T(s_n - s_{n-1})x_{n-1} - (s_n - s_{n-1})Bx_{n-1}| \le (s_n - s_{n-1})\varepsilon$;
(iii)′ $|\hat{x}_n - T(s_n - s_{n-1})\hat{x}_{n-1} - (s_n - s_{n-1})B\hat{x}_{n-1}| \le (s_n - s_{n-1})\hat{\varepsilon}$;
(iv) $\varphi(x_n) \le e^{a(s_n - s_{n-1})}(\varphi(x_{n-1}) + (b+\varepsilon)(s_n - s_{n-1}))$;
(iv)′ $\varphi(\hat{x}_n) \le e^{a(s_n - s_{n-1})}(\varphi(\hat{x}_{n-1}) + (b+\hat{\varepsilon})(s_n - s_{n-1}))$;
(v) $V(T(s_n - s_{n-1})x_{n-1} + (s_n - s_{n-1})Bx_{n-1}, T(s_n - s_{n-1})\hat{x}_{n-1} + (s_n - s_{n-1})B\hat{x}_{n-1}) \le e^{w(\beta)^+(s_n - s_{n-1})}[V(x_{n-1}, \hat{x}_{n-1}) + (s_n - s_{n-1})\delta]$;
(vi) $|x_n - T(s_n)x_0| \le s_n(M+1)$;
(vi)′ $|\hat{x}_n - T(s_n)\hat{x}_0| \le s_n(M+1)$;
(vii) $\varphi(x_n) \le e^{a(s_n+h)}(\varphi(x) + (s_n+h)(b+\varepsilon))$;
(vii)′ $\varphi(\hat{x}_n) \le e^{a(s_n+\hat{h})}(\varphi(\hat{x}) + (b+\hat{\varepsilon})(s_n+\hat{h}))$;
(viii) $x_n \in B(x,r) \cap D$;
(viii)′ $\hat{x}_n \in B(\hat{x},\hat{r}) \cap D$.

It is easy to see from our hypotheses, by setting $s_0 = 0$, $x_0 = y$, $\hat{x}_0 = \hat{y}$, that (i) and (vi) through (viii)′ are satisfied for $n = 0$. Suppose now that $(s_n)_{n=0}^N$, $(x_n)_{n=0}^N$ and $(\hat{x}_n)_{n=0}^N$ have been constructed in such a way that properties (i) and (iii) through (viii)′ hold.

Let $\overline{h}_N$ be the supremum of the positive numbers ξ such that $s_N + \xi \le \eta$ and

$$e^{-w(\beta)\xi} V(T(\xi)x_N + \xi Bx_N, T(\xi)\hat{x}_N + \xi B\hat{x}_N) \le V(x_N, \hat{x}_N) + \xi\delta. \tag{5.8}$$

It is seen from (ST) that $\overline{h}_N > 0$. Let $h_N \in (\overline{h}_N/2, \overline{h}_N)$ and set $s_{N+1} = s_N + h_N$. Since $\varphi(x_N) < \beta$ and $\varphi(\hat{x}_N) < \beta$, one may apply Lemma 4.1 and construct $x_{N+1}, \hat{x}_{N+1} \in D$ such that

$$|x_{N+1} - T(s_{N+1} - s_N)x_N - (s_{N+1} - s_N)Bx_N| \le (s_{N+1} - s_N)\varepsilon; \tag{5.9}$$

$$|\hat{x}_{N+1} - T(s_{N+1} - s_N)\hat{x}_N - (s_{N+1} - s_N)B\hat{x}_N| \le (s_{N+1} - s_N)\hat{\varepsilon} \tag{5.9′}$$

and

$$\varphi(x_{N+1}) \le e^{a(s_{N+1} - s_N)}(\varphi(x_N) + (b+\varepsilon)(s_{N+1} - s_N)); \tag{5.10}$$

$$\varphi(\hat{x}_{N+1}) \le e^{a(s_{N+1} - s_N)}(\varphi(\hat{x}_N) + (b+\hat{\varepsilon})(s_{N+1} - s_N)), \tag{5.10′}$$

that is, (iii), (iii)′, (iv), (iv)′ are satisfied for $n = N+1$. From (5.8), it is seen that (v) is satisfied for $n = N+1$ and using the estimations in (iv) and (iv)′ one may obtain by using an easy induction argument that (vii) and (vii)′ are also satisfied for $n = N+1$. From (5.1), (5.9) and (vi) for $n = N$ one may see that

$$\begin{aligned}|x_{N+1} - T(s_{N+1})x_0| &\le (s_{N+1} - s_N)M + |x_N - T(s_N)x_0| + (s_{N+1} - s_N)\varepsilon \\ &\le s_{N+1}(M+1)\end{aligned}$$

and similarly $|\hat{x}_{N+1} - T(s_{N+1})\hat{x}_0| < s_{N+1}(M+1)$, so (vi) and (iv)′ are also satisfied for $n = N+1$. Also, (vii) and (vii)′ easily imply that $x_{N+1}, \hat{x}_{N+1} \in D_\beta$. One then has

$$\begin{aligned}|x_{N+1} - x| &\le |x_{N+1} - T(s_{N+1})x| + |T(s_{N+1}+h)x - x| \\ &\le (s_{N+1}+h)(M+1) + |T(s_{N+1}+h)x - x| \le r(x,\beta,\varepsilon)\end{aligned}$$

and similarly $\hat{x}_{N+1} \in B(\hat{x}, \hat{r})$, so (viii) and (viii)′ are satisfied for $n = N + 1$. From Lemma 4.4, one may see that sequences $(x_n)_{n\geq 0}$ and $(\hat{x}_n)_{n\geq 0}$ are convergent in X to z and to $\hat{z}$, respectively. We now prove that $\lim_{n\to\infty} s_n = \eta$.

Suppose by contradiction that $\lim_{n\to\infty} s_n < \eta$. From (vii) and (vii)′, it is seen that $\varphi(z) < \beta$ and $\varphi(\hat{z}) < \beta$. Using (S), one finds $\xi \in (0, \eta - s)$ such that

$$\mathrm{e}^{-\omega(\beta)\xi} V(T(\xi)z + \xi Bz, T(\xi)\hat{z} + \xi B\hat{z}) \leq |z - \hat{z}| + (1/2)\xi\delta. \tag{5.11}$$

Let $N \geq 1$ such that $s - s_n \leq \xi/2$ for all $n \geq N$ and let $\xi_n = s - s_n + \xi$. Then $s_n + \xi_n = s + \xi < \eta$ and $\overline{h}_n < 2h_n < 2(s - s_n) < \xi_n$. It then follows from (5.8) that

$$\mathrm{e}^{-\omega(\beta)\xi_n} V(T(\xi_n)x_n + \xi_n Bx_n, T(\xi_n)\hat{x}_n + \xi_n B\hat{x}_n) > V(x_n, \hat{x}_n) + \xi_n\delta$$

and passing to limit as $n \to \infty$ in the above relation one obtains that

$$\mathrm{e}^{-\omega(\beta)\xi} V(T(\xi)z + \xi Bz, T(\xi)\hat{z} + \xi B\hat{z}) \geq V(z, \hat{z}) + \xi\delta,$$

which contradicts (5.11). We then infer from the above that $\lim_{n\to\infty} s_n = \eta$.

Using Lemma 4.3, (5.1), (5.3) and (4.3) one may deduce (5.5) (and similarly (5.5′)). Estimations (5.6) and (5.6′) may be deduced from (vii) and (vii)′. □

By the above lemma, it is also seen that condition (S) is equivalent to its seemingly stronger form

(S‴) For $\alpha > 0$ there is $\omega(\alpha) \in \mathbb{R}$ such that

$$\limsup_{h\downarrow 0} h^{-1}[V(T(h)x + hBx, T(h)y + hBy) - V(x, y)] \leq \omega(\alpha)V(x, y)$$

for all $x, y \in D_\alpha$.

We are now ready to establish our key estimation of the difference between two approximate solutions corresponding to different error parameters ε and $\hat{\varepsilon}$.

Theorem 5.1. *Let $x \in D$, $R > 0$, $\varepsilon_0 \in (0, 1/3)$ and $\beta > \varphi(x)$. Let $M > 0$ and $\tau > 0$ be such that $|By| \leq M$ for $y \in D_\beta \cap B(x, R)$, $\tau(M + 1) + \sup_{t\in[0,\tau]} |T(t)x - x| \leq R$ and $\mathrm{e}^{a\tau}(\varphi(x) + (b + \varepsilon_0)\tau) < \beta$. Let $\varepsilon, \hat{\varepsilon} \in (0, \varepsilon_0)$ and suppose that for $\varepsilon, \hat{\varepsilon}$ there are sequences $(t_i^\varepsilon)_{i=0}^{N_\varepsilon}$, $(x_i^\varepsilon)_{i=0}^{N_\varepsilon}$, and $(t_i^{\hat{\varepsilon}})_{i=0}^{N_{\hat{\varepsilon}}}$, $(x_i^{\hat{\varepsilon}})_{i=0}^{N_{\hat{\varepsilon}}}$, respectively, with the properties given in Lemma 4.2.*

Let $P = P_\varepsilon \cup P_{\hat{\varepsilon}}$, $P_\varepsilon = \{t_0^\varepsilon, t_1^\varepsilon, \ldots, t_{N_\varepsilon}^\varepsilon\}$, $P_{\hat{\varepsilon}} = \{t_0^{\hat{\varepsilon}}, t_1^{\hat{\varepsilon}}, \ldots, t_{N_{\hat{\varepsilon}}}^{\hat{\varepsilon}}\}$. Let $s_0 = 0$ and define a new partition $(s_k)_{k=0}^N$ of $[0, \tau]$ by $s_k = \min\{P \setminus \{s_0, s_1, \ldots, s_{k-1}\}\}$ for $k \geq 1$. Then there is a double sequence $(z_k, \hat{z}_k)_{k=0}^N$ such that

(i) *if $s_k = t_i^\varepsilon$, then $z_k = x_i^\varepsilon$; if $s_k = t_j^{\hat{\varepsilon}}$, then $\hat{z}_k = x_j^{\hat{\varepsilon}}$;*
(ii) *$\sum_{u=q}^k |z_u - T(s_u - s_{u-1})z_{u-1} - (s_u - s_{u-1})Bz_{u-1}| < 2\varepsilon + 4\varepsilon \sum_{t_u^\varepsilon\in\{s_q,\ldots,s_k\}} (t_u^\varepsilon - t_{u-1}^\varepsilon)$ for $1 \leq q \leq k$;*
(ii)′ *$\sum_{u=q}^k |\hat{z}_u - T(s_u - s_{u-1})\hat{z}_{u-1} - (s_u - s_{u-1})B\hat{z}_{u-1}| < 2\hat{\varepsilon} + 4\hat{\varepsilon} \sum_{t_u^{\hat{\varepsilon}}\in\{s_q,\ldots,s_k\}} (t_u^{\hat{\varepsilon}} - t_{u-1}^{\hat{\varepsilon}})$ for $1 \leq q \leq k$;*
(iii) *$\varphi(z_k) \leq \mathrm{e}^{as_k}(\varphi(x) + (b + \varepsilon)s_k)$;*
(iii)′ *$\varphi(\hat{z}_k) \leq \mathrm{e}^{as_k}(\varphi(x) + (b + \hat{\varepsilon})s_k)$;*
(iv) *$V(z_k, \hat{z}_k) \leq \mathrm{e}^{w(\beta)^+ s_k}[2L(\varepsilon + \hat{\varepsilon})s_k + \eta_k(\varepsilon, \hat{\varepsilon})]$, where $\eta_k(\varepsilon, \hat{\varepsilon}) = 4L(\varepsilon \sum_{t_u^\varepsilon\in\{s_1,\ldots,s_k\}} (t_u^\varepsilon - t_{u-1}^\varepsilon) + \hat{\varepsilon} \sum_{t_u^{\hat{\varepsilon}}\in\{s_1,\ldots,s_k\}} (t_u^{\hat{\varepsilon}} - t_{u-1}^{\hat{\varepsilon}}))$.*

Proof. Let $z_0 = \hat{z_0} = x$. From our hypotheses, it is easy to see that (i), (iii), (iii)′ and (iv) hold for $k = 0$. Suppose now that the finite sequence $(z_k, \hat{z}_k)_{k=0}^{l-1}$ has been constructed in such a way that (i) through (iv) hold for $0 \leq k \leq l-1$. Let i, j be such that $t_{i-1}^\varepsilon < s_l \leq t_i^\varepsilon$, $t_{j-1}^{\hat{\varepsilon}} < s_l \leq t_j^{\hat{\varepsilon}}$. We first prove that

$$|z_{l-1} - T(s_{l-1} - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon| \leq (s_{l-1} - t_{i-1}^\varepsilon)(M + 1) \tag{5.12}$$

and

$$\varphi(z_{l-1}) \leq \mathrm{e}^{a(s_{l-1}-t_{i-1}^\varepsilon)}(\varphi(x_{i-1}^\varepsilon) + (b + \varepsilon)(s_{l-1} - t_{i-1}^\varepsilon)). \tag{5.13}$$

If $s_{l-1} = t^{\varepsilon}_{i-1}$, the above relations are trivially satisfied. Suppose now that $s_{l-1} > t^{\varepsilon}_{i-1}$. Then $t^{\varepsilon}_{i-1} = s_p$ for some $p < l-1$. One sees that

$$\begin{aligned}|z_{p+1} - T(s_{p+1} - s_p)x^{\varepsilon}_{i-1}| &\le |z_{p+1} - T(s_{p+1} - s_p)x^{\varepsilon}_{i-1} - (s_{p+1} - s_p)Bx^{\varepsilon}_{i-1}| + (s_{p+1} - s_p)|Bx^{\varepsilon}_{i-1}| \\ &< 2(s_{p+1} - s_p)\varepsilon + M(s_{p+1} - s_p) < (M+1)(s_{p+1} - s_p),\end{aligned}$$

from which we deduce that $z_{p+1} \in B(x^{\varepsilon}_{i-1}, r^{\varepsilon}_{i-1})$ and so $|Bz_{p+1} - Bx^{\varepsilon}_{i-1}| < \varepsilon/4$. Similarly,

$$\begin{aligned}|z_{p+k} - T(s_{p+k} - s_p)x^{\varepsilon}_{i-1}| &\le |z_{p+k} - T(s_{p+k} - s_{p+k-1})z_{p+k-1} - (s_{p+k} - s_{p+k-1})Bz_{p+k-1}| \\ &\quad + (s_{p+k} - s_{p+k-1})(|Bx^{\varepsilon}_{i-1}| + |Bz_{p+k-1} - Bx^{\varepsilon}_{i-1}|) \\ &\quad + |T(s_{p+k} - s_{p+k-1})z_{p+k-1} - T(s_{p+k} - s_p)x^{\varepsilon}_{i-1}| \\ &< 2(s_{p+1} - s_p)\varepsilon + (s_{p+1} - s_p)/4 + |z_{p+k-1} - T(s_{p+k-1} - s_p)x^{\varepsilon}_{i-1}| \\ &\quad + M(s_{p+k} - s_{p+k-1})\end{aligned}$$

for $p + k \le l - 1$. By an induction argument one deduces that

$$|z_{p+k} - T(s_{p+k} - s_p)x^{\varepsilon}_{i-1}| < (M+1)(s_{p+k} - s_p)$$

for $p + k \le l - 1$ and so (5.12) is proved. By a similar argument it is seen that the exponential estimate (5.13) holds. It may also be seen that

$$|\hat{z}_{l-1} - T(s_{l-1} - t^{\hat{\varepsilon}}_{j-1})x^{\hat{\varepsilon}}_{j-1}| \le (s_{l-1} - t^{\hat{\varepsilon}}_{j-1})(M+1) \tag{5.12$'$}$$

and

$$\varphi(\hat{z}_{l-1}) \le \mathrm{e}^{a(s_{l-1} - t^{\hat{\varepsilon}}_{j-1})}(\varphi(x^{\hat{\varepsilon}}_{j-1}) + (b + \hat{\varepsilon})(s_{l-1} - t^{\hat{\varepsilon}}_{j-1})). \tag{5.13$'$}$$

Note that from the exponential estimates (5.13) and (5.13′) one may get respectively that

$$\varphi(z_{l-1}) \le \mathrm{e}^{as_{l-1}}(\varphi(x) + (b + \varepsilon)s_{l-1})$$

and

$$\varphi(\hat{z}_{l-1}) \le \mathrm{e}^{as_{l-1}}(\varphi(x) + (b + \hat{\varepsilon})s_{l-1}).$$

One may then apply Lemma 5.1 for $x = x^{\varepsilon}_{i-1}$, $\hat{x} = x^{\hat{\varepsilon}}_{j-1}$, $y = z_{l-1}$, $\hat{y} = \hat{z}_{l-1}$, $h = s_{l-1} - t^{\varepsilon}_{i-1}$, $\hat{h} = s_{l-1} - t^{\hat{\varepsilon}}_{j-1}$, $\eta = s_{l+1} - s_l$, $\delta = L(\varepsilon + \hat{\varepsilon})$ and find $y_l \in D_\beta \cap B(x^{\varepsilon}_{i-1}, r^{\varepsilon}_{i-1})$, $\hat{y}_l \in D_\beta \cap B(x^{\hat{\varepsilon}}_{j-1}, r^{\hat{\varepsilon}}_{j-1})$ such that

$$|y_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| \le 2(s_l - s_{l-1})\varepsilon; \tag{5.14}$$

$$|\hat{y}_l - T(s_l - s_{l-1})\hat{z}_{l-1} - (s_l - s_{l-1})B\hat{z}_{l-1}| \le 2(s_l - s_{l-1})\hat{\varepsilon}; \tag{5.14$'$}$$

$$V(y_l, \hat{y}_l) \le \mathrm{e}^{w(\beta)^+(s_l - s_{l-1})}[V(z_{l-1}, \hat{z}_{l-1}) + (s_l - s_{l-1})2L(\varepsilon + \hat{\varepsilon})]$$

and

$$\varphi(y_l) \le \mathrm{e}^{a(s_l - s_{l-1})}(\varphi(z_{l-1}) + (b + \varepsilon)(s_l - s_{l-1})); \tag{5.15}$$

$$\varphi(\hat{y}_l) \le \mathrm{e}^{a(s_l - s_{l-1})}(\varphi(\hat{z}_{l-1}) + (b + \hat{\varepsilon})(s_l - s_{l-1})). \tag{5.15$'$}$$

We now define elements z_l and $\hat{z}_l$ by $z_l = \begin{cases} y_l, & \text{if } s_l < t^{\varepsilon}_i \\ x^{\varepsilon}_i, & \text{if } s_l = t^{\varepsilon}_i \end{cases}$, and by $\hat{z}_l = \begin{cases} \hat{y}_l, & \text{if } s_l < t^{\hat{\varepsilon}}_j \\ x^{\hat{\varepsilon}}_j, & \text{if } s_l = t^{\hat{\varepsilon}}_j \end{cases}$, respectively, and attempt to prove that (ii) is satisfied for $k = l$. It is seen that

$$\begin{aligned}\sum_{u=q}^{l} |z_u - T(s_u - s_{u-1})z_{u-1} - (s_u - s_{u-1})Bz_{u-1}| &= \sum_{u=q}^{l-1} |z_u - T(s_u - s_{u-1})z_{u-1} - (s_u - s_{u-1})Bz_{u-1}| \\ &\quad + |z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| \\ &< 2\varepsilon(s_{l-1} - s_{q-1}) + 4\varepsilon \sum_{t^{\varepsilon}_u \in \{s_q, \ldots, s_{l-1}\}} (t^{\varepsilon}_u - t^{\varepsilon}_{u-1}) \\ &\quad + |z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}|.\end{aligned}$$

If $s_l < t_i^\varepsilon$, then

$$|z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| < 2\varepsilon(s_l - s_{l-1})$$

and

$$\sum_{t_u^\varepsilon \in \{s_q,\ldots,s_{l-1}\}} (t_u^\varepsilon - t_{u-1}^\varepsilon) = \sum_{t_u^\varepsilon \in \{s_q,\ldots,s_l\}} (t_u^\varepsilon - t_{u-1}^\varepsilon)$$

and so (ii) is satisfied for $k = l$. Suppose now that $s_l = t_i^\varepsilon$. One then has

$$\begin{aligned}|z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| &= |x_i^\varepsilon - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| \\ &\le |y_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| + |y_l - x_i^\varepsilon| \\ &\le 2(s_l - s_{l-1})\varepsilon + |y_l - T(s_l - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon - (s_l - t_{i-1}^\varepsilon)Bx_{i-1}^\varepsilon| \\ &\quad + |x_i^{\hat{\varepsilon}} - T(s_l - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon - (s_l - t_{i-1}^\varepsilon)Bx_{i-1}^\varepsilon|.\end{aligned}$$

Also,

$$\begin{aligned}|y_l - T(s_l - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon - (s_l - t_{i-1}^\varepsilon)Bx_{i-1}^\varepsilon| &\le \sum_{u=p}^{l-1} |\overline{z}_{u+1} - T(s_{u+1} - s_u)\overline{z}_u - (s_{u+1} - s_u)B\overline{z}_u| \\ &\quad + \sum_{u=p}^{l-1} (s_{u+1} - s_u)|B\overline{z}_u - Bx_{i-1}^\varepsilon| \\ &\quad + \sum_{u=p}^{l-1} (s_{u+1} - s_u)|T(s_l - s_{u+1})Bx_{i-1}^\varepsilon - Bx_{i-1}^\varepsilon|,\end{aligned}$$

where $\overline{z}_u = \begin{cases} y_l, & \text{if } u = l \\ z_u, & p \le u \le l-1 \end{cases}$, which yields

$$|y_l - T(s_l - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon - (s_l - t_{i-1}^\varepsilon)Bx_{i-1}^\varepsilon| \le (5/2)\varepsilon(s_l - t_{i-1}^\varepsilon)$$

and therefore

$$|z_l - T(s_l - s_{l-1})z_{l-1} - (s_l - s_{l-1})Bz_{l-1}| \le 2(s_l - s_{l-1})\varepsilon + 4(s_l - t_{i-1}^\varepsilon),$$

from which we infer that (ii) is also satisfied in this case. Condition (ii)$'$ follows in the same manner. Summarizing the above proof, we observe that we have also proved that $|y_l - x_i^\varepsilon| \le 4\varepsilon(s_l - t_i^\varepsilon)$ and $|\hat{y}_l - x_i^{\hat{\varepsilon}}| \le 4\hat{\varepsilon}(s_l - t_i^{\hat{\varepsilon}})$, from which (iv) follows by an easy induction argument.

A construction related to ours was performed in [20, Theorem 3] by means of a procedure which uses a maximum principle in ordered metric spaces rather than our supremum technique. □

6. The existence of the mild solution

As indicated in Section 4, for a given error parameter $\varepsilon > 0$ one may define an approximate solution $u_\varepsilon : [0, \tau) \to X$ by

$$u_\varepsilon(t) = T(t - t_i^\varepsilon)x_i^\varepsilon + (t - t_i^\varepsilon)Bx_i^\varepsilon \quad \text{for } t \in [t_i^\varepsilon, t_{i+1}^\varepsilon),\ 0 \le i \le N_\varepsilon - 1, \tag{6.1}$$

the sequences $(t_i)_{i=0}^{N_\varepsilon}$, $(x_i)_{i=0}^{N_\varepsilon}$ and the positive real number τ being given by Lemma 4.2. Using our key estimate for the difference between two approximate solutions u_ε and $u_{\hat{\varepsilon}}$ corresponding to different error parameters ε and $\hat{\varepsilon}$ given in Theorem 5.1, we obtain the uniform convergence of $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \downarrow 0$ on compact subintervals in $[0, \tau)$. The local mild solution on $[0, \tau)$ is then defined as being the uniform limit of $(u_\varepsilon)_{\varepsilon>0}$.

Theorem 6.1. *Let $x \in D$, $R > 0$, $\varepsilon_0 \in (0, 1/3)$ and $\beta > \varphi(x)$. Let $M > 0$ and $\tau > 0$ be such that $|Bx| \le M$ for $y \in D_\beta \cap B(x, R)$, $\tau(M + 1) + \sup_{t\in[0,\tau]} |T(t)x - x| \le R$ and $e^{a\tau}(\varphi(x) + (b + \varepsilon_0)\tau) < \beta$. Let $\varepsilon \in (0, \varepsilon_0)$ and suppose that the sequences $(t_i)_{i=0}^{N_\varepsilon}$ and $(x_i)_{i=0}^{N_\varepsilon}$ have been constructed with the properties given by Lemma 4.2. Let $u_\varepsilon : [0, \tau) \to X$ be the approximate solution given by (4.6). Then there is a continuous function*

$u : [0, \tau) \to X$ *such that* $\sup_{t\in[0,\tau)} |u_\varepsilon(t) - u(t)| \to 0$ *as* $\varepsilon \downarrow 0$. *Moreover,* u *is a mild solution to* (SP; x) *on* $[0, \tau)$ *satisfying* (QL) *and* (GC).

Proof. Let $\varepsilon, \hat{\varepsilon} \in (0, \varepsilon_0)$. Let $t \in [0, \tau)$ and let $k \geq 1$ be such that $t \in [s_{k-1}, s_k)$, where $(s_k)_{k=0}^n$ is defined as in Theorem 5.1. Let i, j be such that $t_{i-1}^\varepsilon \leq s_{k-1} < s_k \leq t_i^\varepsilon$, $t_{j-1}^{\hat{\varepsilon}} \leq s_{k-1} < s_k \leq t_j^{\hat{\varepsilon}}$. One then has

$$\begin{aligned} V(u^\varepsilon(t), u^{\hat{\varepsilon}}(t)) &\leq V(z_{k-1}, \hat{z}_{k-1}) + L(|T(t - t_{i-1}^\varepsilon)x_{i-1}^\varepsilon + (t - t_{i-1}^\varepsilon)Bx_{i-1}^\varepsilon - z_{k-1}| \\ &\quad + |T(t - t_{j-1}^{\hat{\varepsilon}})x_{j-1}^{\hat{\varepsilon}} + (t - t_{j-1}^{\hat{\varepsilon}})Bx_{j-1}^{\hat{\varepsilon}} - \hat{z}_{k-1}|) \\ &\leq V(z_{k-1}, \hat{z}_{k-1}) + L(M+1)(s_{k-1} - t_{i-1}^\varepsilon) + L(M+1)(s_{k-1} - t_{j-1}^{\hat{\varepsilon}}). \end{aligned}$$

It is then seen that

$$\begin{aligned} V(u^\varepsilon(t), u^{\hat{\varepsilon}}(t)) &\leq e^{w(\beta)^+ s_{k-1}} \left[2L(\varepsilon + \hat{\varepsilon}) + 4L \left(\varepsilon \sum_{t_u^\varepsilon \in \{s_1,\ldots,s_k\}} (t_u^\varepsilon - t_{u-1}^\varepsilon) + \hat{\varepsilon} \sum_{t_u^{\hat{\varepsilon}} \in \{s_1,\ldots,s_k\}} (t_u^{\hat{\varepsilon}} - t_{u-1}^{\hat{\varepsilon}}) \right) \right] \\ &\quad + L(M+1)(t_{k-1} - t_{i-1}^\varepsilon) + L(M+1)(t_{k-1} - t_{j-1}^{\hat{\varepsilon}}) \\ &\leq 6Le^{w(\beta)^+ \tau}(\varepsilon + \hat{\varepsilon})\tau + L(M+1)(\varepsilon + \hat{\varepsilon}). \end{aligned}$$

It now follows from (V2) and (V3) that $(u_\varepsilon)_{\varepsilon>0}$ converges to a function $u(\cdot)$ uniformly on $[0, \tau)$. Define now $\gamma_\varepsilon = t_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $v_\varepsilon : [0, \tau) \to X$ by

$$v_\varepsilon(t) = T(t)x + \int_0^t T(t-s)Bu_\varepsilon(\gamma_\varepsilon(s))\mathrm{d}s. \tag{6.2}$$

It is easily seen that v_ε is continuous on $[0, \tau)$. For $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$, with the help of Lemma 4.3, one obtains that

$$\begin{aligned} u_\varepsilon(t) - v_\varepsilon(t) &= \sum_{k=0}^{i-1} T(t - t_{k+1}^\varepsilon)[x_{k+1}^\varepsilon - T(t_{k+1}^\varepsilon - t_k^\varepsilon)x_k^\varepsilon - (t_{k+1}^\varepsilon - t_k^\varepsilon)Bx_k^\varepsilon] \\ &\quad - \sum_{k=0}^{i-1} \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} [T(t-s)Bx_k^\varepsilon - T(t - t_{k+1}^\varepsilon)Bx_k^\varepsilon]\mathrm{d}s - \int_{t_i^\varepsilon}^t [T(t-s)Bx_i^\varepsilon - Bx_i^\varepsilon]\mathrm{d}s \end{aligned} \tag{6.3}$$

and so

$$|u_\varepsilon(t) - v_\varepsilon(t)| \leq (5/4)t\varepsilon. \tag{6.4}$$

Also,

$$\begin{aligned} |u_\varepsilon(\gamma_\varepsilon(t)) - u_\varepsilon(t)| &= |x_i^\varepsilon - T(t - t_i^\varepsilon)x_i^\varepsilon + (t - t_i^\varepsilon)Bx_\varepsilon^i| \\ &\leq (t - t_i^\varepsilon)M + |T(t - t_i^\varepsilon)x_i^\varepsilon - x_i^\varepsilon| \leq \varepsilon \end{aligned} \tag{6.5}$$

for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$, $i = 0, \ldots N_\varepsilon - 1$. It is also seen that

$$\varphi(u_\varepsilon(\gamma_\varepsilon(t))) = \varphi(x_i^\varepsilon) \leq e^{at_i^\varepsilon}(\varphi(x) + (b+\varepsilon)t_i^\varepsilon). \tag{6.6}$$

Since $u_\varepsilon(\gamma_\varepsilon(t))$ and $u(t)$ belong to D_β for each $t \in [0, \tau)$, the continuity of B on D_β asserts that

$$Bu_\varepsilon(\gamma_\varepsilon(t)) \to Bu(t) \quad \text{as } \varepsilon \to 0, \text{ uniformly on } [0, \tau). \tag{6.7}$$

From (6.2), (6.4), (6.5) and (6.7) it is seen that (M) is satisfied, while from (6.6) and the lower semicontinuity of φ it is seen that (GC) is satisfied. Estimation (QL) will be proved through the use of the next lemma, which also insures the uniqueness of the mild solution to (SP; x).

Lemma 6.1. *Suppose that the semilinear stability condition* (S) *holds. Let* $\alpha > 0$ *and* $x, y \in D_\alpha$. *Let* $u(\cdot)$ *and* $v(\cdot)$ *be mild solutions to* (SP; x) *and to* (SP; y)*, respectively, which satisfy* (GC). *Then*

$$V(u(t), v(t)) \leq e^{\omega(\beta)t}V(x, y) \tag{6.8}$$

for $\tau > 0$, $\beta > \mathrm{e}^{a\tau}(\alpha + b\tau)$ and all $t \in [0, \tau)$.

Proof. Let $t \in [0, \tau)$. Using the definition of a mild solution to (SP), one sees that

$$\begin{aligned} & h^{-1}[V(u(t+h), v(t+h)) - V(u(t), v(t))] \\ & \quad \le h^{-1}[V(T(h)u(t) + hBu(t), T(h)u(t) + hBv(t)) - V(u(t), v(t))] \\ & \qquad + h^{-1}L \int_t^{t+h} |T(t+h-s)Bu(s) - Bu(t)|\mathrm{d}s \\ & \qquad + h^{-1}L \int_t^{t+h} |T(t+h-s)Bv(s) - Bv(t)|\mathrm{d}s \end{aligned} \tag{6.9}$$

for $0 < h < \tau - t$. We note that

$$\varphi(u(s)) \le \mathrm{e}^{as}(\varphi(u(0)) + bs) \le \beta$$

and so $u(s) \in D_\beta$ for $s \in [0, \tau)$. Similarly, it is seen that $v(s) \in D_\beta$ for $s \in [0, \tau)$. Passing to the inferior limit in (6.9) as $h \downarrow 0$ and using the continuity of B on D_β we obtain that

$$D_+V(u(t), v(t)) \le \omega(\beta)V(u(t), v(t)) \quad \text{for } t \in [0, \tau)$$

from which we infer that (6.8) holds. □

From the above lemma we infer that (QL) is satisfied with $w_1(\alpha, \tau) = w(\beta)$. Using (V2) and Lemma 6.1, one also obtains that the mild solution to (SP; x) given by Theorem 6.1 is unique. The global existence of the mild solution to (SP; x) and its semigroup property now follow from a classical continuability argument (see, for instance, [3, Theorem 6.2] or [8, Proposition 5.2]). Theorem 6.1 is then completely proved. □

7. An example

In this section we discuss a simple example which can be treated in an operator theoretic fashion within the framework described above. See also [4,5,18] for related constructions.

Our purpose is to study the positivity of the solutions to the semilinear initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + g(u(t, x)), & t \ge 0, x \in \mathbb{R}; \\ u(0, x) = \mathrm{u}_0(x), & x \in \mathbb{R}, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian and bounded function for which $g(0) = 0$ and u_0 is a nonnegative initial datum such that $\mathrm{u}_0(x) \to 0$ as $|x| \to \infty$.

Let $X = C_0(\mathbb{R}) = \{\mathrm{u} \in C(\mathbb{R}); \mathrm{u}(x) \to 0 \text{ as } |x| \to \infty\}$, endowed with the classical supremum norm, defined by $\|\mathrm{u}\| = \sup_{x\in\mathbb{R}} |\mathrm{u}(x)|$, let $V : X \times X \to \mathbb{R}$, $V(\mathrm{u}, \mathrm{v}) = \|\mathrm{u} - \mathrm{v}\|$ and let $\varphi : X \to \mathbb{R}$, $\varphi(\mathrm{u}) = \|\mathrm{u}\|$.

Let $A : D(A) \subset X \to X$, $A\mathrm{u} = \mathrm{u}''$, where $D(A) = \{\mathrm{u} \in C^2(\mathbb{R}); \mathrm{u}, \mathrm{u}', \mathrm{u}'' \in X\}$. It is easy to see that A generates a positivity-preserving contraction semigroup $T = \{T(t); t \ge 0\}$ on X, given by

$$(T(t)\mathrm{u})(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4t}} \mathrm{u}(y)\mathrm{d}y.$$

Let $D = \{\mathrm{u} \in C_0(\mathbb{R}); \mathrm{u}(x) \ge 0 \text{ for all } x \in \mathbb{R}\}$ and let $B : D \to X$, $(B\mathrm{u})(x) = g(\mathrm{u}(x))$. It is also easy to see that B is a locally Lipschitzian operator on D. With the above notation, our problem can be rewritten as a semilinear problem in X, of the form

$$(\mathrm{SP}; \mathrm{u}_0)\ \mathrm{u}'(t) = (A + B)\mathrm{u}(t), \quad t > 0; \quad \mathrm{u}(0) = \mathrm{u}_0 \in D,$$

where $\mathrm{u}(t) = u(t, \cdot)$. Since T is a contraction semigroup on X, it is seen that

$$h^{-1}(\|T(h)\mathrm{u} + hB\mathrm{u} - T(h)\mathrm{v} - hB\mathrm{v}\| - \|\mathrm{u} - \mathrm{v}\|) \le w(\alpha)\|\mathrm{u} - \mathrm{v}\|$$

for all $h > 0$ and $\mathrm{u}, \mathrm{v} \in D_\alpha$, where $w(\alpha)$ is the Lipschitz constant of B on D_α, which implies that (S) is satisfied.

Let $\mathrm{u} \in D$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that $|t| \le \delta$ implies $|g(t)| \le \varepsilon$. Also, since g is bounded on $\mathbb{R}$, there is $\eta > 0$ such that $\inf_{x\in\mathbb{R}} g(\mathrm{u}(x)) + \delta/(2h) \ge 0$ and $\|T(h)\mathrm{u} - \mathrm{u}\| \le \delta/2$ for all $h \in (0, \eta)$.

Fix $x \in \mathbb{R}$. If $\mathrm{u}(x) \le \delta$, then

$$h^{-1}[(T(h)\mathrm{u})(x) + hg(\mathrm{u}(x))] \ge g(\mathrm{u}(x)) \ge -\varepsilon.$$

If $\mathrm{u}(x) > \delta$, then

$$\begin{aligned} h^{-1}[(T(h)\mathrm{u})(x) + hg(\mathrm{u}(x))] &= h^{-1}[(T(h)\mathrm{u})(x) - \mathrm{u}(x)] + h^{-1}[\mathrm{u}(x) + g(\mathrm{u}(x))] \\ &\ge -h^{-1}\|T(h)\mathrm{u} - \mathrm{u}\| + h^{-1}\delta + g(\mathrm{u}(x)) \\ &\ge (2h)^{-1}\delta + g(\mathrm{u}(x)) \ge 0. \end{aligned}$$

Let us define $\mathrm{u}_h : \mathbb{R} \to \mathbb{R}$ by $\mathrm{u}_h(x) = [(T(h)\mathrm{u})(x) + hg(\mathrm{u}(x))]^+$ for $x \in \mathbb{R}$. Then $\mathrm{u}_h \in D$, $\|T(h)\mathrm{u} + hB\mathrm{u} - \mathrm{u}_h\| \le \varepsilon$ and $|\mathrm{u}_h(x)| \le |(T(h)\mathrm{u})(x) + hg(\mathrm{u}(x))| \le \|\mathrm{u}\| + h\|g\|$ for $x \in \mathbb{R}$, which implies that (ST) is also satisfied. In view of the above, it is seen that there is a nonlinear semigroup $S = \{S(t); t \ge 0\}$ on D associated with (SP), which means that for any nonnegative initial datum $\mathrm{u}_0 \in C_0(\mathbb{R})$ the function $t \mapsto S(t)\mathrm{u}_0$ is a nonnegative mild solution to (SP; u_0).

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