# GLOBAL EXISTENCE FOR MILD SOLUTIONS TO SEMILINEAR EVOLUTION EQUATIONS UNDER "GENERALIZED" DISSIPATIVITY CONDITIONS

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Abstract. Let $X$ be a real Banach space, let $A : D(A) \subset X \to X$ be the generator of a $(C_0)$-contraction semigroup on $X$ and let $B : D \subset [0,T) \times X \to X$ be a continuous operator. Under a combination of Pavel's subtangential condition, a semilinear stability condition defined in terms of a uniqueness function $w : [0,T) \times \mathbb{R} \to \mathbb{R}$ and suitable connectedness and closedness asumptions on the domain $D$ of the operator $B$, we prove the global existence of the mild solution to the equation $u' = Au + B(t,u)$. In our setting, no dissipativity property is assumed for the operator $B$.

## 1. Introduction. Statement of the main result

Let $X$ be a real Banach space with norm $|\cdot|$. We define the semi-inner products $[\cdot,\cdot]_-$ and $[\cdot,\cdot]_+$ on $X$ by $[x,y]_- = \lim_{h\uparrow 0}(|x+hy|-|x|)/h$, respectively by $[x,y]_+ = \lim_{h\downarrow 0}(|x+hy|-|x|)/h$. Given $r>0$ and $(t,x) \in \mathbb{R}\times X$, we define $S_r(t,x) = \{(s,y) \in \mathbb{R}\times X; |t-s| \le r \text{ and } |y-x| \le r\}$. For $x \in X$ and $S \subset X$, we also define the distance between $x$ and $S$ by $d(x,S) = \inf\{|y-x|\,; y \in S\}$.

We consider the semilinear problem

$$\begin{cases} u'(t) = Au(t) + B(t,u(t)), & 0 \le s < t < T \le +\infty; \\ u(s) = u_0. \end{cases} \qquad (\text{SP}; s, u_0)$$

It is assumed that $A$ and $B$ satisfy the following hypotheses:

(A) $A$ generates a $(C_0)$-contraction semigroup $T = \{T(t); t \ge 0\}$ on $X$;

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(B) $B : D \to X$ is a continuous operator,

$D$ being a subset of $[0,T) \times X$ which satisfies hypothesis $(D)$ below:

(D) a) $D(t) = \{x \in X; (t,x) \in D\} \neq \emptyset$ for all $t \in [0,T)$;
b) If $(t_n, x_n) \in D, t_n \uparrow t$ in $[0,T)$ and $x_n \to x$ in $X$ as $n \to \infty$, then $(t,x) \in D$;
c) $D$ is connected.

Given a function $w : [0,T) \times \mathbb{R} \to \mathbb{R}$, it is said that $w$ is a uniqueness function if it satisfies the following condition:

(U) $w(t,0) = 0$ for $t \in [0,T)$ and $r \equiv 0$ is the unique solution of the initial value problem

$$\begin{cases} r'(t) = w(t, r(t)), \quad 0 < t < T; \\ r(0) = 0. \end{cases}$$

We also assume that Pavel's subtangential condition is satisfied, that is,

$$\liminf_{h \downarrow 0} (1/h) d\left(T(h)x + hB(t,x), D(t+h)\right) = 0, \quad \text{for all } (t,x) \in D, \tag{ST}$$

together with the semilinear stability condition

$$\liminf_{h \downarrow 0} (1/h)(|T(h)(x-y) + h(B(t,x) - B(t,y))| - |x-y|) \leq w(t, |x-y|), \text{ for all } (t,x), (t,y) \in D, \tag{S}$$

where $w$ is a continuous and separately nondecreasing uniqueness function.

The semilinear stability condition (S) was first employed (for $B(t,x) \equiv B(x)$ and $w(t,x) \equiv wx$) by Iwamiya, Oharu and Takahashi in [4]. It is possible to prove, using essentially the same argument as in [4, Proposition 3.1], that if (ST) holds and $[B(t,x) - B(t,y), x-y]_- \leq w(t, |x-y|)$ for all $(t,x)$, $(t,y) \in D$ (that is, $B$ is dissipative with respect to a uniqueness function $w$), then (S) is satisfied for the same choice of uniqueness function $w$. Also, if (S) holds, it may be shown that, for all $(t,x), (t,y) \in D$ such that $x, y \in D(A)$,

$$\lim_{h \downarrow 0} (1/h)(|T(h)(x-y) + h(B(t,x) - B(t,y))| - |x-y|) = [x-y, (Ax + B(t,x)) - (Ay + B(t,y))]_+ ,$$

that is, the semilinear operator $A + B$ is strongly quasidisspative with respect to the uniqueness function $w$. See [4, Proposition 3.2] for a similar result regarding the autonomous case.

Let $I$ be a subinterval of $[0, T)$, $I = [s, c]$ or $I = [s, c)$. A continuous function from $I$ into $X$ is said to be a mild solution for (SP;$s, u_0$) on $I$ if it satisfies

$$u(t) = T(t-s)\,u_0 + \int_s^t T(t-\xi)\,B(\xi, u(\xi))\,d\xi \quad \text{for all } t \in I.$$

Our main result may now be stated as follows.

**Theorem 1.1.** *Suppose that conditions (A), (B), (D) are satisfied, together with the subtangential condition (ST) and the semilinear stability condition (S), and that $w$ is a continuous, separately nondecreasing function which satisfies (U). Then for each $(s, u_0) \in D$ the semilinear problem (SP;$s, u_0$) has a unique mild solution $u(\cdot; s, u_0)$ on $[s, T)$. Moreover, for any $(s, u_0)$ and $(s, \overline{u}_0) \in D$ and $\xi \in [s, \tau(s, |u_0 - \overline{u}_0|))$, one has*

$$|u(\xi; s, u_0) - u(\xi; s, \overline{u}_0)| \le m(\xi; s, |u_0 - \overline{u}_0|),$$

*where $m(\cdot; s, x)$ is the maximal solution of the initial value problem*

$$\begin{cases} r'(t) = w(t, r(t)), & s < t < T; \\ r(s) = x \end{cases}$$

*and $[s, \tau(s, |u_0 - \overline{u}_0|))$ is its maximal interval of existence.*

We briefly outline the main points of our argument. First, our argument involves the construction of discrete schemes consistent with our semilinear problem. In order to make full use of our hypotheses, we investigate the subtangential condition (ST) and show that it holds uniformly in a certain sense. Then, given a small parameter $\varepsilon$, we construct a time-discretizing sequence $(t_i)_{0\le i\le N}$ and a solution-discretizing sequence $(x_i)_{0\le i\le N}$ enjoying a number of fundamental quantitative properties. These sequences are then used to define the corresponding approximate solution $u_\varepsilon$ as a piecewise continuous function.

In order to estimate the difference between two approximate solutions corresponding to different small parameters $\varepsilon$ and $\widehat{\varepsilon}$, we establish a lemma which, applied repeatedly, enables us to construct discrete sequences which "intermediate" between $u_\varepsilon$ and $u_{\widehat{\varepsilon}}$ and whose difference can be estimated using an inductive argument. The convergence of a sequence of approximate solutions corresponding to a null sequence of parameters is then established using estimations which are derived via a well-known comparison principle for solutions of initial value problems associated to ordinary differential equations. The limit function is then shown to be a (local) mild solution of our semilinear problem, and the existence in the large follows from a "Lipschitz-like" estimation deduced using our semilinear stability condition (S) together with a result due to Iwamiya.

Previous related results were obtained by Iwamiya, Oharu and Takahashi in [4] for the autonomous case (that is, $B(t,x) \equiv B(x)$ and $w(t,x) \equiv wx$), by Georgescu and Oharu in [1] for $B(t,x) \equiv B(x)$ and $w(t,x) \equiv wx$, the continuity of the operator $B$ being localized by means of a lower semicontinuous functional $\varphi$, by Georgescu and Shioji in [2] for $B(t,x) \equiv B(x)$ and $w(t,x) \equiv w(x)$, $w$ being an increasing uniqueness function, by Pavel in [7] for $B(t,\cdot)$ $g(t)$-dissipative, $g : [0,T) \to \mathbb{R}$ being a nondecreasing function and by Iwamiya in [3] for $B$ satisfying $[B(t,x) - B(t,y), x-y]_- \leq w(t,|x-y|)$ for all $(t,x),(t,y) \in D$. The present paper is strongly connected to these works.

## 2. Comparison theorems

For convenience of future reference in the rest of the paper, we state some comparison results for solutions of initial value problems for ordinary differential equations.

Let $w : [0,T) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $(s,x) \in [0,T) \times X$, we shall denote by $m_\delta(t;s,x)$ the maximal solution of the initial value problem

$$\begin{cases} r'(t) = w(t,r(t)) + \delta, & s < t < T; \\ r(s) = x \end{cases}$$

and by $[s, \tau_\delta(s,x))$ its largest interval of existence. When $\delta = 0$, we shall sometimes omit the subscript $\delta$, since there is no danger of confusion.

The following basic comparison result ([6, Theorem 1.6.1]) holds.

**Lemma 2.1.** *Suppose that $\Omega \subset \mathbb{R}^2$ is open and $g \in C(\Omega)$. Let $(t_0,u_0) \in \Omega$ and let $[t_0, \tau(t_0,u_0))$ be the largest interval of existence on which the maximal solution $m(t;t_0,u_0)$ of the initial value problem*

$$\begin{cases} u'(t) = g(t,u(t)), & t > t_0; \\ u(t_0) = u_0 \end{cases}$$

*exists. Let* $x \in C([t_0, \tau(t_0, u_0)))$ *be such that* $(t, x(t)) \in \Omega$ *for* $t \in [t_0, \tau(t_0, u_0))$, $x(t_0) \leq u_0$ *and* $Dx(t) \leq g(t, x(t))$ *for some fixed Dini derivative* $D$ *and for* $t \in [t_0, \tau(t_0, u_0)) \setminus N$, $N$ *being an at most countable set. Then* $x(t) \leq m(t; t_0, u_0)$ *for* $t \in [t_0, \tau(t_0, u_0))$.

As a consequence, the following fundamental properties of the nonextendable maximal solution $m_\delta(t; s, x)$ may be obtained using an argument which is similar to the one employed in [5, Lemma 5.1].

**Lemma 2.2.** *Let* $\delta_0, \alpha_0 \geq 0$ *and let* $0 \leq t_0 < T$. *Then the following properties (i) through (iii) hold:*

*(i) If* $\alpha \geq \alpha_0$ *and* $\delta \geq \delta_0$, *then* $\tau_\delta(t_0, \alpha) \leq \tau_{\delta_0}(t_0, \alpha_0)$ *and* $m_\delta(t; t_0, \alpha) \geq m_{\delta_0}(t; t_0, \alpha_0)$ *for* $t \in [t_0, \tau_\delta(t_0, \alpha))$.

*(ii) If* $\alpha \downarrow \alpha_0$ *and* $\delta \downarrow \delta_0$, *then* $\tau_\delta(t_0, \alpha) \uparrow \tau_{\delta_0}(t_0, \alpha_0)$ *and* $m_\delta(t; t_0, \alpha) \downarrow m_{\delta_0}(t; t_0, \alpha_0)$ *uniformly on every compact subinterval of* $[0, \tau_\delta(t_0, \alpha))$.

*(iii) If* $0 \leq s < \tau_{\delta_0}(t_0, \alpha_0)$, *then* $\tau_{\delta_0}(t_0, \alpha_0) \leq \tau_{\delta_0}(s, m_{\delta_0}(s; s_0, \alpha_0))$ *and*

$$m_{\delta_0}(t; s, m_{\delta_0}(s; \alpha_0)) = m_{\delta_0}(t; s_0, \alpha_0) \qquad \text{for } t \in [s, \tau_{\delta_0}(t_0, \alpha_0)).$$

**Remark 2.1.** *The function* $g$ *in Lemma 2.1 (and consequently the function* $w$ *in Lemma 2.2) needs not be neither a uniqueness function nor separately nondecreasing; it suffices to be continuous. However, if* $w$ *is a continuous uniqueness function, then* $m(t; t_0, 0)$ *is defined on* $[t_0, T)$ *and* $m(t; t_0, 0) \equiv 0$.

Let us now particularize $w$ to be an uniqueness function, not necessarily increasing. Given $K > 0$, we define

$$w^K(t, x) = \begin{cases} w(t, x) & \text{if} \quad t \in [0, \tau) \text{ and } x \in [0, K]; \\ w(t, K) & \text{if} \quad t \in [0, \tau) \text{ and } x > K. \end{cases}$$

We shall also denote by $m_\delta^K(t; t_0, \alpha)$ the maximal solution of the initial value problem

$$\begin{cases} r'(t) = w^K(t, r(t)) + \delta, \quad t > t_0; \\ r(t_0) = \alpha. \end{cases}$$

We note that, since $w^K(\cdot, \cdot)$ is bounded on $[0, c] \times \mathbb{R}$ for any $0 < c < T$, $\tau_\delta^K(t_0, \alpha) = T$ for any $\delta, \alpha \in \mathbb{R}_+$. It is then seen that the following property, which is similar to [5, Lemma 3.5] holds.

**Lemma 2.3.** *Suppose that* $w$ *is an uniqueness function and let* $K > 0$. *The following properties (i) and (ii) hold.*

*(i) If* $0 \leq t_0 < T$, $\alpha \downarrow \alpha_0$ *and* $\delta \downarrow \delta_0$, *then* $m_\delta^K(t; t_0, \alpha) \downarrow m_{\delta_0}^K(t; t_0, \alpha_0)$ *uniformly on any compact subinterval of* $[0, T)$.

*(ii)* $m_0^K(t; t_0, 0) = 0$ *for* $t_0 \in [0, T)$ *and* $t_0 \leq t < T$.

Now, suppose that $w$ is a continuous function which is separately nondecreasing. One may see that the following result holds.

**Lemma 2.4.** *Let* $w : [0, T) \times \mathbb{R} \to \mathbb{R}$ *be a continuous function which is separately nondecreasing. Then* $m(t; t_0, u_0) + \alpha \leq m(t; t_0, u_0 + \alpha)$ *for all* $t_0 \in [t_0, T)$, $u_0, \alpha \in \mathbb{R}_+$ *and* $t \in [t_0, \tau(t_0, u_0))$.

*Proof.* Let us denote $u_1(t) = m(t; t_0, u_0) + \alpha$ and $u_2(t) = m(t; t_0, u_0 + \alpha)$. One then has

$$\begin{aligned} u_1'(t) &= m'(t; t_0, u_0) = w(t, m(t; t_0, u_0)) \leq w(t, u_1(t)); \\ u_2'(t) &= m'(t; t_0, u_0 + \alpha) = w(t, m(t; t_0, u_0 + \alpha)) = w(t, u_2(t)). \end{aligned}$$

Since $u_1(0) = u_2(0) = u_0 + \alpha$, one deduces the conclusion from Lemma 2.1. □

## 3. The construction of the approximate solutions

A first step towards the proof of our global existence result is to establish that the subtangential condition (ST) holds uniformly in a local sense.

**Theorem 3.1.** *Let* $(t, x) \in D, \varepsilon \in (0, 1)$ *and let* $r = r(t, x, \varepsilon)$ *be chosen such that* $|B(s, y) - B(t, x)| \leq \varepsilon/4$, $\sup_{\sigma \in [0,r]} |T(\sigma)B(t, x) - B(t, x)| \leq \varepsilon/4$ *and* $|B(s, y)| \leq M$ *for any* $(s, y) \in D \cap S_r(t, x)$ *and some* $M > 0$. *Define* $h(t, x, \varepsilon) = \sup\{h \in (0, T - t); h(M + 1) + \sup_{\sigma \in [0,h]} |T(\sigma)x - x| \leq r\}$ *and let* $h \in [0, h(t, x, \varepsilon))$, $y \in D(t + h)$ *satisfying* $|y - T(h)x| \leq M + 1$. *Then for each* $\eta > 0$ *with* $h + \eta \leq h(t, x, \varepsilon)$ *there is* $z \in D(t + \eta)$ *such that* $(t + \eta, z) \in D \cap S_r(t, x)$ *and* $|z - T(\eta)y - \eta B(t, y)| \leq \eta\varepsilon$.

*Proof.* See [3, Proposition 5.1]. Note that Iwamiya's extra assumption on the operator $B$ is not used in the proof of this result. □

**Remark 3.1.** *We note that the existence of* $y$ *in the above theorem is insured by condition (S). Also, if we let* $h = 0$ *and* $y = x$ *in the above theorem, it is seen that for every* $0 < \eta \leq h(t, x, \varepsilon)$ *there is* $z \in D(t + \eta)$ *such that* $(t + \eta, z) \in D \cap S_r(t, x)$ *and* $|z - T(\eta)x - \eta B(t, x)| \leq \eta\varepsilon$.

*This implies that our subtangential condition (ST) is actually equivalent to its apparently stronger form*

$$\limsup_{h\downarrow 0}(1/h)d\left(T\left(h\right)x+hB\left(t,x\right),D\left(t+h\right)\right)=0 \quad \text{for all } (t,x)\in D.$$

We now state two auxiliary results ([3, Lemma 5.1] and [3, Lemma 5.2]), which will be used to establish various convergence results throughout this paper.

**Lemma 3.1.** *Let $(\overline{s}_n, y_n)_{n\geq 0}$ be a sequence in $D$ such that $\overline{s}_n \leq \overline{s}_{n+1}$. The following identity holds:*

$$y_n = T\left(\overline{s}_n-\overline{s}_0\right)y_0+\sum_{k=0}^{n-1}\left(\overline{s}_{k+1}-\overline{s}_k\right)T\left(\overline{s}_n-\overline{s}_{k+1}\right)B(\overline{s}_k,y_k)$$
$$+\sum_{k=0}^{n-1}T\left(\overline{s}_n-\overline{s}_{k+1}\right)\left[y_{k+1}-T\left(\overline{s}_{k+1}-\overline{s}_k\right)y_k-\left(\overline{s}_{k+1}-\overline{s}_k\right)B(\overline{s}_k,y_k)\right].$$

**Lemma 3.2.** *Let $\varepsilon>0$ and $M>0$. Let $(\overline{s}_n, y_n)_{n\geq 0}$ be a sequence in $D$ such that $\overline{s}_n \leq \overline{s}_{n+1}$, $|B(\overline{s}_n,y_n)|\leq M$ and*

$$\left|y_{n+1}-T\left(\overline{s}_{n+1}-\overline{s}_n\right)y_n-\left(\overline{s}_{n+1}-\overline{s}_n\right)B(\overline{s}_n,y_n)\right|\leq\left(\overline{s}_{n+1}-\overline{s}_n\right)\varepsilon$$

*for $n\geq 0$. If $\overline{s}_n\uparrow s$ as $n\to\infty$, then the sequence $(y_n)_{n\geq 0}$ is a Cauchy sequence in $X$ and $\lim_{n\to\infty}(\overline{s}_n,y_n)=(s,y)\in D$.*

We now turn our attention to the construction of the approximate solution for (SP;$s,x$). First, for a given small parameter $\varepsilon$ we construct time-discretizing sequences, respectively solution-discretizing sequences, $(t_i)_{0\leq i\leq N}$ and $(x_i)_{0\leq i\leq N}$ enjoying a number of fundamental properties which will be used to establish our convergence estimations. Our approximate solutions will then be defined as piecewise continuous functions whose expressions involve the sequences $(t_i)_{0\leq i\leq N}$ and $(x_i)_{0\leq i\leq N}$.

**Theorem 3.2.** *Suppose that the subtangential condition (ST) is satisfied. Let $(t,x)\in D$ and assume that $R>0$ and $M>0$ are such that $t+R<T$ and $|B\left(s,y\right)|\leq M$ for $(s,y)\in D\cap S_R\left(t,x\right)$. Let $\tau>0$ small enough to satisfy $\tau\left(M+1\right)+\sup_{\sigma\in[0,\tau]}\left|T\left(\sigma\right)x-x\right|\leq R$.*

*Then for each $\varepsilon\in(0,1)$ there exist sequences $(t_i)_{0\leq i\leq N}$ and $(x_i)_{0\leq i\leq N}$ such that*

(i) $t_0=t$, $x_0=x$, $t_N=t+\tau$;

(ii) $0 < t_{i+1} - t_i \leq \varepsilon$ *for* $0 \leq i \leq N-1$ ;

(iii) $(t_i, x_i) \in D \cap S_R(t, x)$ *for* $0 \leq i \leq N$ ;

(iv) $|x_{i+1} - T(t_{i+1} - t_i)\, x_i - (t_{i+1} - t_i)\, B(t_i, x_i)| \leq (t_{i+1} - t_i)\, \varepsilon$ *for* $0 \leq i \leq N-1$ ;

(v) $|x_i - T(t_i)\, x| \leq t_i (M+1)$ *for* $0 \leq i \leq N$ ;

(vi) *For* $0 \leq i \leq N-1$ *there is* $r_i \in (0, \varepsilon]$ *such that* $|B(s, y) - B(t_i, x_i)| \leq \varepsilon/4$ *for* $(s, y) \in S_{r_i}(t_i, x_i) \cap D$, $\sup_{\sigma \in [0, r_i]} |T(\sigma)\, B(t_i, x_i) - B(t_i, x_i)| \leq \varepsilon/4$ *and* $(t_{i+1} - t_i)(M+1) + \sup_{\sigma \in [0, t_{i+1} - t_i]} |T(\sigma)\, x_i - x_i| \leq r_i$.

*Proof.* Set $t_0 = t$ and $x_0 = x$. Suppose that $(t_i)_{0 \leq i \leq n}$ and $(x_i)_{0 \leq i \leq n}$ have been constructed in such a way that conditions (i) through (vi) are fulfilled. We then define

$$r_n = \sup\Big\{ r \in (0, \varepsilon]\, ;\, |B(s, y) - B(t_n, x_n)| \leq \varepsilon/4 \text{ for } y \in D \cap S_r(t_n, x_n) \text{ and } \sup_{\sigma \in [0, r]} |T(\sigma)\, B(t_n, x_n) - B(t_n, x_n)| \leq \varepsilon/4 \Big\}. \tag{3.1}$$

and

$$\eta_n = \sup\Big\{ t > 0;\, t(M+1) + \sup_{\sigma \in [0, t]} |T(\sigma)\, x_n - x_n| \leq r_n \Big\}. \tag{3.2}$$

We define $h_n = \min(t + \tau - t_n, \eta_n)$ and $t_{n+1} = t_n + h_n$. Applying Theorem 3.1 with $h = 0$, $\eta = h_n$, $y = x = x_n$ and $r = r_n$, one finds $x_{n+1}$ such that $(t_{n+1}, x_{n+1}) \in D \cap S_{r_n}(t_n, x_n)$ and

$$|x_{n+1} - T(h_n)\, x_n - h_n B(t_n, x_n)| \leq \varepsilon h_n\ .$$

By our induction hypotheses, it is seen that

$$\begin{aligned} |x_{n+1} - T(t_{n+1})\, x| &\leq |x_{n+1} - T(h_n)\, x_n - h_n B(t_n, x_n)| \\ &\quad + |T(h_n)(T(t_n)\, x_n - x_n)| + h_n |B(t_n, x_n)| \\ &\leq \varepsilon h_n + t_n (M+1) + h_n (M+1) \\ &< t_{n+1} (M+1), \end{aligned}$$

which implies

$$|x_{n+1} - x| \leq t_{n+1} (M+1) + |T(t_{n+1})\, x - x|$$

$$\leq \tau(M+1) + \sup_{s\in[0,\tau]} |T(s)x - x| \leq R,$$

and hence the sequences $(t_i)_{0\leq i\leq n+1}$ and $(x_i)_{0\leq i\leq n+1}$satisfy (ii) through (vi) and the first part of (i). It now remains to show that $t+\tau$ can be attained in a finite number of steps.

We argue by contradiction and suppose that $t_i < t+\tau$ for all $i \geq 0$ (which implies that $h_i = \eta_i$ for all $i \geq 0$). Then $(t_i)_{i\geq 0}$ is convergent to some $t' \leq t+\tau$, and hence $(x_i)_{i\geq 0}$ is convergent to some $x'$, by Lemma 3.2. Since $B$ is continuous, it is seen that $B(t_i, x_i) \to B(t', x')$ as $i \to +\infty$. Interpreting the definition of $r_i$ and using the continuity of $B$, one obtains that there is $c > 0$ such that $\eta_i > c$ for all $i \geq 0$ and hence $(t_i)_{i\geq 0}$ diverges, which is a contradiction. We therefore obtain that there is $N \geq 1$ such that $t_N = t+\tau$, which finishes the proof. □

**Remark 3.2.** *Let us denote by* $(\bar{t}_j)_{0\leq j\leq \overline{N}}$ *an arbitrary partition of the interval* $[t, t+\tau]$. *By defining* $\overline{h}_n = \min(t+\tau - t_n, \eta_n, \bar{t}_{j+1})$, *if* $t_n \in [\bar{t}_j, \bar{t}_{j+1})$, *putting at each step* $t_{n+1} = t_n + \overline{h}_n$ *instead of* $t_{n+1} = t_n + h_n$ *and noting that if* $t_i < t+\tau$ *for all* $i \geq 0$, *then* $h_i = \eta_i$ *for* $i$ *greater than some* $N_0$ *since* $(t_i)_{i\geq N_0}$ *will remain in some interval* $[t_{j_0}, t_{j_0+1})$, *one obtains using the same argument by contradiction that* $(t_i)_{0\leq i\leq N}$ *may be constructed in such a way that* $\{\bar{t}_j; 0 \leq j \leq \overline{N}\} \subset \{t_i; 0 \leq i \leq N\}$.

For a given small parameter $\varepsilon$, using the previously constructed finite sequences $(t_i)_{0\leq i\leq N}$ and $(x_i)_{0\leq i\leq N}$, we may define an approximate solution $u_\varepsilon : [t, t+\tau] \to X$ by

$$u_\varepsilon(\xi) = \begin{cases} T(\xi - t_i)x_i + (\xi - t_i)B(t_i, x_i) & \text{for } \xi \in [t_i, t_{i+1}), \\ & \quad 0 \leq i \leq N-1 \\ T(t+\tau - t_{N-1})x_{N-1} & \\ \quad +(t+\tau - t_{N-1})B(t_{N-1}, x_{N-1}) & \text{for } \xi = t+\tau. \end{cases} \tag{3.3}$$

Using (3.3) and property (vi) in the previous lemma, one may show that $|x_i - u_\varepsilon(\xi)| \leq \varepsilon$ for any $\xi \in [t_i, t_{i+1})$, $0 \leq i \leq N-1$ and $|x_N - u_\varepsilon(t+\tau)| \leq \varepsilon$.

## 4. A convergence estimate

Given a null sequence of parameters $(\varepsilon_n)_{n\geq 1}$, we now show that the corresponding sequence of approximate solutions $(u_{\varepsilon_n})_{n\geq 1}$ is uniformly convergent to a continuous function $u$, which will in turn be a mild solution of (SP;$t,x$). Since it is difficult to estimate directly the difference between two approximate solutions corresponding to different small parameters $\varepsilon$ and $\hat{\varepsilon}$, we provide the following lemma which, used repeatedly, will allow us to construct two sequences of "intermediate" elements by means of which $|u_\varepsilon - u_{\hat{\varepsilon}}|$ can be estimated using an inductive argument.

**Lemma 4.1.** *Suppose that conditions (S) and (ST) are satisfied and that $w$ is a separately nondecreasing continuous function. Let $(t,x) \in D$ and $\varepsilon \in (0,1/3)$. Assume that $r = r(t,x,\varepsilon)$ is a real number such that $0 < r \le \varepsilon$,*

$$|B(s,y) - B(t,x)| \le \varepsilon/4, |B(s,y)| \le M(t,x,\varepsilon) \text{ for any } (s,y) \in D \cap S_r(t,x) \tag{4.1}$$

*and*

$$\sup_{\sigma\in[0,r]} |T(\sigma)B(t,x) - B(t,x)| \le \varepsilon/4, \tag{4.2}$$

*where $M(t,x,\varepsilon)$ is a real number.*

*Denote*

$h(t,x,\varepsilon) = \sup\{h \in (0,T-t); h(M+1) + \sup_{\sigma\in[0,h]} |T(\sigma)x - x| \le r\}$.

*Let $h \in [0, h(t,x,\varepsilon))$ and let $y \in D(t+h)$ such that $|y - T(h)x| \le M+1$. Let $\widehat{x} \in D(t+h)$ and $\widehat{\varepsilon} \in (0,1/3)$. Assume that $\widehat{r} = \widehat{r}(t+h,\widehat{x},\widehat{\varepsilon})$ is a real number such that $0 < \widehat{r} \le \widehat{\varepsilon}$,*

$$|B(s,y) - B(t+h,\widehat{x})| \le \widehat{\varepsilon}/4, |B(s,y)| \le \widehat{M}(t+h,\widehat{x},\widehat{\varepsilon}) \text{ for any } (s,y) \in D \cap S_{\widehat{r}}(t+h,\widehat{x}) \tag{4.3}$$

*and*

$$\sup_{\sigma\in[0,\widehat{r}]} |T(\sigma)B(t+h,\widehat{x}) - B(t+h,\widehat{x})| \le \widehat{\varepsilon}/4, \tag{4.4}$$

*where $\widehat{M}(t+h,\widehat{x},\widehat{\varepsilon})$ is a real number. Denote $\widehat{h}(t+h,\widehat{x},\widehat{\varepsilon}) = \sup\{\overline{h} \in (0,T-t-h); \overline{h}(M+1) + \sup_{\sigma\in[0,\overline{h}]} |T(\sigma)x - x| \le \widehat{r}\}$. Then for each $\delta > 0$ and $\eta > 0$ such that $h + \eta \le h(t,x,\varepsilon)$ and $\eta \le \widehat{h}(t+h,\widehat{x},\widehat{\varepsilon})$ there exist $z, \widehat{z} \in D(t+h+\eta)$ such that $(t+h+\eta, z) \in D \cap S_r(t,x)$, $(t+h+\eta,\widehat{z}) \in D \cap S_{\widehat{r}}(t+h,\widehat{x})$ and*

$$|z - T(\eta)y - \eta B(t+h,y)| \le 2\eta\varepsilon, \tag{4.5}$$

$$|\widehat{z} - T(\eta)\widehat{x} - \eta B(t+h,\widehat{x})| \le 2\eta\widehat{\varepsilon}, \tag{4.6}$$

$$|z - \widehat{z}| \le m_{\delta+\varepsilon+\widehat{\varepsilon}}^{|x-\widehat{x}|+\varepsilon+\widehat{\varepsilon}}(t+h+\eta; t+h, |y-\widehat{x}|). \tag{4.7}$$

*Proof.* First, we see that $(t+h,y)\in D\cap S_r(t,x)$, since

$$|y-x|\le|y-T(h)x|+|T(h)x-x|\le h(M+1)+|T(h)x-x|\le r$$

and $h\le r$. We now construct sequences $(s_n)_{n\ge0}$, $(x_n)_{n\ge0}$ and $(\hat{x}_n)_{n\ge0}$ satisfying

(i) $s_0=0,\ x_0=y,\ \hat{x}_0=\hat{x}$;

(ii) $0<s_n<s_{n+1},(t+h+s_n,x_n),(t+h+s_n,\hat{x}_n)\in D$ and $\lim\limits_{n\to\infty}s_n=\eta$;

(iii) $|x_n-T(s_n-s_{n-1})x_{n-1}-(s_n-s_{n-1})B(t+h+s_{n-1},x_{n-1})| \le(s_n-s_{n-1})\varepsilon$;

(iv) $|\hat{x}_n-T(s_n-s_{n-1})\hat{x}_{n-1}-(s_n-s_{n-1})B(t+h+s_{n-1},\hat{x}_{n-1})| \le(s_n-s_{n-1})\hat{\varepsilon}$;

(v) $|T(s_n-s_{n-1})(x_{n-1}-\hat{x}_{n-1})+(s_n-s_{n-1})[B(t+h+s_{n-1},x_{n-1}) -B(t+h+s_{n-1},\hat{x}_{n-1})]|\le|x_{n-1}-\hat{x}_{n-1}|+(s_n-s_{n-1})\delta +(s_n-s_{n-1})w(t+h+s_{n-1},|x_{n-1}-\hat{x}_{n-1}|)$;

(vi) $|x_n-T(s_n)x_0|\le s_n(M+1)$;

(vii) $|\hat{x}_n-T(s_n)\hat{x}_0|\le s_n(M+1)$;

(viii) $(t+s_n+h,x_n)\in S_r(t+h,x)\cap D$;

(ix) $(t+s_n+h,\hat{x}_n)\in S_{\hat{r}}(t+h,\hat{x})\cap D$,

for each $n\ge0$, properties (iii) through (v) being not formulated for $n=0$. We set $s_0=0$, $x_0=y$, $\hat{x}_0=\hat{x}$, so that the remaining properties (i) and (vi) through (ix) are satisfied for $n=0$. Assume that $(s_n)_{0\le n\le N}$, $(x_n)_{0\le n\le N}$ and $(\hat{x}_n)_{0\le n\le N}$ have been constructed in such a way that (i) and (iii) through (ix) are satisfied, together with the first half of (ii). We denote

$$\begin{aligned}\overline{h}_N=\sup\{\xi>0;s_N+\xi\le\eta;|T(\xi)(x_N-\hat{x}_N)\\ +\xi(B(t+h+s_N,x_N)-B(t+h+s_N,\hat{x}_N))|\le|x_N-\hat{x}_N|\\ +\xi w(t+h+s_N,|x_N-\hat{x}_N|)+\xi\delta\}.\end{aligned}$$

Now, condition (S) implies that $\overline{h}_N>0$. We choose $h_N\in\left(\overline{h}_N/2,\overline{h}_N\right)$ and set $s_{N+1}=s_N+h_N$, which insures the validity of (v) for $n=N+1$.

Using Theorem 3.1, one may find $x_{N+1}, \hat{x}_{N+1} \in D(t+h+s_{N+1})$ such that

$$\begin{aligned}&|x_{N+1} - T(s_{N+1}-s_N)x_N - (s_{N+1}-s_N)B(t+h+s_N, x_N)| \\ &\qquad \le (s_{N+1}-s_N)\varepsilon; \\ &|\hat{x}_{N+1} - T(s_{N+1}-s_N)\hat{x}_N - (s_{N+1}-s_N)B(t+h+s_N, \hat{x}_N)| \\ &\qquad \le (s_{N+1}-s_N)\hat{\varepsilon},\end{aligned}$$

and so (iii) and (iv) are satisfied for $n = N+1$.

Also, using (iii) one sees that

$$\begin{aligned}|x_{N+1} - T(s_{N+1})x_0| &\le (s_{N+1}-s_N)M + |x_N - T(s_N)x_0| \\ &\quad + (s_{N+1}-s_N)\varepsilon \\ &\le s_{N+1}(M+1)\end{aligned}$$

and similarly $|\hat{x}_{N+1} - T(s_{N+1})\hat{x}_0| < s_{N+1}(M+1)$, so that (vi) and (vii) are also satisfied for $n = N+1$. From (vi) it may be obtained that

$$|x_{N+1} - T(s_{N+1}+h)x| \le (s_{N+1}+h)(M+1),$$

which yields

$$|x_{N+1} - x| < (s_{N+1}+h)(M+1) + |T(s_{N+1}+h)x - x| \le r(t, x, \varepsilon)$$

and therefore $(x_{N+1}, t+s_{N+1}) \in S_r(t, x)$. The validity of (vii) and (ix) may be proved in a similar manner. Now, (iii) and (iv) together with Lemma 3.2 imply that $(x_n)_{n\ge 0}$ and $(\hat{x}_n)_{n\ge 0}$ are convergent to some $z$, respectively $\widehat{z}$. It remains to show that $\lim_{n\to\infty} s_n = \eta$.

Suppose that $\lim_{n\to\infty} s_n = \overline{\eta} < \eta$. From (ST), one may find $\xi \in (0, \eta)$ such that

$$\begin{aligned}&|T(\xi)(z-\hat{z}) + \xi(B(t+\overline{\eta}, z) - B(t+\overline{\eta}, \hat{z}))| \\ &\qquad \le |z-\hat{z}| + \xi w(t+\overline{\eta}, |z-\hat{z}|) + (1/2)\xi\delta. \qquad (4.8)\end{aligned}$$

Since $s_n \to s$ as $n \to \infty$, one may choose $N \ge 1$ so that $s - s_n \le \xi/2$ for all $n \ge N$ . Define $\xi_n = s - s_n + \xi$. Since $s_n + \xi_n < \eta$ and $\xi_n > \overline{h}_n$ for each $n \ge N$, one sees that

$$\begin{aligned}&|T(\xi_n)(x_n - \hat{x}_n) + \xi_n(B(t+h+s_n, x_n) - B(t+h+s_n, \hat{x}_n))| \\ &\quad > |x_n - \hat{x}_n| + \xi_n w(t+h+s_n, |x_n - \hat{x}_n|) + \xi_n\delta\end{aligned}$$

for all $n \ge N$, and passing to limit as $n \to \infty$ we obtain that

$$|T(\xi)(z-\hat{z}) + \xi(B(t+\overline{\eta}, z) - B(t+\overline{\eta}, \hat{z}))|$$

$$\geq |z-\hat{z}| + \xi w\left(t+\overline{\eta}, |z-\hat{z}|\right) + \xi\delta,$$

which contradicts (4.8). Using Lemma 3.1 for $y_n = x_n$ and $\overline{s}_n = t + s_n$, together with (iii), one deduces that $|x_n - T(s_n)\, y - s_n B(t+h, y)| \leq 7s_n/4$, which implies that $|z - T(\eta)\, y - \eta B(t+h, y)| \leq 2\eta\varepsilon$. One may also obtain in a similar manner that $|\widehat{z} - T(\eta)\,\widehat{x} - \eta B(t+h, \widehat{x})| \leq 2\eta\widehat{\varepsilon}$, that is, estimations (4.5) and (4.6) are valid.

Also, it is easy to see that

$$\begin{aligned}
&|x_{n+1} - \widehat{x}_{n+1}| \\
&\quad \leq |x_{n+1} - T(s_{n+1}-s_n)\, x_n - (s_{n+1}-s_n)\, B(t+h+s_n, x_n)| \\
&\qquad + |\hat{x}_{n+1} - T(s_{n+1}-s_n)\, \hat{x}_n - (s_{n+1}-s_n)\, B(t+h+s_n, \hat{x}_n)| \\
&\qquad + |T(s_{n+1}-s_n)\,(x_n - \hat{x}_n) + (s_{n+1}-s_n)\,[B(t+h+s_n, x_n) \\
&\qquad\quad - B(t+h+s_n, \hat{x}_n)]|\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
|x_{n+1} - \widehat{x}_{n+1}| \leq |x_n - \widehat{x}_n| &+ (s_{n+1}-s_n)\, w\left(t+h+s_n, |x_n - \hat{x}_n|\right) \\
&+ (s_{n+1}-s_n)\left(\delta + \varepsilon + \widehat{\varepsilon}\right).
\end{aligned}$$

We also note that, from (viii) and (ix), $|x_n - \widehat{x}_n| \leq |x - \widehat{x}| + r + \widehat{r}$ , so $w\left(t+h+s_n, |x_n - \hat{x}_n|\right) = w^{|x-\widehat{x}|+r+\widehat{r}}\left(t+h+s_n, |x_n - \hat{x}_n|\right)$. We denote

$$\begin{aligned}
u_1(\xi) &= |x_n - \widehat{x}_n| + (\xi - t - h - s_n)\, w^{|x-\widehat{x}|+r+\widehat{r}}\left(t+h+s_n, |x_n - \hat{x}_n|\right) \\
&\quad + (\xi - t - h - s_n)\left(\delta + \varepsilon + \widehat{\varepsilon}\right); \\
u_2(\xi) &= m^{|x-\widehat{x}|+r+\widehat{r}}_{\delta+\varepsilon+\widehat{\varepsilon}}\left(\xi; t+h+s_n, |x_n - \widehat{x}_n|\right), \; \xi \geq t+h+s_n.
\end{aligned}$$

It is seen that

$$\begin{aligned}
u_1'(\xi) &= |x_n - \widehat{x}_n| + w^{|x-\widehat{x}|+r+\widehat{r}}\left(t+h+s_n, |x_n - \hat{x}_n|\right) + \left(\delta + \varepsilon + \widehat{\varepsilon}\right) \\
&\leq w^{|x-\widehat{x}|+r+\widehat{r}}\left(\xi, u_1(\xi)\right) + \left(\delta + \varepsilon + \widehat{\varepsilon}\right); \\
u_2'(\xi) &= w^{|x-\widehat{x}|+r+\widehat{r}}\left(\xi, u_2(\xi)\right) + \left(\delta + \varepsilon + \widehat{\varepsilon}\right), \; \xi \geq t+h+s_n.
\end{aligned}$$

Since $u_1(t+h+s_n) = u_2(t+h+s_n) = |x_n - \widehat{x}_n|$, one obtains that $u_1(\xi) \leq u_2(\xi)$ for $\xi \geq t+h+s_n$, and setting $\xi = t+h+s_{n+1}$ it is inferred that

$$|x_{n+1} - \widehat{x}_{n+1}| \leq m^{|x-\widehat{x}|+r+\widehat{r}}_{\delta+\varepsilon+\widehat{\varepsilon}}\left(t+h+s_{n+1}; t+s_n, |x_n - \widehat{x}_n|\right).$$

Using Lemma 2.3, we easily deduce that

$$|x_{n+1} - \widehat{x}_{n+1}| \leq m^{|x-\widehat{x}|+r+\widehat{r}}_{\delta+\varepsilon+\widehat{\varepsilon}}\left(t+h+s_{n+1}; t, |y - \widehat{x}|\right).$$

Passing to limit as $n \to \infty$ in the above we obtain estimation (4.7). $\square$

## 5. The local existence of the solution

We now apply the estimations obtained in the previous lemma in order to obtain the existence of a (unique) local mild solution to (SP;$t, x$) via a limiting argument.

**Theorem 5.1.** *Suppose that conditions (ST), (S) and (U) are satisfied. Let $(t, x) \in D$ and let $R > 0$, $M > 0$ and $\tau > 0$ be such that $t + R < T$, $|B(s, y)| \leq M$ for $(s, y) \in D \cap S_R(t, x)$ and $\tau(M + 1) + \sup_{\sigma \in [0,\tau]} |T(\sigma)x - x| \leq R$. Then there exists a unique mild solution $u(\cdot)$ to (SP;$t, x$) on $[t, t + \tau]$ satisfying the initial condition $u(t) = x$.*

*Proof.* Let $\varepsilon_0 \in (0, 1/3)$ and let $(\varepsilon_n)_{n\geq 1}$ be a null sequence in $(0, \varepsilon_0)$. Our proof will consist in constructing a corresponding sequence of approximate solutions $(u_{\varepsilon_n})_{n\geq 1}$ with the help of (3.3) and Theorem 3.2, proving its uniform convergence as $n \to \infty$ using some estimations provided by Lemma 4.1 and showing that the uniform limit $u$ is actually a mild solution of (SP;$t, x$).

Using Theorem 3.2, we construct a sequence of partitions $(P_n)_{n\geq 1} = \left((t_i^n)_{0\leq i\leq N_n}\right)_{n\geq 1}$ of $[t, t + \tau]$ and a sequence of solution-discretizing elements $\left((x_i^n)_{0\leq i\leq N_n}\right)_{n\geq 1}$ in $S_R(t, x)$ which enjoy properties (i) through (vi) mentioned in the statement of Theorem 3.2. As seen in Remark 3.2, one may actually construct $\left((t_i^n)_{0\leq i\leq N_n}\right)_{n\geq 1}$ in such a way that $P_{n+1} = \left(t_i^{n+1}\right)_{0\leq i\leq N_{n+1}}$ refines $P_n = (t_i^n)_{0\leq i\leq N_n}$ for all $n$. We also construct a sequence of approximate solutions $(u_n)_{n\geq 1}$ on $[t, t + \tau]$ using the formula indicated in (3.3), that is,

$$u_n(\xi) = \begin{cases} T(\xi - t_i^n)x_i^n + (\xi - t_i^n)B(t_i^n, x_i^n) & \text{for } \xi \in \left[t_i^n, t_{i+1}^n\right), \\ & \quad 0 \leq i \leq N_n - 1 \\ T(t + \tau - t_{N_n-1}^n)x_{N_n-1}^n & \\ \quad +(t + \tau - t_{N_n-1}^n)B(t_{N_n-1}^n, x_{N_n-1}^n) & \text{for } \xi = t + \tau. \end{cases}$$

Let $1 \leq n < m$ and let $s \in (t, t + \tau)$ (if $s = t + \tau$, then our desired convergence estimate can be obtained in essentially the same manner). Then there are $0 \leq i \leq N_n - 1$ and $0 \leq j \leq N_m - 1$ such that $t \in \left[t_i^n, t_{i+1}^n\right) \cap \left[t_j^m, t_{j+1}^m\right)$. We need now to find an uniform estimate for $|u_n(s) - u_m(s)|$.

Since $P_m$ is finer than $P_n$, we first note that each node of the first partition is also a node for the second partition. Let us define a partition $(s_l)_{l=0}^{j+1}$ of $[t, s]$ by $s_l = t_l^m$ for $0 \leq l \leq j$ and $s_{j+1} = s$. We plan to estimate $|u_n(s) - u_m(s)|$ by means of Lemma 4.1, using a recurrent argument. To study the applicability of Lemma 4.1, take an arbitrary $l$, $0 \leq l \leq j$.

If $s_l$ is a common point for $P_m$ and $P_n$, that is, $s_l = t_k^n$ for some $k$, one sees that Lemma 4.1 is immediately applicable for $t = s_l$, $x = x_k^n$, $\hat{x} = x_l^m$, $y = x$, $h = 0$, $\eta = s_{l+1} - s_l$, $\delta = \varepsilon_m$ and finds $z_{l+1}$ and $\hat{z}_{l+1}$ satisfying

$$|z_{l+1} - T(s_{l+1} - s_l)\, x_k^n - (s_{l+1} - s_l)\, B(s_l, x_k^n)| < 2(s_{l+1} - s_l)\,\varepsilon_n, \quad (5.1)$$

$$|\hat{z}_{l+1} - T(s_{l+1} - s_l)\, x_l^m - (s_{l+1} - s_l)\, B(s_l, x_l^m)| < 2(s_{l+1} - s_l)\,\varepsilon_m, \quad (5.2)$$

$$|z_{l+1} - \hat{z}_{l+1}| \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}(s_{l+1}; s_l, |x_k^n - x_l^m|). \quad (5.3)$$

We now study the case in which $s_l$ is not a common point for $P_m$ and $P_n$. Let $s_l = t_l^m \in (t_k^n, t_{k+1}^n)$ and suppose that $s_{l-1} = t_k^n$, that is, $s_l$ is the first uncommon point in $(t_k^n, t_{k+1}^n)$. We let $t = t_k^n$, $x = x_k^n$, $\hat{x} = x_l^m$, $y = z_l$, $h = s_l - t_k^n$, $\eta = s_{l+1} - s_l$ and $\delta = \varepsilon_m$. Since

$$|z_{l_0+1} - T(s_{l_0+1} - t_k^n)\, x_k^n - (s_{l_0+1} - t_k^n)\, Bx_k^n| \leq 2(s_{l_0+1} - t_k^n)\,\varepsilon_n,$$

one may infer that

$$\begin{aligned}|z_{l_0+1} - T(s_{l_0+1} - t_k^n)\, x_k^n| &\leq (s_{l_0+1} - t_k^n)(M + 2\varepsilon_n)\\ &< (s_{l_0+1} - t_k^n)(M + 1)\end{aligned}$$

and Lemma 4.1 is also applicable in this situation. Suppose now that $s_{l-p} = t_k^n$ for some $p > 0$, that is, $s_l$ is the $p$-th uncommon point in $(t_k^n, t_{k+1}^n)$. If $2 < a \leq p$ and the auxiliary elements $z_{l-p+1}, \ldots, z_{l-p+a-1}$ are constructed by means of Lemma 4.1, satisfying the required property $|z_{l-p+b} - T(s_{l-p+b} - t_k^n)\, x_k^n| \leq (s_{l-p+b} - t_k^n)(M + 1)$ for $1 \leq b \leq a - 1$, then

$$\begin{aligned}&|z_{l-p+a} - T(s_{l-p+a} - t_k^n)\, x_k^n|\\ &\quad \leq (s_{l-p+a} - s_{l-p+a-1})(M + 2\varepsilon_n) + |z_{l-p+a-1} - T(s_{l-p+a-1} - t_k^n)|\end{aligned}$$

and therefore

$$|z_{l-p+a} - T(s_{l-p+a} - t_k^n)\, x_k^n| < (s_{l-p+a} - t_k^n)(M + 1).$$

Reasoning inductively, one deduces that

$$|z_l - T(s_l - t_k^n)\, x_k^n| < (s_l - t_k^n)(M + 1)$$

and Lemma 4.1 is again applicable.

In view of the above, we can again apply Lemma 4.1 in the case in which $s_l$ is not a common point for $P_m$ and $P_n$ and find $z_{l+1}$ and $\widehat{z}_{l+1}$ satisfying

$$|z_{l+1} - T\left(s_{l+1} - s_l\right) z_l - \left(s_{l+1} - s_l\right) B\left(s_l, z_l\right)| < 2\left(s_{l+1} - s_l\right) \varepsilon_n, \quad (5.4)$$

$$|\hat{z}_{l+1} - T\left(s_{l+1} - s_l\right) x_l^m - \left(s_{l+1} - s_l\right) B\left(s_l, x_l^m\right)| < 2\left(s_{l+1} - s_l\right) \varepsilon_m, \quad (5.5)$$

$$|z_{l+1} - \hat{z}_{l+1}| \leq m_{2\varepsilon_m + \varepsilon_n}^{2R+1}\left(s_{l+1}; s_l, |z_l - x_l^m|\right). \quad (5.6)$$

We now estimate $|u_n(s) - u_m(s)|$. One sees that

$$\begin{aligned} |u_n(s) - u_m(s)| \leq{}& |u_n(s) - z_{j+1}| + |z_{j+1} - \widehat{z}_{j+1}| \\ &+ |\widehat{z}_{j+1} - u_m(s)|. \end{aligned} \quad (5.7)$$

Since $s_{j+1} = s$ and $s_j = t_j^m$, from (5.2) or (5.5) we obtain that

$$\begin{aligned} |\widehat{z}_{j+1} - u_m(s)| &= \left|\widehat{z}_{j+1} - T\left(s - t_j^m\right) x_j^m + \left(s - t_j^m\right) B\left(t_j^m, x_j^m\right)\right| \\ &\leq 2\left(s - t_j^m\right) \varepsilon_m. \end{aligned} \quad (5.8)$$

To estimate the first term in (5.7), one should consider whether or not $t_j^m$ is a common point for partitions $P_m$ and $P_n$.

If $t_j^m$ is a common point, it may be obtained that $|u_n(s) - z_{j+1}| \leq 2\left(s - t_j^m\right) \varepsilon_n$ reasoning as above. If $t_j^m$ is not a common point, suppose that $t_i^n = s_{i_0}$ for some $i_0$. Then

$$\begin{aligned} |u_n(s) - z_{j+1}| &= |z_{j+1} - T\left(s_{j+1} - t_i^n\right) x_i^n - \left(s_{j+1} - t_i^n\right) B\left(s_{i_0}, x_i^n\right)| \\ &\leq \left|z_{i_0+1} - T\left(s_{i_0+1} - s_{i_0}\right) x_i^n - \left(s_{i_0+1} - s_{i_0}\right) B\left(s_{i_0}, x_i^n\right)\right| \\ &\quad + \sum_{l=i_0+1}^{j} |z_{l+1} - T\left(s_{l+1} - s_l\right) z_l - \left(s_{l+1} - s_l\right) B\left(s_l, z_l\right)| \\ &\quad + \sum_{l=i_0}^{j} \left(s_{l+1} - s_l\right) |B\left(s_l, z_l\right) - B\left(s_{i_0}, x_i^n\right)| \\ &\quad + \sum_{l=i_0}^{j} \left(s_{l+1} - s_l\right) |T\left(s_{j+1} - s_{l+1}\right) B\left(s_{i_0}, x_i^n\right) - B\left(s_{i_0}, x_i^n\right)| \end{aligned}$$

and using (5.1), (5.4) together with (4.1) and (4.2) we can infer that

$$|u_n(s) - z_{j+1}| \leq 3\left(s_{j+1} - t_n^i\right) \varepsilon_n. \quad (5.9)$$

Let now $\left[t_k^n, t_{k+1}^n\right]$ be a generic interval for $P_n$. Since $P_m$ is finer than $P_n$, there are $l_0, l_1 \in \mathbb{N}$ such that $\left[t_k^n, t_{k+1}^n\right] = \left[s_{l_0}, s_{l_1}\right]$. Using the same argument displayed for the derivation of (5.9), one obtains that

$$\left|z_{l_1} - x_{k+1}^n\right| \leq 4\left(t_{k+1}^n - t_k^n\right)\varepsilon_n. \tag{5.10}$$

Let $s_{l_2}$ be a generic point for $P_m$. From (5.1) and (5.4) and from (iv) in Theorem 3.2, one finds that

$$\left|z_{l_2} - x_{l_2}^m\right| \leq 3\left(s_{l_2} - s_{l_2-1}\right)\varepsilon_m. \tag{5.11}$$

We now start estimating $|z_{l+1} - \hat{z}_{l+1}|$. Let $l \in \mathbb{N}$ , $0 \leq l \leq j$. If $s_l$ is a common point for $P_m$ and $P_n$, relations (5.3), (5.10) and (5.11) yield that

$$\begin{aligned} &|z_{l+1} - \hat{z}_{l+1}| \\ &\quad \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_{l+1}; s_l, |z_l - \hat{z}_l| + 3\left(s_l - s_{l-1}\right)\varepsilon_m + 4\left(t_k^n - t_{k-1}^n\right)\varepsilon_n\right). \end{aligned} \tag{5.12}$$

If $s_l$ is not a common point, relations (5.6) and (5.11) yield that

$$|z_{l+1} - \hat{z}_{l+1}| \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_{l+1}; s_l, |z_l - \hat{z}_l| + 3\left(s_l - s_{l-1}\right)\varepsilon_m\right). \tag{5.13}$$

We note that

$$|z_1 - \hat{z}_1| \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_1; 0, 0\right) \tag{5.14}$$

and

$$m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(t; s_1, m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_1; s_2, \alpha_1\right) + \alpha_2\right) \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(t; s_2, \alpha_1 + \alpha_2\right) \tag{5.15}$$

for any $t, s_1, s_2$ and $\alpha_1, \alpha_2 \geq 0$.

From (5.12), (5.13), (5.14) and (5.15) we obtain using a recurrent argument that

$$|z_{j+1} - \hat{z}_{j+1}| \leq m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_{j+1}; 0, 4\varepsilon_n t_{i+1}^n + 3\varepsilon_m t_{j+1}^m\right). \tag{5.16}$$

Using (5.7), (5.8), (5.9) and (5.16), it is then deduced that

$$\begin{aligned} |u_n(s) - u_m(s)| \leq\ & m_{2\varepsilon_m+\varepsilon_n}^{2R+1}\left(s_{j+1}; 0, 4\varepsilon_n t_{i+1}^n + 3\varepsilon_m t_{j+1}^m\right) \\ & + 2\left(s - t_j^m\right)\varepsilon_m + 3\left(t_{j+1}^m - t_n^i\right)\varepsilon_n. \end{aligned}$$

Now, our convergence result, Lemma 2.3, implies that $(u_m)_{m\geq 1}$ is uniformly convergent on $[t, t+\tau]$ to a function satisiying $u(t) = x$.

Let $\sigma \in [t, t+\tau)$. Then for each $n \geq 1$ there is $i_n$ such that $\sigma \in \left[t_{i_n}^n, t_{i_n+1}^n\right)$. Since $\left|x_{i_n}^n - u_n(\sigma)\right| \leq \varepsilon_n$, it is seen that $x_{i_n}^n \to u(\sigma)$ as $n \to \infty$ and since $t_{i_n}^n \uparrow t$, one obtains from (D) that $(t, u(t)) \in D$. The case $\sigma = \tau$ may be treated in the same manner.

Let us define $\gamma_n : [t, t+\tau] \to [t, t+\tau]$ by

$$\gamma_n(\xi) = \begin{cases} t_i^n & \text{for } \xi \in \left[t_i^n, t_{i+1}^n\right), 0 \le i \le N_n - 1 \\ t_{N_n-1}^n & \text{for } \xi = t+\tau \end{cases}, \tag{5.17}$$

and $v_n : [t, t+\tau] \to X$ by

$$v_n(\xi) = T(\xi - t)\, x + \int_t^\xi T(\xi - \sigma)\, B(\sigma, u_n(\gamma_n(\sigma)))d\sigma \text{ for } \xi \in [t, t+\tau]. \tag{5.18}$$

One may see that

$$|u_n(t) - v_n(t)| < 5/4\,(\xi - t)\,\varepsilon_n \quad \text{for } \xi \in [t, t+\tau],$$

and

$$|u_n(\gamma_n(\xi)) - u_n(\xi)| \le \varepsilon_n, \quad |\gamma_n(\xi) - \xi| \le \varepsilon_n$$

(see [1, pag. 163] for a related argument). Passing to limit as $n \to \infty$ in (5.18) and noting that $B$ is continuous, one obtains that $u$ is a mild solution for (SP;$t, x$). The uniqueness of the mild solution will follow from the next lemma. □

## 6. The global existence of the solution and its uniqueness

Let us now study the global existence of the mild solution and its uniqueness. We first indicate a lemma which insures a local "Lipschitz-like" dependence of the solution with respect to the initial data.

**Lemma 6.1.** *Let $u, v$ be mild solutions of (SP;$t, x$) and (SP;$t, y$) defined on a common interval of existence $[t, t+\tau]$. Then*

$$|u(\xi) - v(\xi)| \le |x - y| + \int_t^\xi w(\xi, |u(\xi) - v(\xi)|)d\xi \quad \textit{for } \xi \in [t, t+\tau]. \tag{6.1}$$

*Also,*

$$|u(\xi) - v(\xi)| \le m(\xi; t, |x - y|) \quad \textit{for } \xi \in [t, t+\tau] \tag{6.2}$$

*and if $x \equiv y$ then $u \equiv v$.*

*Proof.* Let $s \in [t, t+\tau)$ and $h > 0$ such that $s + h \le t + \tau$. One has

$$\begin{aligned}&(1/h)\left(|u(s+h) - v(s+h)| - |u(s) - v(s)|\right)\\ &\quad \le (1/h)\left(|T(h)(u(s) - v(s)) + h\left(B(s, u(s)) - B(s, v(s))\right)|\right.\\ &\qquad \left. - |u(s) - v(s)|\right)\end{aligned}$$

$$+ (1/h) \int_s^{s+h} |T(s+h-\xi) B(\xi, u(\xi)) - B(s, u(s))| \, ds$$

$$+ (1/h) \int_s^{s+h} |T(s+h-\xi) B(\xi, v(\xi)) - B(s, v(s))| \, ds. \quad (6.3)$$

Passing to inferior limit as $h \downarrow 0+$, we obtain that $D_+(|u(s) - v(s)|) \leq w(s; |u(s) - v(s)|)$ and then (6.2) follows from Lemma 2.1. Also, (6.2) immediately implies the uniqueness of the mild solution for given initial data.

Define $\varphi : [t, t+\tau] \to \mathbb{R}$ by

$$\varphi(\xi) = |u(\xi) - v(\xi)| - \int_t^{\xi} w(\xi, |u(s) - v(s)|) d\xi.$$

It is easy to see that $\varphi \in C([t, t+\tau])$ and

$(D_+\varphi)(\xi) = D_+(|u(\xi) - v(\xi)|) - w(\xi; |u(\xi) - v(\xi)|) \leq 0$ for $\xi \in [t, t+\tau)$.

Hence $\varphi$ is decreasing on $[t, t+\tau]$, which implies (6.1). □

Our main result may now be stated as follows.

**Theorem 6.1.** *Suppose that conditions (A), (B), (D) are satisfied, together with the subtangential condition (ST) and the semilinear stability condition (S), and that $w$ is a continuous, separately nondecreasing function which satisfies (U). Then for each $(s, u_0) \in D$ the semilinear problem (SP;$s, u_0$) has a unique mild solution $u(\cdot; s, u_0)$ on $[s, T)$. Moreover, for any $(s, u_0)$ and $(s, \overline{u}_0) \in D$ and $\xi \in [s, T)$, one has*

$$|u(\xi; s, u_0) - u(\xi; s, \overline{u}_0)| \leq m(\xi; s, |u_0 - \overline{u}_0|). \quad (6.4)$$

*Proof.* The local existence of $u(\cdot; s, u_0)$ was proved in Theorem 5.1. Its global existence follows as in [3, Proposition 8.1], noting that our local existence result, Theorem 5.1, together with Lemma 6.1 may replace Iwamiya's Theorem 7.1 and Proposition 4.1, which are used in the proof of [3, Proposition 8.1], and that the rest of his proof does not require any dissipativity assumption on the operator $B$. Finally, (6.4) results from Lemma 6.1. □

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