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A NOTE ON THE SHARPNESS OF NONOPTIMAL COMPARISON RATES FOR C_0-SEMIGROUPS

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Abstract. If $\alpha \in (0,1)$ is given and $T_1 = \{T_1(t); t \geq 0\}$, $T_2 = \{T_2(t); t \geq 0\}$ are C_0-semigroups defined on a real Banach space $(X, |\cdot|)$, we indicate sufficient conditions for the existence of $x_\alpha \in X$ such that $|T_1(t)x_\alpha - T_2(t)x_\alpha| = O(t^\alpha)$ and $|T_1(t)x_\alpha - T_2(t)x_\alpha| \neq o(t^\alpha)$ as $t \to 0+$, that is, for which the approximation rate $O(t^\alpha)$ is sharp.

1. Introduction

Let $T_1 = \{T_1(t); t \geq 0\}$ and $T_2 = \{T_2(t); t \geq 0\}$ be C_0-semigroups on a real Banach space $(X, |\cdot|)$. Since the pioneering paper of D. W. Robinson [1], many works have been devoted to finding necessary and sufficient conditions so that $\|T_1(t) - T_2(t)\| = O(t)$ as $t \to 0+$; we quote here only Desch and Schappacher [2], Diekmann, Gyllenberg and Thieme [3], Piskarev and Shaw [4], who employed various approaches in order to study this problem. However, the sharpness of the approximation rates is not discussed there, and the nonoptimal case, that is, the case in which $\|T_1(t) - T_2(t)\| = O(t^\alpha)$, $0 < \alpha < 1$, has attracted much less attention. Also, the condition in this case are sufficient only. A discussion on the necessity may be found in Davies [5, Theorem 3.25].

First, we prove that the best possible nontrivial comparison rate in the operatorial norm for T_1 and T_2 as $t \to 0+$ is $O(t)$, since if $\|T_1(t) - T_2(t)\| = o(t)$, then $T_1 \equiv T_2$; see Lemma 1 below. Once seen that the best nontrivial comparison rate in the operatorial norm for the semigroups T_1 and T_2 as $t \to 0+$ is $O(t)$, it may be interesting to investigate the existence of elements which provide sharp nonoptimal comparison rates, that is, if $\alpha \in (0,1)$, the existence of $x_\alpha \in X$ such that $|T_1(t)x_\alpha - T_2(t)x_\alpha| = O(t^\alpha)$ and $|T_1(t)x_\alpha - T_2(t)x_\alpha| \neq o(t^\alpha)$ as $t \to 0+$. If $T_2 \equiv I$, Davydov has proved in [6], as a consequence of his deep quantitative

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resonance principle, that the unboundedness of the infinitesimal generator A_1 of the C_0-semigroup T_1 is a sufficient condition for the existence of such x_α. It is also easy to see that in this case the unboundedness of A_1 is also a necessary condition for the existence of such x_α.

In the following, using a condensation principle obtained by Davydov in [6], we shall prove a general result which enables us to indicate sufficient conditions for the existence of elements which provide sharp nonoptimal comparison rates, that is, the existence of $x_\alpha \in X$ such that $|T_1(t)x_\alpha - T_2(t)x_\alpha| = O(t^\alpha)$ and $|T_1(t)x_\alpha - T_2(t)x_\alpha| \neq o(t^\alpha)$ as $t \to 0+$. Some examples for which our conditions are satisfied are given in Section 4.

2. Preliminaries and notations

Throughout this paper, we denote by $(X, |\cdot|)$ a real Banach space. Given a continuous function $v : (0,1] \to (0,\infty)$ such that $v(t) \to 0$ as $t \to 0+$ and a C_0-semigroup $T = \{T(t); t \geq 0\}$ on X, we define

$$F_v^T = \left\{x \in X;\ \sup_{t\in(0,1]} \frac{|T(t)x - x|}{v(t)} < \infty\right\}.$$

Note that if $v(t)/t \to 0$ as $t \to 0+$, then $F_v^T \subset \{x \in D(A); Ax = 0\}$, where A is the infinitesimal generator of T, and so F_v^T is not likely to verify the denseness assumptions which will be used in the following. If $v(t) = t^\alpha$ with $\alpha \in (0,1]$, then F_v^T is just the Favard class of fractional order α, defined as follows:

$$F_\alpha^T = \{x \in X;\ |T(t)x - x| = O(t^\alpha) \text{ as } t \to 0+\}.$$

For further properties of the Favard classes, see Butzer and Berens [7].

A functional p on X will be called a seminorm if it satisfies $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) \leq |\alpha| p(x)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. If p is a seminorm on X, we shall denote $\|p\| = \sup\{p(x); |x| \leq 1\}$. It is easy to see that a seminorm p is continuous if and only if $\|p\| < \infty$. Given a family H of continuous seminorms on X, we denote

$$\mathcal{N}_H = \left\{x \in X;\ \lim_{h\in H, \|h\|\to\infty} h(x) = 0\right\}.$$

We now state Davydov's condensation principle [6, Theorem 1], which plays an important role in the proof of our result.

THEOREM A. *Let H be an unbounded family of continuous seminorms. Assume that $\mathcal{N}_H$ is dense in X. Then there exists an element $x \in X$ such that*

$$\sup_{h\in H} h(x) \leq 1 \quad \text{and} \quad \limsup_{h\in H,\, \|h\|\to\infty} h(x) = 1.$$

3. The existence of nonoptimal comparison rates

Given two C_0-semigroups $T_1 = \{T_1(t);t \geq 0\}$ and $T_2 = \{T_2(t);t \geq 0\}$ on X, it is easy to show that the quantity $\|T_1(t) - T_2(t)\|$ cannot be arbitrarily small as $t \to 0+$, but still positive. More precisely, one obtains the following result:

LEMMA 1. *Let $T_1 = \{T_1(t);t \geq 0\}$ and $T_2 = \{T_2(t);t \geq 0\}$ be C_0-semigroups on X. Then $\|T_1(t) - T_2(t)\| = o(t)$ as $t \to 0+$ if and only if $T_1(t) \equiv T_2(t)$ for each $t \geq 0$.*

Proof. One implication is trivial. For the other, let us denote by A_1 and A_2 the generators of T_1, respectively of T_2, and take $x \in D(A_1)$. From the estimation

$$\left|\frac{T_2(t)x - x}{t} - A_1x\right| \leq \frac{\|T_1(t) - T_2(t)\|}{t}|x| + \left|\frac{T_1(t)x - x}{t} - A_1x\right|,$$

one deduces that $x \in D(A_2)$ and $A_1x = A_2x$; therefore, A_2 is an extension of A_1. In the same way we can prove that A_1 is an extension of A_2, which yields that $A_1 \equiv A_2$. Since a C_0-semigroup on X is uniquely determined by its generator, one obtains that $T_1(t) \equiv T_2(t)$ for each $t \geq 0$, which finishes the proof. ■

We can now state our main result.

THEOREM 1. *Let $w, w_1, w_2 : (0,1] \to (0,\infty)$ be continuous functions and $T_1 = \{T_1(t);t \geq 0\}$, $T_2 = \{T_2(t);t \geq 0\}$ be C_0-semigroups satisfying*

$$\text{(C.1)} \limsup_{t\to 0+}\frac{\|T_1(t) - T_2(t)\|}{w_1(t)} > 0, \quad \text{(C.2) } F_{w_2}^{T_1} \cap F_{w_2}^{T_2} \textit{ is dense in } X$$

and

$$\text{(C.3)} \lim_{t\to 0+} w_2(t) = 0, \quad \lim_{t\to 0+}\frac{w_1(t)}{w(t)} = +\infty \quad \text{and} \quad \lim_{t\to 0+}\frac{w_2(t)}{w(t)} = 0.$$

Then there exists $x \in X$ such that

$$|T_1(t)x - T_2(t)x| \leq w(t) \textit{ for all } t \in (0,1] \quad \textit{and} \quad \limsup_{t\to 0+}\frac{|T_1(t)x - T_2(t)x|}{w(t)} = 1.$$

Proof. Let $H = \{h_t; t \in (0,1]\}$ be a family of continuous seminorms on X defined as follows:

$$h_t(x) = \frac{|T_1(t)x - T_2(t)x|}{w(t)} \quad \text{for each } t \in (0,1] \text{ and } x \in X.$$

From (C.1), there exist $\varepsilon_0 > 0$ and $(t_n)_{n\geq 1}$, $t_n \to 0+$ as $n \to \infty$ such that $\|T_1(t_n) - T_2(t_n)\| \geq \varepsilon_0 w_1(t_n)$, which implies that $\|h_{t_n}\| \geq \varepsilon w_1(t_n)/w(t_n)$ for all

$n \in \mathbb{N}$. Then (C.3) yields that H is unbounded. Also, since there exists $M > 0$ such that $\|T_1(t)\|, \|T_2(t)\| \le M$ for all $t \in (0,1]$, one sees that $\|h_t\| \le 2M/w(t)$ for all $t \in (0,1]$, and so $\|h_t\|$ may diverge to $+\infty$ only if $t \to 0+$.

Let $x \in F_{w_2}^{T_1} \cap F_{w_2}^{T_2}$. Then $|T_1(t)x - x| = O(w_2(t)), |T_2(t)x - x| = O(w_2(t))$ as $t \to 0+$, and this implies that $|T_1(t)x - T_2(t)x| \le Cw_2(t)$ for $t \in (0,\delta]$, $\delta > 0$ small enough and some $C \in \mathbb{R}$. We have then $h_t(x) \le Cw_2(t)/w(t)$ for $t \in (0,\delta]$. Therefore, condition (C.3) implies that $\lim_{t\to 0+} h_t(x) = 0$. Since

$$\left\{x \in X; \lim_{t\to 0+} h_t(x) = 0\right\} \subseteq \left\{x \in X; \lim_{\|h_t\|\to\infty} h(x) = 0\right\},$$

we infer that $F_{w_2}^{T_1} \cap F_{w_2}^{T_2} \subseteq \mathcal{N}_H$, and so $\mathcal{N}_H$ is dense in X. From Theorem A, the conclusion follows. ■

REMARK 1. Let us define $H = \{h_t; t \in (0,1]\}$ by

$$h_t(x) = \sup_{s\in(0,t]} \frac{|T_1(s)x - T_2(s)x|}{w(t)} \quad \text{for } x \in X.$$

If we suppose, in addition to (C.1), (C.2) and (C.3), that w_2 is increasing, then by the same argument we obtain the existence of $x \in X$ such that

$$\sup_{0<s\le t} |T_1(s)x - T_2(s)x| \le w(t) \ \text{ for all } t \in (0,1] \quad \text{and} \quad \limsup_{t\to 0+} h_t(x) = 1.$$

For $w(t) = t^\alpha$, $w_1(t) = t^\beta$, $w_2(t) = t^\gamma$, $0 \le \beta < \alpha < \gamma \le 1$, we obtain the following result, which addresses the problem mentioned in the introduction:

COROLLARY 1. *Let $\alpha \in (0,1)$ and let $T_1 = \{T_1(t); t \ge 0\}$, $T_2 = \{T_2(t); t \ge 0\}$ be C_0-semigroups on X such that*

(C.1)$'$ $\limsup_{t\to 0+} \dfrac{\|T_1(t) - T_2(t)\|}{t^\beta} > 0$ for some β with $0 \le \beta < \alpha$,

(C.2)$'$ $F_\gamma^{T_1} \cap F_\gamma^{T_2}$ is dense in X for some γ with $\alpha < \gamma \le 1$.

Then there exists $x_\alpha \in X$ such that

$$|T_1(t)x_\alpha - T_2(t)x_\alpha| = O(t^\alpha) \text{ and } |T_1(t)x_\alpha - T_2(t)x_\alpha| \ne o(t^\alpha) \text{ as } t \to 0+.$$

4. Examples and concluding remarks

We now indicate some situations in which (C.1)$'$ and (C.2)$'$ are satisfied.

EXAMPLE 1. Conditions $(C.1)'$ and $(C.2)'$ are satisfied in their strongest forms, that is, with $\beta = 0$, respectively $\gamma = 1$, if T_1 is a uniformly continuous semigroup and T_2 is a C_0-semigroup which is not uniformly continuous. In this situation, $\limsup_{t\to 0+} \|T_2(t) - T_1(t)\| > 0$. Also, $F_1^{T_1} \cap F_1^{T_2} = F_1^{T_2} \supset D(A_2)$, where A_2 is the infinitesimal generator of T_2, and so $F_1^{T_1} \cap F_1^{T_2}$ is dense in X.

As a consequence of Corollary 1, we then obtain the following result.

COROLLARY 2. *Let $\alpha \in (0,1)$ and let $T_1 = \{T_1(t); t \geq 0\}$, $T_2 = \{T_2(t); t \geq 0\}$ be a uniformly continuous semigroup, respectively a C_0-semigroup which is not uniformly continuous. Then there exists $x_\alpha \in X$ such that*

$$|T_1(t)x_\alpha - T_2(t)x_\alpha| = O(t^\alpha) \text{ and } |T_1(t)x_\alpha - T_2(t)x_\alpha| \neq o(t^\alpha) \text{ as } t \to 0+.$$

For $T_1 \equiv I$, we obtain Davydov's result on the existence of nonoptimal approximation rates for C_0-semigroups which are not uniformly continuous ([6, Corollary]).

Actually, condition $(C.1)'$ states that $T_1(t)$ and $T_2(t)$ may have different regularities as $t \to 0+$, but it is not necessary for T_1 to be uniformly continuous to satisfy $(C.1)'$; it suffices, for instance, to be analytic.

LEMMA 2. *Let $T_1 = \{T_1(t); t \geq 0\}$ be a uniformly bounded analytic semigroup and let $T_2 = \{T_2(t); t \geq 0\}$ be a C_0-semigroup such that $\|T_1(t) - T_2(t)\| \to 0$ as $t \to 0+$. Then T_2 is analytic.*

Proof. Since T_1 is analytic, using [8, Theorem 2.5.6], there exist $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $k, \delta > 0$ such that $|(\xi I - T_1(t))x| \geq (1/k)x$ for all $x \in X$ and $t \in (0, \delta]$. Since $|(\xi I - T_2(t))x| \geq |(\xi I - T_1(t))x| - |T_1(t)x - T_2(t)x| \geq (1/2k)|x|$ for $t > 0$ small enough, we conclude using the same theorem that T_2 is analytic. ∎

EXAMPLE 2. As seen from the preceding lemma, condition $(C.1)'$ is satisfied with $\beta = 0$ if $T_1 = \{T_1(t); t \geq 0\}$ is a uniformly bounded analytic semigroup, while $T_2 = \{T_2(t); t \geq 0\}$ is a uniformly bounded C_0-semigroup which is not analytic.

EXAMPLE 3. Let $T_1 = \{T_1(t); t \geq 0\}$, $T_2 = \{T_2(t); t \geq 0\}$ be C_0-semigroups and let $w_1 : (0,1] \to (0,\infty)$ be a continuous function. One may see that condition (C.1) is satisfied if $F_{w_1}^{T_1} \neq F_{w_1}^{T_2}$. Namely, suppose that $F_{w_1}^{T_1} \neq F_{w_1}^{T_2}$ and fix $x \in F_{w_1}^{T_1} \setminus F_{w_1}^{T_2}$. Then there are $\delta > 0$ small enough and $M \geq 0$ such that $|T_1(t)x - x|/w_1(t) \leq M$ for $t \in (0,\delta]$, and $\limsup_{t\to 0+} (|T_2(t)x - x|/w_1(t)) =$

$+\infty$. Since

$$\frac{\|T_2(t)-T_1(t)\|\,|x|}{w_1(t)} \geq \frac{|T_2(t)x-x|}{w_1(t)} - \frac{|T_1(t)x-x|}{w_1(t)},$$

(C.1) is satisfied.

The noncoincidence condition $F_{w_1}^{T_1} \neq F_{w_1}^{T_2}$ is not implied by the noncoincidence of T_1 and T_2, since if $w_1 : (0,1] \to (0,\infty)$ is a continuous function such that $\{t/w_1(t); 0 < t \leq 1\}$ is bounded, $A_2 = A_1 + B$, where A_2 is the generator of T_2, A_1 is the generator of T_1 and B is a linear and bounded operator on X, then $\|T_2(t) - T_1(t)\| = O(t)$ as $t \to 0+$ (see [8, Corollary 3.1.3]), from which we easily obtain that $F_{w_1}^{T_1} = F_{w_1}^{T_2}$. Note that in this case one has $F_{\beta}^{T_1} = F_{\beta}^{T_2}$ for all $\beta \in [0,1]$.

We now conclude with a result concerning the existence of nonoptimal approximation rates for families of semigroups.

THEOREM 2. *Let $(T_n)_{n\geq 0}$ be a sequence of C_0-semigroups, at least one not being uniformly continuous, and assume that there exist $M, w \in \mathbb{R}$ such that $\|T_n(t)\| \leq Me^{wt}$ for all $t \geq 0$ and $n \in \mathbb{N}$. Assume also that $T_n(t)x \to T_0(t)x$ as $n \to \infty$ for every $x \in X$ and $t \geq 0$. Then for each $\alpha \in (0,1)$ there exists $(x_n^\alpha)_{n\geq 0}$ such that $x_n^\alpha \to x_0^\alpha$ as $n \to \infty$, $\sup_{n\geq 0} |T_n(t)x_n^\alpha - x_n^\alpha| = O(t^\alpha)$, and $\sup_{n\geq 0} |T_n(t)x_n^\alpha - x_n^\alpha| \neq o(t^\alpha)$ as $t \to 0+$.*

Proof. Let $\mathcal{X} = \{x = (x_n)_{n\geq 0}\,; x_n \to x_0 \text{ as } n \to \infty\}$ be the Kisyński space consisting of all the sequences which are convergent to their first component. We endow X with the supremum norm and define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by $\mathcal{T} = \{(T_n(t))_{n\geq 0}\,; t \geq 0\}$. By Trotter-Kato's theorem (see [8, Theorem 3.4.2]), $\mathcal{T}$ is a C_0-semigroup on $\mathcal{X}$, since the convergence of $T_n(t)x$ to $T(t)x$ as $n \to \infty$ is actually uniform on compact t-subintervals of $[0,\infty)$. It is easy to see that $\mathcal{T}$ is not uniformly continuous, and the use of Corollary 2 yields now the desired result. ■

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