Nume şi prenume: ______________________________

Grupa: ________________

ANALIZĂ MATEMATICĂ II
MODEL DE EXAMEN

(20p) 1. Să se calculeze: **a)** $\int 5x^4 \, \mathrm{d}x$; **b)** $\int\limits_0^{\pi/2} x \cdot \sin(x) \, \mathrm{d}x$.

(10p) 2. Să se calculeze: $\int\limits_1^2 \frac{3x^2+2x}{x^3+x^2+1} \, \mathrm{d}x$.

(20p) 3. Să se calculeze următoarele integrale curbilinii, ştiind că $(C): \ y = x^2, \ x \in [0,1]$:

a) $\int\limits_C \frac{y^2}{x} \, \mathrm{d}x + xy \, \mathrm{d}y$; **b)** $\int\limits_C x \, \mathrm{d}s$.

(10p) 4. Să se calculeze $\iint\limits_D (2x+2y) \, \mathrm{d}x \, \mathrm{d}y$, unde $D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, \ x < y < 3x\}$.

(20p) 5. Să se calculeze $\iint\limits_S z \, \mathrm{d}S$, unde

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 4, \ x \geq 0, \ y \geq 0, \ z \geq 0\}.$$

(10p) 6. Să se calculeze $\iiint\limits_V \left(2x+3y^2+4z^3\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$, ştiind că $V = [0,1] \times [1,2] \times [2,3]$.

ANALIZĂ MATEMATICĂ II
MODEL DE EXAMEN REZOLVAT

(10p) 1. Să se calculeze: **a)** $\int 5x^4 \, \mathrm{d}x$; **b)** $\int\limits_0^{\pi/2} x \cdot \sin(x) \, \mathrm{d}x$.

Rezolvare:

a) Vom folosi formula $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + \mathcal{C}$:

$$\int 5x^4 \, \mathrm{d}x = 5 \int x^4 \, \mathrm{d}x = 5 \cdot \frac{x^5}{5} + \mathcal{C} = x^5 + \mathcal{C}.$$

□

b) Vom folosi metoda de integrare prin părţi, adică formula:

$$\int\limits_a^b f(x) \cdot g'(x) \, \mathrm{d}x = \big(f(x) \cdot g(x)\big)\Big|_a^b - \int\limits_a^b f'(x) \cdot g(x) \, \mathrm{d}x$$
$$= f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f'(x) \cdot g(x) \, \mathrm{d}x,$$

cu

$$\begin{cases} f(x) = x \\ \\ g'(x) = \sin(x) \end{cases} \Rightarrow \begin{cases} f'(x) = 1 \\ \\ g(x) = -\cos(x). \end{cases}$$

Prin urmare, avem:

$$\int\limits_0^{\pi/2} x \cdot \sin(x) \, \mathrm{d}x = \big(x \cdot (-\cos(x))\big)\Big|_0^{\pi/2} - \int\limits_0^{\pi/2} 1 \cdot (-\cos(x)) \, \mathrm{d}x$$
$$= \frac{\pi}{2} \cdot \left(-\cos\left(\frac{\pi}{2}\right)\right) - 0 \cdot \Big(-\cos(0)\Big) + \int\limits_0^{\pi/2} \cos(x) \, \mathrm{d}x$$
$$= \frac{\pi}{2} \cdot 0 + \int_0^{\pi/2} \cos(x) \, \mathrm{d}x$$
$$= 0 + \big(\sin(x)\big)\Big|_0^{\pi/2}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.$$

□

(10p) 2. Să se calculeze: $\displaystyle\int_1^2 \frac{3x^2+2x}{x^3+x^2+1}\,\mathrm{d}x.$

Rezolvare: Fie $t = x^3 + x^2 + 1$. Atunci $\mathrm{d}t = (3x^2+2x)\,\mathrm{d}x$ (obţinem această relaţie prin derivare). Mai mult, avem:

x	1	2
t	$1^3+1^2+1=3$	$2^3+2^2+1=13$

Prin urmare:

$$\int_1^2 \frac{3x^2+2x}{x^3+x^2+1}\,\mathrm{d}x = \int_1^2 \frac{1}{x^3+x^2+1}\cdot(3x^2+2x)\,\mathrm{d}x = \int_3^{13}\frac{1}{t}\,\mathrm{d}t = \ln(t)\Big|_3^{13} = \ln(13)-\ln(3).$$

$\square$

(20p) 3. Să se calculeze următoarele integrale curbilinii, ştiind că $(C):\ y = x^2,\ x\in[0,1]$:

$$\textbf{a)}\ \int_C \frac{y^2}{x}\,\mathrm{d}x + xy\,\mathrm{d}y; \qquad \textbf{b)}\ \int_C x\,\mathrm{d}s.$$

Rezolvare: Cum $(C):\ y = x^2,\ x\in[0,1]$, atunci o parametrizare a curbei C este:

$$\begin{cases} x = x(t) = t, \\ y = y(t) = t^2, \end{cases} \quad t\in[0,1] \Rightarrow \begin{cases} x'(t) = 1, \\ y'(t) = 2t. \end{cases}$$

Vom folosi următoarele formule de calcul a integralelor curbilinii de primul şi al doilea tip:

$$\int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \int_a^b \Big(P(x(t),y(t))\cdot x'(t) + Q(x(t),y(t))\cdot y'(t)\Big)\,\mathrm{d}t$$

şi

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t),y(t))\cdot\sqrt{\big(x'(t)\big)^2+\big(y'(t)\big)^2}\,\mathrm{d}t,$$

unde parametrizarea curbei C este $x = x(t)$, $y = y(t)$ şi $t\in[a,b]$.

a)

$$\begin{aligned}
\int_C \frac{y^2}{x}\,\mathrm{d}x + xy\,\mathrm{d}y &= \int_0^1 \left(\frac{y^2(t)}{x(t)}\cdot x'(t) + x(t)\cdot y(t)\cdot y'(t)\right)\mathrm{d}t \\
&= \int_0^1 \left(\frac{(t^2)^2}{t}\cdot 1 + t\cdot t^2\cdot 2t\right)\mathrm{d}t = \int_0^1 (t^3+2t^4)\,\mathrm{d}t = \frac{t^4}{4}\Big|_{t=0}^{t=1} + 2\cdot\frac{t^5}{5}\Big|_{t=0}^{t=1} \\
&= \left(\frac{1^4}{4}-\frac{0^3}{3}\right) + 2\cdot\left(\frac{1^5}{5}-\frac{0^5}{5}\right) = \left(\frac{1}{4}-\frac{0}{4}\right) + 2\cdot\left(\frac{1}{5}-\frac{0}{5}\right) = \frac{1}{4}+\frac{1}{5} = \frac{9}{20}.
\end{aligned}$$

$\square$

b)

$$\int_C y\ \mathrm{d}s = \int_0^1 y(t)\cdot\sqrt{\big(x'(t)\big)^2+\big(y'(t)\big)^2}\ \mathrm{d}t$$
$$= \int_0^1 t\cdot\sqrt{1^2+(2t)^2}\ \mathrm{d}t = \int_0^1 t\sqrt{1+4t^2}\ \mathrm{d}t.$$

Vom efectua următoarea schimbare de variabilă: $u=1+4t^2$. Atunci $\mathrm{d}u=8t\ \mathrm{d}t$,

t	0	1
$u=4t^2+1$	$4\cdot 0^2+1=1$	$4\cdot 1^2+1=5$

de unde obţinem:

$$\int_C y\ \mathrm{d}s = \frac{1}{8}\int_0^1 \sqrt{1+4t^2}\cdot 8t\ \mathrm{d}t = \frac{1}{8}\int_1^5 \sqrt{u}\ \mathrm{d}u = \frac{1}{8}\int_1^5 u^{1/2}\ \mathrm{d}u$$
$$= \frac{1}{8}\cdot\frac{u^{1/2+1}}{1/2+1}\Bigg|_{u=1}^{u=5} = \frac{1}{8}\cdot\frac{u^{3/2}}{\frac{3}{2}}\Bigg|_{u=1}^{u=5} = \frac{1}{8}\cdot\frac{2\sqrt{u^3}}{3}\Bigg|_{u=1}^{u=5}$$
$$= \frac{\sqrt{u^3}}{12}\Bigg|_{u=1}^{u=5} = \frac{\sqrt{5^3}}{12}-\frac{\sqrt{1^3}}{12} = \frac{\sqrt{125}-1}{12} = \frac{5\sqrt{5}-1}{12}.$$

$\square$

(10p) 4. Să se calculeze $\displaystyle\iint_D (2x+2y)\ \mathrm{d}x\ \mathrm{d}y$, unde $D=\{(x,y)\in\mathbb{R}^2 \mid 0<x<1,\ x<y<3x\}$.

Rezolvare: Vom folosi următoarea formulă de integrare:

Teoremă. Fie f o funcţie integrabilă pe domeniul D simplu în raport cu Oy astfel încât pentru orice $x\in[a,b]$ există integrala $F(x)=\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy$. Atunci există şi integrala $\displaystyle\int_a^b F(x)\ dx$ şi avem:

$$\iint_D f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\ \mathrm{d}y\right]\mathrm{d}x.$$

În cazul nostru, avem $a=0$, $b=1$, $g_1(x)=x$ şi $g_2(x)=3x$, pentru orice $x\in[0,1]$. Atunci:

$$\iint_D (2x+2y)\ \mathrm{d}x\mathrm{d}y = \int_0^1\left[\int_x^{3x}(2x+2y)\ \mathrm{d}y\right]\mathrm{d}x = \int_0^1\left[(2xy+y^2)\Big|_{y=x}^{y=3x}\right]\mathrm{d}x$$
$$= \int_0^1 \big(2x\cdot(3x)+(3x)^2\big)-(2x\cdot x+x^2)\ \mathrm{d}x = \int_0^1 (6x^2+9x^2)-3x^2\ \mathrm{d}x$$
$$= \int_0^1 12x^2\ \mathrm{d}x = 12\cdot\frac{x^3}{3}\Big|_0^1 = 12\left(\frac{1^3}{3}-\frac{0^3}{3}\right) = 12\cdot\frac{1}{3} = 4.$$

$\square$

(20p) 5. Să se calculeze $\iint\limits_S z \; \mathrm{d}S$, unde

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}.$$

Rezolvare: Deoarece S este dată de $x^2 + y^2 + z^2 = 4 = 2^2$, obţinem că S este o submulţime a sferei de rază 2, centrată în origine. Deoarece x, y şi z sunt pozitive, obţinem parametrizarea:

$$\begin{cases} x = x(u, v) = 2\cos u \cos v, \\ y = y(u, v) = 2\cos u \sin v, \\ z = z(u, v) = 2\sin u, \end{cases} \quad \text{unde } u \in \left[0, \frac{\pi}{2}\right],\ v \in \left[0, \frac{\pi}{2}\right].$$

Prin urmare, avem:

$$\overrightarrow{r}(u, v) = \big(2\cos u \cos v, 2\cos u \sin v, 2\sin u\big),$$

de unde obţinem:

$$\begin{cases} \overrightarrow{r_u}(u, v) = \dfrac{\partial \overrightarrow{r}}{\partial u}(u, v) = (-2\sin u \cos v, -2\sin u \sin v, 2\cos u) \\ \quad \text{şi} \\ \overrightarrow{r_v}(u, v) = \dfrac{\partial \overrightarrow{r}}{\partial v}(u, v) = (-2\cos u \sin v, 2\cos u \cos v, 0\big). \end{cases}$$

Atunci, folosindu-ne de formulele

$$\begin{cases} E(u, v) = \left\|\overrightarrow{r_u}(u, v)\right\|^2 \\ F(u, v) = \overrightarrow{r_u}(u, v) \cdot \overrightarrow{r_v}(u, v) =< \overrightarrow{r_u}(u, v), \overrightarrow{r_v}(u, v) > \\ G(u, v) = \left\|\overrightarrow{r_v}(u, v)\right\|^2 \end{cases}$$

obţinem:

$$\begin{aligned} E(u, v) &= \big\|(-2\sin u \cos v, -2\sin u \sin v, 2\cos u)\big\|^2 = (-2\sin u \cos v)^2 + (-2\sin u \sin v)^2 + (2\cos u)^2 \\ &= 4\sin^2 u \cos^2 v + 4\sin^2 u \sin^2 v + 4\cos^2 u = 4\sin^2 u\big(\cos^2 v + \sin^2 v\big) + 4\cos^2 u \\ &= 4\sin^2 u \cdot 1 + 4\cos^2 u = 4\sin^2 u + 4\cos^2 u = 4 \cdot \big(\sin^2 u + \cos^2 u\big) = 4, \end{aligned}$$

apoi

$$\begin{aligned} F(u, v) &= (-2\sin u \cos v, -2\sin u \sin v, 2\cos u) \cdot (-2\cos u \sin v, 2\cos u \cos v, 0) \\ &= (-2\sin u \cos v) \cdot (-2\cos u \sin v) + (-2\sin u \sin v) \cdot (2\cos u \cos v) + 2\cos u \cdot 0 \\ &= 4\sin u \cos u \sin v \cos v - 4\sin u \cos u \sin v \cos v + 0 = 0 \end{aligned}$$

şi

$$\begin{aligned} G(u, v) &= \big\|(-2\cos u \sin v, 2\cos u \cos v, 0)\big\|^2 = (-2\cos u \sin v)^2 + (2\cos u \cos v)^2 + 0^2 \\ &= 4\cos^2 u \sin^2 v + 4\cos^2 u \cos^2 v = 4\cos^2 u \cdot \big(\sin^2 v + \cos^2 v\big) = 4\cos^2 u \cdot 1 = 4\cos^2 u. \end{aligned}$$

Ne vom folosi apoi de următoarea formulă: *dacă suprafaţa S este descrisă de parametrizarea $\overrightarrow{r}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$, cu $u \in [a, b]$ şi $v \in [c, d]$, atunci*

$$\iint\limits_S f(x, y, z) \; \mathrm{d}S = \int\limits_a^b \int\limits_c^d f(x(u, v), y(u, v), z(u, v)) \cdot \sqrt{E(u, v) \cdot G(u, v) - F^2(u, v)} \; \mathrm{d}v \; \mathrm{d}u.$$

În cazul nostru, obţinem:

$$
\begin{aligned}
\iint\limits_S z \, \mathrm{d}S &= \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} z(u,v)\cdot\sqrt{E(u,v)\cdot G(u,v)-F^2(u,v)}\,\mathrm{d}v\,\mathrm{d}u \\
&= \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} 2\sin u\cdot\sqrt{4\cdot 4\cos^2 u-0^2}\,\mathrm{d}v\,\mathrm{d}u = \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} 2\sin u\cdot\sqrt{16\cos^2 u}\,\mathrm{d}v\,\mathrm{d}u \\
&= \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} 2\sin u\cdot 4\cdot|\cos u|\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned}
$$

Deoarece $u\in\left[0,\dfrac{\pi}{2}\right]$, atunci avem $\cos u>0$, de unde obţinem $|\cos u|=\cos u$. Obţinem astfel:

$$
\begin{aligned}
\iint\limits_S z \, \mathrm{d}S &= \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} 8\sin u\cos u\,\mathrm{d}v\,\mathrm{d}u \\
&= \int\limits_0^{\pi/2} \big(8\sin u\cos u\cdot v\big)\Big|_{v=0}^{v=\pi/2}\,\mathrm{d}u \\
&= \int\limits_0^{\pi/2} \Big(8\sin u\cos u\cdot\frac{\pi}{2}-8\sin u\cos u\cdot 0\Big)\,\mathrm{d}u \\
&= \int\limits_0^{\pi/2} 4\pi\cdot\sin u\cos u\,\mathrm{d}u = 4\pi\int\limits_0^{\pi/2}\sin u\cos u\,\mathrm{d}u.
\end{aligned}
$$

Vom efectua schimbarea de variabilă $t=\sin u$, de unde obţinem $\mathrm{d}t=\cos u\,\mathrm{d}u$,

u	0	$\pi/2$
$t=\sin u$	$\sin 0=0$	$\sin\big(\pi/2\big)=1$

şi

$$
\int\limits_S z\,\mathrm{d}S = 4\pi\int\limits_0^1 t\,\mathrm{d}t = 4\pi\cdot\frac{t^2}{2}\Big|_{t=0}^{t=1} = 4\pi\cdot\left(\frac{1^2}{2}-\frac{0^2}{2}\right) = 4\pi\cdot\frac{1}{2} = 2\pi.
$$

□

(10p) 6. Să se calculeze $\iiint\limits_V \left(2x + 3y^2 + 4z^3\right) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, ştiind că $V = [0,1] \times [1,2] \times [2,3]$.

Rezolvare: Avem:

$$\begin{aligned}
\iiint\limits_V \left(2x + 3y^2 + 4z^3\right) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \int_1^2 \int_2^3 \left(2x + 3y^2 + 4z^3\right) \,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_1^2 \left(2xz + 3y^2 z + z^4\right)\Bigg|_{z=2}^{z=3} \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_1^2 \left(\left(6x + 9y^2 + 3^4\right) - \left(4x + 6y^2 + 2^4\right)\right) \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_1^2 \left(2x + 3y^2 + (81 - 16)\right) \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_1^2 \left(2x + 3y^2 + 65\right) \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left(2xy + y^3 + 65y\right)\Bigg|_{y=1}^{y=2} \mathrm{d}x \\
&= \int_0^1 \left(\left(4x + 8 + 130\right) - \left(2x + 1 + 65\right)\right) \mathrm{d}x \\
&= \int_0^1 \left(2x + 7 + 65\right) \mathrm{d}x \\
&= \int_0^1 \left(2x + 72\right) \mathrm{d}x \\
&= \left(x^2 + 72x\right)\Bigg|_{x=0}^{x=1} = 1^2 + 72 - 0^2 - 72 \cdot 0 = 73.
\end{aligned}$$

$\square$