SEMINAR NR. 5, REZOLVĂRI
Algebră liniară şi Geometrie analitică

4. SPAŢII LINIARE EUCLIDIENE
4.1. Definiţii. Exemple

Definiţie. Fie $(\mathbb{X},+,\cdot,\mathbb{R})$ un spaţiu liniar real. O funcţie
$\langle\cdot,\cdot\rangle : \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ este *produs scalar real* dacă:
(SP_1) $\forall(\mathbf{u},\mathbf{v},\mathbf{w})\in\mathbb{X}^3 : \langle\mathbf{u}+\mathbf{w},\mathbf{v}\rangle = \langle\mathbf{u},\mathbf{v}\rangle + \langle\mathbf{w},\mathbf{v}\rangle$;
precum $(\mathbf{u}+\mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{v}$
(liniaritate în primul argument)
(SP_2) $\forall(\alpha,\mathbf{u},\mathbf{v})\in\mathbb{R}\times\mathbb{X}^2 : \langle\alpha\mathbf{u},\mathbf{v}\rangle = \alpha\langle\mathbf{u},\mathbf{v}\rangle$;
precum $(\alpha\mathbf{u})\cdot\mathbf{v} = \alpha(\mathbf{u}\cdot\mathbf{v})$
(omogeneitate în primul argument)
(SP_3) $\forall(\mathbf{u},\mathbf{v})\in\mathbb{X}^2 : \langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle$;
(simetrie)
(SP_4) $\forall\mathbf{u}\in\mathbb{X} : \langle\mathbf{u},\mathbf{u}\rangle\geq 0$;
(pozitivitate)
(SP_5) $\forall\mathbf{u}\in\mathbb{X} : [\langle\mathbf{u},\mathbf{u}\rangle = 0_{\mathbb{R}} \Leftrightarrow \mathbf{u} = \boldsymbol{\theta}_{\mathbb{X}}]$.
Definiţie. Fie $(\mathbb{X},+,\cdot,\mathbb{R})$ un spaţiu liniar real şi $\langle\cdot,\cdot\rangle : \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ un produs scalar real. Perechea ordonată $(\mathbb{X},\langle\cdot,\cdot\rangle)$ se numeşte *spaţiu liniar euclidian.*
Observaţie. Spaţiile liniare euclidiene sunt cele pe care se va studia o geometrie a vectorilor (lungime, unghi) şi o analiză matematică a vectorilor (convergenţă).
Observaţie. Fie $(\mathbb{X},\langle\cdot,\cdot\rangle)$ un spaţiu liniar euclidian. Atunci
$\left(SP_1^1\right)$ $\forall(\mathbf{u},\mathbf{v},\mathbf{w})\in\mathbb{X}^3 : \langle\mathbf{u},\mathbf{v}+\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{v}\rangle + \langle\mathbf{u},\mathbf{w}\rangle$;
$\left(SP_2^1\right)$ $\forall(\alpha,\mathbf{u},\mathbf{v})\in\mathbb{R}\times\mathbb{X}^2 : \langle\mathbf{u},\alpha\mathbf{v}\rangle = \alpha\langle\mathbf{u},\mathbf{v}\rangle$;
(SP_6) $\forall\mathbf{u}\in\mathbb{X} : \langle\mathbf{u},\boldsymbol{\theta}_{\mathbb{X}}\rangle = \langle\boldsymbol{\theta}_{\mathbb{X}},\mathbf{u}\rangle = 0_{\mathbb{R}}$.
Exemple de spaţii liniare euclidiene standard :
1. $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ este un spaţiu liniar euclidian, unde
$\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $\langle\mathbf{x},\mathbf{y}\rangle = \boxed{\langle(x_1,x_2,...,x_n),(y_1,y_2,...,y_n)\rangle = x_1y_1+x_2y_2+...+x_ny_n}$
Renotăm $(\mathcal{M}_{n\times 1}(\mathbb{R}),+,\cdot,\mathbb{R})$ cu $(\mathbb{R}_n,+,\cdot,\mathbb{R})$. $(\mathbb{R}_n,\langle\cdot,\cdot\rangle)$ este un spaţiu liniar euclidian, unde
$$\langle\cdot,\cdot\rangle : \mathbb{R}_n\times\mathbb{R}_n\to\mathbb{R},\ \langle\mathbf{x},\mathbf{y}\rangle = \left\langle\begin{pmatrix}x_1\\x_2\\...\\x_n\end{pmatrix},\begin{pmatrix}y_1\\y_2\\...\\y_n\end{pmatrix}\right\rangle = x_1y_1+x_2y_2+...+x_ny_n.$$
2. $(\mathcal{M}_n(\mathbb{R}),\langle\cdot,\cdot\rangle)$ este un spaţiu liniar euclidian, unde
$\langle\cdot,\cdot\rangle : \mathcal{M}_n(\mathbb{R})\times\mathcal{M}_n(\mathbb{R})\to\mathbb{R}$, $\boxed{\langle\mathbf{A},\mathbf{B}\rangle = \mathrm{Tr}\left(\mathbf{A}^T\cdot\mathbf{B}\right)}$.
3. $(\mathcal{C}([a,b];\mathbb{R}),\langle\cdot,\cdot\rangle)$ este un spaţiu liniar euclidian, unde
$\langle\cdot,\cdot\rangle : \mathcal{C}([a,b];\mathbb{R})\times\mathcal{C}([a,b];\mathbb{R})\to\mathbb{R}$, $\langle\mathbf{f},\mathbf{g}\rangle = \int_a^b\mathbf{f}(x)\,\mathbf{g}(x)\,dx$.
În particular, $\left(\mathbb{R}_n^{[a,b]}[x],\langle\cdot,\cdot\rangle\right)$ este un spaţiu liniar euclidian, unde
$\langle\cdot,\cdot\rangle : \mathbb{R}_n^{[a,b]}[x]\times\mathbb{R}_n^{[a,b]}[x]\to\mathbb{R}$, $\langle\mathbf{p},\mathbf{q}\rangle = \int_a^b\mathbf{p}(x)\,\mathbf{q}(x)\,dx$.

Exerciţiul 1. În spaţiile liniare precizate se definesc aplicaţiile de mai jos. Să se precizeze care din

ele sunt produse scalare:
a) În $(\mathbb{R}^2,+,\cdot,\mathbb{R})$, $\langle\cdot,\cdot\rangle:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R},\forall\mathbf{x}=(x_1,x_2),\mathbf{y}=(y_1,y_2)\in\mathbb{R}^2:$
$\langle\mathbf{x},\mathbf{y}\rangle=\langle(x_1,x_2),(y_1,y_2)\rangle=3x_1y_1-x_1y_2-x_2y_1+2x_2y_2;$
Rezolvare. Fie $\mathbb{X}=\mathbb{R}^2$- spaţiul vectorilor (perechi).
$\mathbb{K}=\mathbb{R}$- spaţiul scalarilor.
Se reaminteşte $\langle\mathbf{x},\mathbf{y}\rangle_{\text{standard}}=\langle(x_1,x_2),(y_1,y_2)\rangle=x_1y_1+x_2y_2$.
Se verifică axiomele pentru $\langle\cdot,\cdot\rangle$ definit la exerciţiu:
(SP_1) $\forall(\mathbf{x},\mathbf{y},\mathbf{z})\in(\mathbb{R}^2)^3:\langle\mathbf{x}+\mathbf{z},\mathbf{y}\rangle=\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{z},\mathbf{y}\rangle.$
Fie $\forall(\mathbf{x},\mathbf{y},\mathbf{z})\in(\mathbb{R}^2)^3\Rightarrow$

$$\langle\mathbf{x}+\mathbf{z},\mathbf{y}\rangle=\langle(x_1,x_2)+(z_1,z_2),(y_1,y_2)\rangle\overset{+\text{ între perechi}}{=}$$
$$=\left\langle\left(\underbrace{x_1+z_1}_{\text{un }x_1\text{ pt def pr sc de la ex}},\underbrace{x_2+z_2}_{\text{un }x_2\text{ pt def pr sc de la ex}}\right),(y_1,y_2)\right\rangle\overset{\text{def. pr sc de la ex}}{=}$$
$$=3(x_1+z_1)y_1-(x_1+z_1)y_2-(x_2+z_2)y_1+2(x_2+z_2)y_2=M_1$$
$$\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{z},\mathbf{y}\rangle\overset{\text{def. pr sc de la ex}}{=}(3x_1y_1-x_1y_2-x_2y_1+2x_2y_2)+(3z_1y_1-z_1y_2-z_2y_1+2z_2y_2)=M_2.$$

Se observă că $M_1=M_2\Rightarrow(SP_1)$ este verificată.
(SP_2) $\forall(\alpha,\mathbf{x},\mathbf{y})\in\mathbb{R}\times(\mathbb{R}^2)^2:\langle\alpha\mathbf{x},\mathbf{y}\rangle=\alpha\langle\mathbf{x},\mathbf{y}\rangle.$
Fie $\forall(\alpha,\mathbf{x},\mathbf{y})\in\mathbb{R}\times(\mathbb{R}^2)^2\Rightarrow$

$$\langle\alpha\mathbf{x},\mathbf{y}\rangle=\langle\alpha(x_1,x_2),(y_1,y_2)\rangle=\langle(\alpha x_1,\alpha x_2),(y_1,y_2)\rangle=$$
$$=3(\alpha x_1)y_1-(\alpha x_1)y_2-(\alpha x_2)y_1+2(\alpha x_2)y_2=M_1$$
$$\alpha\langle\mathbf{x},\mathbf{y}\rangle=\alpha(3x_1y_1-x_1y_2-x_2y_1+2x_2y_2)=M_2.$$

Se observă că $M_1=M_2\Rightarrow(SP_2)$ este verificată.
(SP_3) $\forall(\mathbf{x},\mathbf{y})\in(\mathbb{R}^2)^2:\langle\mathbf{x},\mathbf{y}\rangle=\langle\mathbf{y},\mathbf{x}\rangle.$
Fie $\forall(\mathbf{x},\mathbf{y})\in(\mathbb{R}^2)^2\Rightarrow$

$$\langle\mathbf{x},\mathbf{y}\rangle=3x_1y_1-x_1y_2-x_2y_1+2x_2y_2=M_1$$
$$\langle\mathbf{y},\mathbf{x}\rangle=3y_1x_1-y_1x_2-y_2x_1+2y_2x_2=M_2.$$

Se observă că $M_1=M_2\Rightarrow(SP_3)$ este verificată.
(SP_4) $\forall\mathbf{x}\in\mathbb{R}^2:\langle\mathbf{x},\mathbf{x}\rangle\geq0.$
Fie $\forall\mathbf{x}\in\mathbb{R}^2\Rightarrow$

$$\langle\mathbf{x},\mathbf{x}\rangle=3x_1x_1-x_1x_2-x_2x_1+2x_2x_2=3\left(x_1-\tfrac{1}{3}x_2\right)^2+\tfrac{5}{3}(x_2)^2\geq0$$

$\Rightarrow(SP_4)$ este verificată.
(SP_5) $\forall\mathbf{x}\in\mathbb{R}^2:[\langle\mathbf{x},\mathbf{x}\rangle=0_{\mathbb{R}}\Leftrightarrow\mathbf{x}=\boldsymbol{\theta}_{\mathbb{R}^2}].$
Fie $\forall\mathbf{x}\in\mathbb{R}^2\Rightarrow$

$$[\langle\mathbf{x},\mathbf{x}\rangle=0_{\mathbb{R}}\Leftrightarrow3\left(x_1-\tfrac{1}{3}x_2\right)^2+\tfrac{5}{3}(x_2)^2=0\Leftrightarrow\begin{cases}x_1-\frac{1}{3}x_2=0\\x_2=0\end{cases}\Leftrightarrow$$
$$(x_1,x_2)=(0,0)\Leftrightarrow\mathbf{x}=\boldsymbol{\theta}_{\mathbb{R}^2}]$$

$\Rightarrow(SP_5)$ este verificată.
Deci aplicaţia $\langle\cdot,\cdot\rangle$ definită anterior este un produs scalar real pe $\mathbb{R}^2$, altul decât cel standard.
b) În $(\mathbb{R}^2,+,\cdot,\mathbb{R})$, $\langle\cdot,\cdot\rangle:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$,
$\forall\mathbf{x}=(x_1,x_2),\mathbf{y}=(y_1,y_2)\in\mathbb{R}^2:\langle\mathbf{x},\mathbf{y}\rangle=x_1y_1+2x_1y_2+10x_2y_2;$
Rezolvare . Se verifică axiomele :
(SP_1) $\forall(\mathbf{x},\mathbf{y},\mathbf{z})\in(\mathbb{R}^2)^3:\langle\mathbf{x}+\mathbf{z},\mathbf{y}\rangle=\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{z},\mathbf{y}\rangle.$
Fie $\forall(\mathbf{x},\mathbf{y},\mathbf{z})\in(\mathbb{R}^2)^3\Rightarrow$

$$\langle\mathbf{x}+\mathbf{z},\mathbf{y}\rangle=\langle(x_1,x_2)+(z_1,z_2),(y_1,y_2)\rangle=\langle(x_1+z_1,x_2+z_2),(y_1,y_2)\rangle=$$

$= (x_1 + z_1) y_1 + 2 (x_1 + z_1) y_2 + 10 (x_2 + z_2) y_2 = M_1$

$\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = x_1y_1 + 2x_1y_2 + 10x_2y_2 + z_1y_1 + 2z_1y_2 + 10z_2y_2 = M_2.$

Se observă că $M_1 = M_2 \Rightarrow (SP_1)$ este verificată.

(SP_2) $\forall (\alpha, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^2)^2 : \langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle .$

Fie $\forall (\alpha, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^2)^2 \Rightarrow$

$\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \langle \alpha (x_1, x_2), (y_1, y_2) \rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2) \rangle = (\alpha x_1) y_1 + 2 (\alpha x_1) y_2 + 10 (\alpha x_2) y_2 = M_1$

$\alpha \langle \mathbf{x}, \mathbf{y} \rangle = \alpha (x_1y_1 + 2x_1y_2 + 10x_2y_2) = M_2.$

Se observă că $M_1 = M_2 \Rightarrow (SP_2)$ este verificată.

(SP_3) $\forall (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^2 : \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle .$

Fie $\forall (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^2 \Rightarrow$

$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_1y_2 + 10x_2y_2 = M_1$

$\langle \mathbf{y}, \mathbf{x} \rangle = y_1x_1 + 2y_1x_2 + 10y_2x_2 = M_2.$

Se observă că $\exists (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^2$ a.î. $M_1 \neq M_2 \Rightarrow (SP_3)$ nu este verificată.

Deci aplicaţia $\langle \cdot, \cdot \rangle$ definită anterior nu este un produs scalar real pe $\mathbb{R}^2$.

c) În $(\mathbb{R}^2, +, \cdot, \mathbb{R}), \langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},$

$\forall \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{y} \rangle = 9x_1y_1 - 3x_1y_2 - 3x_2y_1 + x_2y_2;$

Rezolvare. Se verifică axiomele :

(SP_1) $\forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^2)^3 : \langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle .$

Fie $\forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^2)^3 \Rightarrow$

$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle (x_1, x_2) + (z_1, z_2), (y_1, y_2) \rangle = \langle (x_1 + z_1, x_2 + z_2), (y_1, y_2) \rangle =$

$= 9 (x_1 + z_1) y_1 - 3 (x_1 + z_1) y_2 - 3 (x_2 + z_2) y_1 + (x_2 + z_2) y_2 = M_1$

$\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = 9x_1y_1 - 3x_1y_2 - 3x_2y_1 + x_2y_2 + 9z_1y_1 - 3z_1y_2 - 3z_2y_1 + z_2y_2 = M_2.$

Se observă că $M_1 = M_2 \Rightarrow (SP_1)$ este verificată.

(SP_2) $\forall (\alpha, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^2)^2 : \langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle .$

Fie $\forall (\alpha, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^2)^2 \Rightarrow$

$\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \langle \alpha (x_1, x_2), (y_1, y_2) \rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2) \rangle =$

$= 9 (\alpha x_1) y_1 - 3 (\alpha x_1) y_2 - 3 (\alpha x_2) y_1 + (\alpha x_2) y_2 = M_1$

$\alpha \langle \mathbf{x}, \mathbf{y} \rangle = \alpha (9x_1y_1 - 3x_1y_2 - 3x_2y_1 + x_2y_2) = M_2.$

Se observă că $M_1 = M_2 \Rightarrow (SP_2)$ este verificată.

(SP_3) $\forall (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^2 : \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle .$

Fie $\forall (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^2 \Rightarrow$

$\langle \mathbf{x}, \mathbf{y} \rangle = 9x_1y_1 - 3x_1y_2 - 3x_2y_1 + x_2y_2 = M_1$

$\langle \mathbf{y}, \mathbf{x} \rangle = 9y_1x_1 - 3y_1x_2 - 3y_2x_1 + y_2x_2 = M_2.$

Se observă că $M_1 = M_2 \Rightarrow (SP_3)$ este verificată.

(SP_4) $\forall \mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{x} \rangle \geq 0.$

Fie $\forall \mathbf{x} \in \mathbb{R}^2 \Rightarrow$

$\langle \mathbf{x}, \mathbf{x} \rangle = 9x_1x_1 - 3x_1x_2 - 3x_2x_1 + x_2x_2 = (3x_1 - x_2)^2 \geq 0$

$\Rightarrow (SP_4)$ este verificată.

(SP_5) $\forall \mathbf{x} \in \mathbb{R}^2 : [\langle \mathbf{x}, \mathbf{x} \rangle = 0_{\mathbb{R}} \Leftrightarrow \mathbf{x} = \boldsymbol{\theta}_{\mathbb{R}^2}] .$

Fie $\forall \mathbf{x} \in \mathbb{R}^2 \Rightarrow$

$[\langle \mathbf{x}, \mathbf{x} \rangle = 0_{\mathbb{R}} \Leftrightarrow (3x_1 - x_2)^2 = 0 \Leftrightarrow$

$\Leftrightarrow \{3x_1 - x_2 = 0 \nRightarrow (x_1, x_2) = (0, 0)$ ca şi unică soluţie.

$\Rightarrow (SP_5)$ nu este verificată.

Deci aplicaţia $\langle \cdot, \cdot \rangle$ definită anterior nu este un produs scalar real pe $\mathbb{R}^2$.

d) În $(\mathbb{R}^2, +, \cdot, \mathbb{R}), \langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},$

$\forall \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{y} \rangle = (x_1)^2 (y_1)^2 + (x_2)^2 (y_2)^2;$

Rezolvare. Se verifică axiomele :

(SP_1) $\forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^2)^3 : \langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle.$

Fie $\forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^2)^3 \Rightarrow$

$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle (x_1, x_2) + (z_1, z_2), (y_1, y_2) \rangle = \langle (x_1 + z_1, x_2 + z_2), (y_1, y_2) \rangle =$

$= (x_1 + z_1)^2 (y_1)^2 + (x_2 + z_2)^2 (y_2)^2 = M_1$

$\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = (x_1)^2 (y_1)^2 + (x_2)^2 (y_2)^2 + (z_1)^2 (y_1)^2 + (z_2)^2 (y_2)^2 = M_2.$

Se observă că $\exists (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^2)^3 : M_1 \neq M_2 \Rightarrow (SP_1)$ nu este verificată.

Deci aplicaţia $\langle \cdot, \cdot \rangle$ definită anterior nu este un produs scalar real pe $\mathbb{R}^2$.

4.2. Norma euclidiană a unui vector. Unghiul dintre doi vectori

Teoremă. Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian. Atunci $(\mathbb{X}, \|\cdot\|)$ este un *spaţiu liniar normat*, *prehilbertian*, unde *norma indusă de produsul scalar* (sau *norma euclidiană*) dată de

$\|\cdot\| : \mathbb{X} \to \mathbb{R}, \|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ verifică

(N_1) $\forall (\mathbf{u}, \mathbf{v}) \in \mathbb{X} : \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|;$

(inegalitatea triunghiulară sau inegalitatea Minkovski)

(N_2) $\forall (\alpha, \mathbf{u}) \in \mathbb{R} \times \mathbb{X} : \|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|;$

(omogeneitate)

(N_3) $\forall \mathbf{u} \in \mathbb{X} : \|\mathbf{u}\| \geq 0$ şi $(\|\mathbf{u}\| = 0_{\mathbb{R}} \Leftrightarrow \mathbf{u} = \boldsymbol{\theta}_{\mathbb{X}})$.

Teoremă. Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian. Atunci:

(N_4) $\forall (\mathbf{u}, \mathbf{v}) \in \mathbb{X} : |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$

(inegalitatea Cauchy-Schwarz-Buniakowski)

Definiţie. Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian.

a) Vectorul $\mathbf{v} \in \mathbb{X}$ cu proprietatea $\|\mathbf{v}\| = 1$ se numeşte *versor* sau *vector unitar*.

b) Fie $\forall \mathbf{v} \in \mathbb{X}$, $\mathbf{v} \neq \boldsymbol{\theta}_{\mathbb{X}}$. Vectorul $\overset{\circ}{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$ se numeşte *versorul lui* $\mathbf{v}$.

Exemple de norme euclidiene standard:

1. În $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ standard, norma euclidiană standard este

   $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}, \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\langle (x_1, x_2, ..., x_n), (x_1, x_2, ..., x_n) \rangle} = \sqrt{(x_1)^2 + ... + (x_n)^2}.$

   În $(\mathbb{R}_n, \langle \cdot, \cdot \rangle)$ standard, norma euclidiană standard este

   $\|\cdot\| : \mathbb{R}_n \to \mathbb{R}, \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{(x_1)^2 + ... + (x_n)^2}.$

2. În $(\mathcal{M}_n(\mathbb{R}), \langle \cdot, \cdot \rangle)$ standard, norma euclidiană standard este

   $\|\cdot\| : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, \|\mathbf{A}\| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \sqrt{\mathrm{Tr}(\mathbf{A}^T \cdot \mathbf{A})}.$

3. În $(\mathcal{C}([a, b]; \mathbb{R}), \langle \cdot, \cdot \rangle)$ standard, norma euclidiană standard este

   $\|\cdot\| : \mathcal{C}([a, b]; \mathbb{R}) \to \mathbb{R}, \|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} = \sqrt{\int_a^b \mathbf{f}^2(x)\, dx}.$

În particular, în $\left(\mathbb{R}_n^{[a,b]}[x], \langle \cdot, \cdot \rangle\right)$ standard, norma euclidiană standard este

$\|\cdot\| : \mathbb{R}_n^{[a,b]}[x] \to \mathbb{R}, \|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_a^b \mathbf{p}^2(x)\, dx}.$

Definiţie. Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian.

a) Fie $(\mathbf{u}, \mathbf{v}) \in (\mathbb{X} \setminus \{\boldsymbol{\theta}_{\mathbb{X}}\})^2$. Soluţia unică în intervalul $[0, \pi]$, notată $\widehat{(\mathbf{u}, \mathbf{v})}$, a ecuaţiei

$$\boxed{\cos \widehat{(\mathbf{u}, \mathbf{v})} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}}$$

se numeşte *unghiul neorientat al perechii ordonate de vectori* $(\mathbf{u}, \mathbf{v})$.

b) Fie $(\mathbf{u},\mathbf{v}) \in \mathbb{X}^2$. Dacă $\langle \mathbf{u},\mathbf{v}\rangle = 0$ atunci *vectorul* $\mathbf{u}$ *este ortogonal cu vectorul* $\mathbf{v}$ şi se notează $\mathbf{u} \perp \mathbf{v}$.

Exerciţiul 5**.** Fie spaţiul liniar euclidian $\left(\mathbb{R}^3, \langle\cdot,\cdot\rangle\right)$, cu produsul scalar standard şi norma euclidiană standard.
a) Să se calculeze unghiul dintre vectorii $\mathbf{x} = (1,1,-1)\,, \mathbf{y} = \left(\sqrt{6},1,1\right)$;
b) Să se arate că vectorii $\mathbf{x} = (1,1,2)\,, \mathbf{y} = (1,1,-1)$ sunt ortogonali;
c) Să se arate că vectorul $\mathbf{x} = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ este un versor.
Rezolvare. Se reaminteşte produsul scalar standard dintre doi vectori din $\mathbb{R}^3$,
$$\langle \mathbf{x},\mathbf{y}\rangle = \langle (x_1,x_2,x_3)\,,(y_1,y_2,y_3)\rangle = x_1y_1 + x_2y_2 + x_3y_3$$
şi norma euclidiană standard a unui vector din $\mathbb{R}^3$,
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} = \sqrt{\langle (x_1,x_2,x_3)\,,(x_1,x_2,x_3)\rangle} = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}.$$
a) $$\cos\widehat{(\mathbf{x},\mathbf{y})} = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{\left\langle (1,1,-1)\,,\left(\sqrt{6},1,1\right)\right\rangle}{\|(1,1,-1)\|\,\left\|\left(\sqrt{6},1,1\right)\right\|} =$$
$$= \frac{1\cdot\sqrt{6} + 1\cdot 1 + (-1)\cdot 1}{\sqrt{1^2+1^2+(-1)^2}\sqrt{\left(\sqrt{6}\right)^2 + 1^2 + 1^2}} = \frac{1}{2} \Rightarrow \widehat{(\mathbf{x},\mathbf{y})} = \frac{\pi}{3}.$$
b) $\langle \mathbf{x},\mathbf{y}\rangle = \langle (1,1,2)\,,(1,1,-1)\rangle = 1\cdot 1 + 1\cdot 1 + 2\cdot(-1) = 0 \Rightarrow \mathbf{x} \perp \mathbf{y}$.
c) $\|\mathbf{x}\| = \left\|\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right\| = \sqrt{0^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1 \Rightarrow \mathbf{x} = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ este un versor.

Teoremă. Fie $(\mathbb{X}, \langle\cdot,\cdot\rangle)$ un spaţiu liniar euclidian. Atunci $(\mathbb{X}, d)$ este un *spaţiu metric*, unde *metrica, distanţa indusă de norma euclidiană* (sau *metrica, distanţa euclidiană*) dată de
$d : \mathbb{X}\times\mathbb{X} \to \mathbb{R}$, $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$ verifică
(M_1) $\forall(\mathbf{u},\mathbf{v},\mathbf{w}) \in \mathbb{X}^3 : d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{v},\mathbf{w})$;
(inegalitatea triunghiulară sau inegalitatea Minkovski)
(M_2) $\forall(\mathbf{u},\mathbf{v}) \in \mathbb{X}^2 : d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$;
(simetrie)
(M_3) $\forall\mathbf{u} \in \mathbb{X} : d(\mathbf{u},\mathbf{v}) \geq 0$ şi $(d(\mathbf{u},\mathbf{v}) = 0_{\mathbb{R}} \Leftrightarrow \mathbf{u} = \mathbf{v})$.
Exemple de metrici euclidiene standard:
1. În $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ standard, metrica euclidiană standard este
$$d : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R},\ d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}.$$
În $(\mathbb{R}_n, \langle\cdot,\cdot\rangle)$ standard, metrica euclidiană standard este
$$d : \mathbb{R}_n\times\mathbb{R}_n \to \mathbb{R},\ d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}.$$
2. În $(\mathcal{M}_n(\mathbb{R})\,, \langle\cdot,\cdot\rangle)$ standard, metrica euclidiană standard este
$$d : \mathcal{M}_n(\mathbb{R})\times\mathcal{M}_n(\mathbb{R}) \to \mathbb{R},\ d(\mathbf{A},\mathbf{B}) = \|\mathbf{A}-\mathbf{B}\| = \sqrt{\mathrm{Tr}\left((\mathbf{A}-\mathbf{B})^T\cdot(\mathbf{A}-\mathbf{B})\right)}.$$
3. În $(\mathcal{C}([a,b];\mathbb{R})\,, \langle\cdot,\cdot\rangle)$ standard, metrica euclidiană standard este
$$d : \mathcal{C}([a,b];\mathbb{R})\times\mathcal{C}([a,b];\mathbb{R}) \to \mathbb{R},\ d(\mathbf{f},\mathbf{g}) = \|\mathbf{f}-\mathbf{g}\| = \sqrt{\int_a^b (\mathbf{f}(x)-\mathbf{g}(x))^2\,dx}.$$
În particular, în $\left(\mathbb{R}_n^{[a,b]}[x], \langle\cdot,\cdot\rangle\right)$ standard, metrica euclidiană standard este
$$d : \mathbb{R}_n^{[a,b]}[x]\times\mathbb{R}_n^{[a,b]}[x] \to \mathbb{R},\ d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \sqrt{\int_a^b (\mathbf{p}(x)-\mathbf{q}(x))^2\,dx}.$$
Observaţie. Fie $(\mathbb{X}, \langle\cdot,\cdot\rangle)$ un spaţiu liniar euclidian, deci liniar normat prehilbertian, deci metric. Atunci $(\mathbb{X}, \mathcal{T})$ este un *spaţiu topologic*, unde topologia dată de metrica euclidiană se va defini la

Analiză Matematică drept cea a mulţimilor deschise (cu sfere). Spaţiul prehilbertian $\mathbb{X}$ se va numi spaţiu prehilbertian complet sau spaţiu Hilbert dacă orice şir Cauchy de vectori din $\mathbb{X}$ va fi un şir convergent.

4.3. Sisteme de vectori ortonormate. Procedeul de ortonormare Gram-Schmidt

Definiţie. Fie $(\mathbb{X}, \langle\cdot,\cdot\rangle)$ un spaţiu liniar euclidian şi $S = (\mathbf{w}_1, ..., \mathbf{w}_m)$, $m \leq \dim_{\mathbb{R}} \mathbb{X}$. Sistemul S este *sistem de vectori ortonormat* dacă :
(i) S este sistem de vectori *ortogonal*,
$\forall (i,j) \in \{1,...,m\}^2, i \neq j, \mathbf{w}_i \perp \mathbf{w}_j$, adică $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$.
(ii) S conţine doar *versori*,
$\forall i \in \{1,...,m\}, \|\mathbf{w}_i\| = 1$.
Exemplu. Fie $(\mathcal{C}([-\pi,\pi];\mathbb{R}), \langle\cdot,\cdot\rangle)$ standard.
a) Sistemul de vectori
$S = (\mathbf{f}_0, ..., \mathbf{f}_{2n})$, unde $\mathbf{f}_i : \mathbb{R} \to \mathbb{R}$,

	$\mathbf{f}_0(x) = 1;$
$\mathbf{f}_1(x) = \sin x;$	$\mathbf{f}_2(x) = \cos x;$
$\mathbf{f}_3(x) = \sin 2x;$	$\mathbf{f}_4(x) = \cos 2x;$
...	...
$\mathbf{f}_{2n-1}(x) = \sin nx;$	$\mathbf{f}_{2n}(x) = \cos nx.$

este un sistem ortogonal.
Într-adevăr,

$$\int_{-\pi}^{\pi} \cos(ix)\cos(jx)\,dx = 0, i \neq j.$$
$$\int_{-\pi}^{\pi} \sin(ix)\sin(jx)\,dx = 0, i \neq j.$$
$$\int_{-\pi}^{\pi} \cos(ix)\sin(jx)\,dx = 0, i \neq j.$$
$$\int_{-\pi}^{\pi} \cos(ix)\cdot 1dx = 0.$$
$$\int_{-\pi}^{\pi} \sin(ix)\cdot 1dx = 0.$$

Nu conţine versori, deoarece:

$$\int_{-\pi}^{\pi} 1\cdot 1dx = 2\pi.$$
$$\int_{-\pi}^{\pi} \cos(ix)\cos(ix)\,dx = \pi.$$
$$\int_{-\pi}^{\pi} \sin(ix)\sin(ix)\,dx = \pi.$$

b) Sistemul de vectori
$S = (\mathbf{g}_0, ..., \mathbf{g}_{2n})$, unde $\mathbf{g}_i : \mathbb{R} \to \mathbb{R}$,

	$\mathbf{g}_0(x) = \frac{1}{\sqrt{2\pi}};$
$\mathbf{g}_1(x) = \frac{1}{\sqrt{\pi}}\sin x;$	$\mathbf{g}_2(x) = \frac{1}{\sqrt{\pi}}\cos x;$
$\mathbf{g}_3(x) = \frac{1}{\sqrt{\pi}}\sin 2x;$	$\mathbf{g}_4(x) = \frac{1}{\sqrt{\pi}}\cos 2x;$
...	...
$\mathbf{g}_{2n-1}(x) = \frac{1}{\sqrt{\pi}}\sin nx;$	$\mathbf{g}_{2n}(x) = \frac{1}{\sqrt{\pi}}\cos nx.$

este un sistem ortonormat (este ortogonal şi conţine doar versori). Acest sistem de vectori se va

utiliza la definirea seriei Fourier ataşată unei funcţii periodice.
Observaţie. Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian.
a) Orice sistem de vectori ortogonal format din vectori nenuli este liniar independent.
b) Dacă $\dim_{\mathbb{K}} \mathbb{X} = n$, atunci orice sistem de n vectori ortogonal format din vectori nenuli este o bază în $\mathbb{X}$.

Exerciţiul 6. Fie spaţiul liniar euclidian $\left(\mathbb{R}^3, \langle \cdot, \cdot \rangle\right)$ cu produsul scalar standard. Să se arate că baza canonică

$$C = (\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1))$$

este un sistem de vectori ortonormat.
Rezolvare. Se reaminteşte produsul scalar standard dintre doi vectori din $\mathbb{R}^3$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

şi norma euclidiană standard a unui vector din $\mathbb{R}^3$,

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\langle (x_1, x_2, x_3), (x_1, x_2, x_3) \rangle} = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}.$$

(i) C este un sistem ortogonal de vectori, deoarece

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle (1,0,0), (0,1,0) \rangle = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0;$$
$$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle = \langle (0,1,0), (0,0,1) \rangle = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0;$$
$$\langle \mathbf{e}_3, \mathbf{e}_1 \rangle = \langle (0,0,1), (1,0,0) \rangle = 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 0.$$

(ii) C conţine doar versori, deoarece

$$\|\mathbf{e}_1\| = \|(1,0,0)\| = \sqrt{1^2 + 0^2 + 0^2} = 1;$$
$$\|\mathbf{e}_2\| = \|(0,1,0)\| = \sqrt{0^2 + 1^2 + 0^2} = 1;$$
$$\|\mathbf{e}_3\| = \|(0,0,1)\| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

Deci C este o bază ortonormată în $\mathbb{R}^3$ (dar mai există şi alte baze ortonormate în $\mathbb{R}^3$).

Exerciţiul 7. Fie spaţiul liniar euclidian $(\mathcal{M}_2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ unde produsul scalar este definit de $\langle \cdot, \cdot \rangle : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$ prin

$$\forall \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}), \forall \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) :$$
$$\langle \mathbf{A}, \mathbf{B} \rangle = a_{11} b_{11} + a_{12} b_{12} + a_{21} b_{21} + a_{22} b_{22} = \operatorname{Tr}\left(\mathbf{A}^T \cdot \mathbf{B}\right).$$

a) Să se arate că

$$S = \left(\mathbf{A}_1 = \tfrac{\sqrt{3}}{3} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{A}_2 = \tfrac{\sqrt{6}}{6} \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}, \mathbf{A}_3 = \tfrac{1}{3} \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}\right)$$

este un sistem de vectori ortonormat.
Rezolvare. Observăm că

$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \sqrt{(a_{11})^2 + (a_{12})^2 + (a_{21})^2 + (a_{22})^2} = \sqrt{\operatorname{Tr}(\mathbf{A}^T \cdot \mathbf{A})}.$$

<u>modul 1.</u>
(i) S este un sistem ortogonal de vectori, deoarece

$$\langle \mathbf{A}_1, \mathbf{A}_2 \rangle = \left\langle \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{6}}{3} \end{pmatrix} \right\rangle = \frac{\sqrt{3}}{3} \cdot \left(-\frac{\sqrt{6}}{6}\right) + \left(-\frac{\sqrt{3}}{3}\right) \cdot \left(-\frac{\sqrt{6}}{6}\right) + \frac{\sqrt{3}}{3} \cdot 0 + 0 \cdot \frac{\sqrt{6}}{3} = 0;$$

$$\langle \mathbf{A}_2, \mathbf{A}_3 \rangle = \left\langle \begin{pmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{6}}{3} \end{pmatrix}, \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \right\rangle = \left(-\frac{\sqrt{6}}{6}\right) \cdot 0 + \left(-\frac{\sqrt{6}}{6}\right) \cdot \frac{2}{3} + 0 \cdot \frac{2}{3} + \frac{\sqrt{6}}{3} \cdot \frac{1}{3} = 0;$$

$$\langle \mathbf{A}_3, \mathbf{A}_1 \rangle = \left\langle \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & 0 \end{pmatrix} \right\rangle = 0 \cdot \left(\frac{\sqrt{3}}{3}\right) + \frac{2}{3} \cdot \left(-\frac{\sqrt{3}}{3}\right) + \frac{2}{3} \cdot \frac{\sqrt{3}}{3} + \frac{1}{3} \cdot 0 = 0.$$

(ii) S conţine doar versori, deoarece

$$\|\mathbf{A}_1\| = \left\| \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & 0 \end{pmatrix} \right\| = \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2 + \left(-\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + (0)^2} = 1;$$

$$\|\mathbf{A}_2\| = \left\| \begin{pmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{6}}{3} \end{pmatrix} \right\| = \sqrt{\left(-\frac{\sqrt{6}}{6}\right)^2 + \left(-\frac{\sqrt{6}}{6}\right)^2 + 0^2 + \left(\frac{\sqrt{6}}{3}\right)^2} = 1;$$

$$\|\mathbf{A}_3\| = \left\| \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \right\| = \sqrt{0^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1.$$

modul 2.

(i) S este un sistem ortogonal de vectori, deoarece

$$\langle \mathbf{A}_1, \mathbf{A}_2 \rangle = \operatorname{Tr}\left(\mathbf{A}_1^T \cdot \mathbf{A}_2\right) = \operatorname{Tr}\left(\frac{\sqrt{3}}{3}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{\sqrt{6}}{6}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}\right) =$$

$$= \operatorname{Tr}\left(\frac{\sqrt{18}}{18}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right) = -\frac{\sqrt{18}}{18} + \frac{\sqrt{18}}{18} = 0;$$

$$\langle \mathbf{A}_2, \mathbf{A}_3 \rangle = \operatorname{Tr}\left(\mathbf{A}_2^T \cdot \mathbf{A}_3\right) = \operatorname{Tr}\left(\frac{\sqrt{6}}{6}\begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}\right) =$$

$$= \operatorname{Tr}\left(\frac{\sqrt{6}}{18}\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}\right) = 0 + 0 = 0;$$

$$\langle \mathbf{A}_3, \mathbf{A}_1 \rangle = \operatorname{Tr}\left(\mathbf{A}_3^T \cdot \mathbf{A}_1\right) = \operatorname{Tr}\left(\frac{1}{3}\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \cdot \frac{\sqrt{3}}{3}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right) =$$

$$= \operatorname{Tr}\left(\frac{\sqrt{3}}{9}\begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix}\right) = \frac{2\sqrt{3}}{9} - \frac{2\sqrt{3}}{9} = 0.$$

(ii) S conţine doar versori, deoarece

$$\|\mathbf{A}_1\| = \left\| \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\| = \sqrt{\operatorname{Tr}\left(\mathbf{A}_1^T \cdot \mathbf{A}_1\right)} = \sqrt{\operatorname{Tr}\left(\frac{\sqrt{3}}{3}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{\sqrt{3}}{3}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right)} =$$

$$= \sqrt{\operatorname{Tr}\left(\frac{\sqrt{9}}{9}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\right)} = \sqrt{\frac{2\sqrt{9}}{9} + \frac{\sqrt{9}}{9}} = 1;$$

$$\|\mathbf{A}_2\| = \left\| \frac{\sqrt{6}}{6}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \right\| = \sqrt{\operatorname{Tr}\left(\mathbf{A}_2^T \cdot \mathbf{A}_2\right)} = \sqrt{\operatorname{Tr}\left(\frac{\sqrt{6}}{6}\begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix} \cdot \frac{\sqrt{6}}{6}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}\right)} =$$

$$= \sqrt{\operatorname{Tr}\left(\frac{\sqrt{36}}{36}\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}\right)} = \sqrt{\frac{1}{6} + \frac{5}{6}} = 1;$$

$$\|\mathbf{A}_3\| = \left\| \frac{1}{3}\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \right\| = \sqrt{\operatorname{Tr}\left(\mathbf{A}_2^T \cdot \mathbf{A}_2\right)} = \sqrt{\operatorname{Tr}\left(\frac{1}{3}\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}\right)} =$$

$$= \sqrt{\operatorname{Tr}\left(\frac{1}{9}\begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}\right)} = \sqrt{\frac{4}{9} + \frac{5}{9}} = 1.$$

Deci $S = (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ este un sistem ortonormat de matrice din $\mathcal{M}_2(\mathbb{R})$.

Teoremă (*procedeul de ortonormare Gram-Schmidt*). Fie $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ un spaţiu liniar euclidian şi $S = (\mathbf{v}_1, ..., \mathbf{v}_m)$, $m \leq \dim_{\mathbb{R}} \mathbb{X}$ un sistem de vectori liniar independent, Atunci există $S_o = (\mathbf{w}_1, ..., \mathbf{w}_m)$ sistem de vectori liniar independent ortonormat astfel încât $[S_o] = [S]$.

Demonstraţie (schiţă).

Etapa 1. Se caută $\overline{S} = (\mathbf{u}_1, ..., \mathbf{u}_m)$ un sistem de vectori liniar independent ortogonal a.î. $\left[\overline{S}\right] = [S]$. Se găseşte

$$\begin{cases} \mathbf{u}_1 = \mathbf{v}_1 \\ \mathbf{u}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\ \mathbf{u}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \dfrac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 \\ \mathbf{u}_4 = \mathbf{v}_4 - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_3 \rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 \\ ... \end{cases}$$

Etapa 2. Se caută $S_o = (\mathbf{w}_1, ..., \mathbf{w}_m)$ un sistem de vectori liniar independent ortogonal care să conţină doar versori, a.î. $[S_o] = [\overline{S}]$. Se găseşte

$$\begin{cases} \mathbf{w}_1 = \dfrac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 \\ \mathbf{w}_2 = \dfrac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2 \\ \mathbf{w}_3 = \dfrac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3 \\ \mathbf{w}_4 = \dfrac{1}{\|\mathbf{u}_4\|}\mathbf{u}_4 \\ ... \end{cases}$$

Exerciţiul 8. Fie spaţiul liniar euclidian $\left(\mathbb{R}_3^{[-1,1]}[x], \langle\cdot,\cdot\rangle\right)$, unde produsul scalar standard este definit prin

$\langle\cdot,\cdot\rangle : \mathbb{R}_3^{[-1,1]}[x] \times \mathbb{R}_3^{[-1,1]}[x] \to \mathbb{R}$, $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} \mathbf{p}(x)\, \mathbf{q}(x)\, dx$.

Să se ortonormeze baza canonică din $\mathbb{R}_3^{[-1,1]}[x]$.

Rezolvare. Fie baza canonică din $\mathbb{R}_3^{[-1,1]}[x], C = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, unde

$\mathbf{v}_1 : [-1,1] \to \mathbb{R}$, $\mathbf{v}_1(x) = 1$,

$\mathbf{v}_2 : [-1,1] \to \mathbb{R}$, $\mathbf{v}_2(x) = x$,

$\mathbf{v}_3 : [-1,1] \to \mathbb{R}$, $\mathbf{v}_3(x) = x^2$,

$\mathbf{v}_4 : [-1,1] \to \mathbb{R}$, $\mathbf{v}_4(x) = x^3$.

Etapa 0. Se observă că C este un sistem liniar independent de vectori.

Etapa 1. Se caută $\overline{S} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ un sistem de vectori liniar independent ortogonal a.î. $[\overline{S}] = [S]$. Se găsesc $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 : [-1,1] \to \mathbb{R}$ funcţii polinomiale cu următoarele legi de asociere

$\mathbf{u}_1(x) = \mathbf{v}_1(x) = 1$.

$\|\mathbf{u}_1\|^2 = \int_{-1}^{1} 1^2 dx = x \mid_{-1}^{1} = 2$;

$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \int_{-1}^{1} x \cdot 1 dx = \frac{x^2}{2} \mid_{-1}^{1} = 0$;

$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = \int_{-1}^{1} x^2 \cdot 1 dx = \frac{x^3}{3} \mid_{-1}^{1} = \frac{2}{3}$;

$\langle \mathbf{v}_4, \mathbf{u}_1 \rangle = \int_{-1}^{1} x^3 \cdot 1 dx = \frac{x^4}{4} \mid_{-1}^{1} = 0$.

$$\mathbf{u}_2(x) = \mathbf{v}_2(x) - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1(x) = x - \frac{0}{2}1 = x.$$

$\|\mathbf{u}_2\|^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \mid_{-1}^{1} = \frac{2}{3}$;

$\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \int_{-1}^{1} x^2 \cdot x dx = \frac{x^4}{4} \mid_{-1}^{1} = 0$;

$\langle \mathbf{v}_4, \mathbf{u}_2 \rangle = \int_{-1}^{1} x^3 \cdot x dx = \frac{x^5}{5} \mid_{-1}^{1} = \frac{2}{5}$.

$$\mathbf{u}_3(x) = \mathbf{v}_3(x) - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1(x) - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2(x) = x^2 - \frac{\frac{2}{3}}{2}1 - \frac{0}{\frac{2}{3}}x = x^2 - \tfrac{1}{3}.$$

$\|\mathbf{u}_3\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx =$

$= \left(\frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{1}{9}x\right)|_{-1}^{1} = \frac{8}{45};$

$\langle \mathbf{v}_4, \mathbf{u}_3 \rangle = \int_{-1}^{1} x^3 \cdot \left(x^2 - \frac{1}{3}\right) dx = \int_{-1}^{1} \left(x^5 - \frac{1}{3}x^3\right) dx = \left(\frac{x^6}{6} - \frac{1}{3}\frac{x^4}{4}\right)|_{-1}^{1} = 0.$

$\mathbf{u}_4(x) = \mathbf{v}_4(x) - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1(x) - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2(x) - \dfrac{\langle \mathbf{v}_4, \mathbf{u}_3 \rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3(x) =$

$= x^3 - \dfrac{0}{2}1 - \dfrac{\frac{2}{5}}{\frac{2}{3}}x - \dfrac{0}{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right) = x^3 - \frac{3}{5}x.$

$\|\mathbf{u}_4\|^2 = \int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 dx = \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx =$

$= \left(\frac{x^7}{7} - \frac{6}{5}\frac{x^5}{5} + \frac{9}{25}\frac{x^3}{3}\right)|_{-1}^{1} = \frac{8}{175}.$

Etapa 2. Se caută $S_o = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ un sistem de vectori liniar independent ortogonal care să conţină doar versori a.î. $[S_o] = [\overline{S}]$. Se găsesc $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 : [-1, 1] \to \mathbb{R}$ funcţii polinomiale cu următoarele legi de asociere

$\mathbf{w}_1(x) = \dfrac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1(x) = \frac{1}{\sqrt{2}}1;$

$\mathbf{w}_2(x) = \dfrac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2(x) = \frac{1}{\sqrt{\frac{2}{3}}}x;$

$\mathbf{w}_3(x) = \dfrac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3(x) = \frac{1}{\sqrt{\frac{8}{45}}}\left(x^2 - \frac{1}{3}\right);$

$\mathbf{w}_4(x) = \dfrac{1}{\|\mathbf{u}_4\|}\mathbf{u}_4(x) = \frac{1}{\sqrt{\frac{8}{175}}}\left(x^3 - \frac{3}{5}x\right).$

S-a găsit $S_o = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ o bază ortonormată în $\mathbb{R}_3^{[-1,1]}[x]$. Polinoamele corespunzătoare funcţiilor polinomiale din această bază se numesc *polinoame Legendre.*

Exerciţiul 9. În spaţiul liniar euclidian $\left(\mathbb{R}_2^{[0,1]}[x], \langle\cdot,\cdot\rangle\right)$ cu produsul scalar standard, să se ortonormeze sistemul de funcţii $S = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ unde

$\mathbf{f}_1 : [0,1] \to \mathbb{R},\ \mathbf{f}_1(x) = 2,$

$\mathbf{f}_2 : [0,1] \to \mathbb{R},\ \mathbf{f}_2(x) = 2 + x,$

$\mathbf{f}_3 : [0,1] \to \mathbb{R},\ \mathbf{f}_3(x) = (2 + x)^2.$

Rezolvare. Fie produsul scalar standard din $\left(\mathbb{R}_2^{[0,1]}[x], \langle\cdot,\cdot\rangle\right)$,

$\langle\cdot,\cdot\rangle : \mathbb{R}_2^{[0,1]}[x] \times \mathbb{R}_2^{[0,1]}[x] \to \mathbb{R},\ \langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 \mathbf{p}(x)\,\mathbf{q}(x)\,dx.$

Reamintim norma euclidiană

$\|\cdot\| : \mathbb{R}_2^{[0,1]}[x] \to \mathbb{R},\ \|\mathbf{p}\| = \sqrt{\int_0^1 \mathbf{p}^2(x)\,dx}$

Etapa 0. Se studiază dacă $S = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ este un sistem liniar independent de vectori. Se caută $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ astfel încât

$\lambda_1\mathbf{f}_1 + \lambda_2\mathbf{f}_2 + \lambda_3\mathbf{f}_3 = \boldsymbol{\theta}_{\mathbb{R}_2^{[0,1]}[x]} \Leftrightarrow$

$\lambda_1\mathbf{f}_1(x) + \lambda_2\mathbf{f}_2(x) + \lambda_3\mathbf{f}_3(x) = \boldsymbol{\theta}(x), \forall x \in [0,1]$

$\lambda_1 2 + \lambda_2(2 + x) + \lambda_3(2 + x)^2 = 0, \forall x \in [0,1] \Leftrightarrow$

$$\begin{cases} 2\lambda_1 + 2\lambda_2 + 4\lambda_3 = 0 \\ 0\lambda_1 + 1\lambda_2 + 4\lambda_3 = 0 \\ 0\lambda_1 + 0\lambda_2 + 1\lambda_3 = 0 \end{cases}$$

Se calculează $\det A = \begin{vmatrix} 2 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow$ sistemul liniar omogen în necunoscutele $\lambda_1, \lambda_2, \lambda_3$este compatibil unic determinat şi admite soluţia nulă $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$ drept unică soluţie $\Rightarrow$

$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ sunt vectori liniar independenţi.
Etapa 1. Se caută $\overline{S} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ un sistem de vectori liniar independent ortogonal a.î. $[\overline{S}] = [S]$. Se găsesc $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 : [0,1] \to \mathbb{R}$ funcţii polinomiale cu următoarele legi de asociere

$\mathbf{u}_1(x) = \mathbf{f}_1(x) = 2.$

$\|\mathbf{u}_1\|^2 = \int_0^1 2^2 dx = 4x \mid_0^1 = 4;$

$\langle \mathbf{f}_2, \mathbf{u}_1 \rangle = \int_0^1 (2+x) \cdot 2dx = 5;$

$\langle \mathbf{f}_3, \mathbf{u}_1 \rangle = \int_0^1 (2+x)^2 \cdot 2dx = \frac{38}{3}.$

$$\mathbf{u}_2(x) = \mathbf{f}_2(x) - \frac{\langle \mathbf{f}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1(x) = (2+x) - \frac{5}{4}2 = x - \tfrac{1}{2}.$$

$\|\mathbf{u}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12};$

$\langle \mathbf{f}_3, \mathbf{u}_2 \rangle = \int_0^1 (2+x)^2 \cdot \left(x - \frac{1}{2}\right) dx = \frac{5}{12}.$

$$\mathbf{u}_3(x) = \mathbf{f}_3(x) - \frac{\langle \mathbf{f}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1(x) - \frac{\langle \mathbf{f}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2(x) =$$

$$= (2+x)^2 - \frac{\frac{38}{3}}{4}2 - \frac{\frac{5}{12}}{\frac{1}{12}}\left(x - \tfrac{1}{2}\right) = x^2 - x + \tfrac{1}{6}.$$

$\|\mathbf{u}_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \frac{1}{180}.$

Etapa 2. Se caută $S_o = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ un sistem de vectori liniar independent care să conţină doar versori a.î. $[S_o] = [\overline{S}]$. Se găsesc $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 : [0,1] \to \mathbb{R}$ funcţii polinomiale cu următoarele legi de asociere

$$\mathbf{w}_1(x) = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1(x) = \tfrac{1}{2}2;$$

$$\mathbf{w}_2(x) = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2(x) = \frac{1}{\sqrt{\frac{1}{12}}}\left(x - \tfrac{1}{2}\right);$$

$$\mathbf{w}_3(x) = \frac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3(x) = \frac{1}{\sqrt{\frac{1}{180}}}\left(x^2 - x + \tfrac{1}{6}\right).$$

Exerciţiul 10. Fie $\left(\mathbb{R}^4, \langle \cdot, \cdot \rangle\right)$ cu produsul scalar standard. Să se determine o bază ortonormată în spaţiul generat de

$S = (\mathbf{v}_1 = (1,2,2-1), \mathbf{v}_2 = (1,1,-5,3), \mathbf{v}_3 = (3,2,8,-7)).$

Rezolvare. Fie produsul scalar standard din $\left(\mathbb{R}^4, \langle \cdot, \cdot \rangle\right)$, $\langle \cdot, \cdot \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ definit prin

$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$

Reamintim norma euclidiană

$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$

Etapa 0. Se studiază dacă $S = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ este un sistem liniar independent de vectori. Se caută $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ astfel încât

$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \boldsymbol{\theta}_{\mathbb{R}^4} \Leftrightarrow$

$\lambda_1(1,2,2-1) + \lambda_2(1,1,-5,3) + \lambda_3(3,2,8,-7) = (0,0,0,0) \Leftrightarrow$

$$\begin{cases} 1\lambda_1 + 1\lambda_2 + 3\lambda_3 = 0 \\ 2\lambda_1 + 1\lambda_2 + 2\lambda_3 = 0 \\ 2\lambda_1 - 5\lambda_2 + 8\lambda_3 = 0 \\ -1\lambda_1 + 3\lambda_2 - 7\lambda_3 = 0 \end{cases}$$

Ultimul sistem este un sistem liniar omogen în necunoscutele $\lambda_1, \lambda_2, \lambda_3$, care admite măcar soluţia nulă $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$. Se studiază dacă admite şi alte soluţii.

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & -5 & 8 & 0 \\ -1 & 3 & -7 & 0 \end{array}\right) \underset{\substack{l_1 \\ l_2-2l_1 \\ l_3-2l_1 \\ l_4+l_1}}{\overset{pas1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & \boxed{-1} & -4 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right) \underset{\substack{l_1 \\ l_2 \\ l_3-7l_2 \\ l_4+4l_2}}{\overset{pas2}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \boxed{30} & 0 \\ 0 & 0 & -20 & 0 \end{array}\right)$$

$\Rightarrow$ rang $A = 3 \Rightarrow$ sistemul este compatibil unic determinat cu soluţia unică $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$ $\Rightarrow \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ sunt vectori liniar independenţi.

Etapa 1. Se caută $\overline{S} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ un sistem de vectori liniar independent ortogonal a.î. $[\overline{S}] = [S]$. Se găsesc

$\mathbf{u}_1 = \mathbf{v}_1 = (1, 2, 2-1)$.

$\|\mathbf{u}_1\|^2 = \|(1,2,2-1)\|^2 = 1^2 + 2^2 + 2^2 + (-1)^2 = 10$;

$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (1,1,-5,3), (1,2,2-1) \rangle = 1 \cdot 1 + 1 \cdot 2 + (-5) \cdot 2 + 3 \cdot (-1) = -10$;

$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = \langle (3,2,8,-7), (1,2,2-1) \rangle = 3 \cdot 1 + 2 \cdot 2 + 8 \cdot 2 + (-7) \cdot (-1) = 30$.

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1,1,-5,3) - \tfrac{-10}{10}(1,2,2-1) = (2,3,-3,2).$$

$\|\mathbf{u}_2\|^2 = \|(2,3,-3,2)\|^2 = 2^2 + 3^2 + (-3)^2 + 2^2 = 26$;

$\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \langle (3,2,8,-7), (2,3,-3,2) \rangle = 3 \cdot 2 + 2 \cdot 3 + 8 \cdot (-3) + (-7) \cdot 2 = -26$.

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 =$$
$$= (3,2,8,-7) - \tfrac{30}{10}(1,2,2-1) - \tfrac{-26}{26}(2,3,-3,2) = (2,-1,-1,-2).$$

$\|\mathbf{u}_3\|^2 = \|(2,-1,-1,-2)\|^2 = 2^2 + (-1)^2 + (-1)^2 + (-2)^2 = 10$.

Etapa 2. Se caută $S_o = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ un sistem de vectori liniar independent ortogonal care să conţină doar versori a.î. $[S_o] = [\overline{S}]$. Se găseşte

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \tfrac{1}{\sqrt{10}}(1,2,2-1);$$
$$\mathbf{w}_2 = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2 = \tfrac{1}{\sqrt{26}}(2,3,-3,2);$$
$$\mathbf{w}_3 = \frac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3 = \tfrac{1}{\sqrt{10}}(2,-1,-1,-2).$$

S-a găsit $S_o = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ o bază ortonormată în $[S]$.

Observaţie. Descompunerea QR.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{2}\sqrt{3} & 0 \\ \frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{2}\sqrt{3} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{2}\sqrt{3} & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{2}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{2}\sqrt{3} \\ 0 & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$