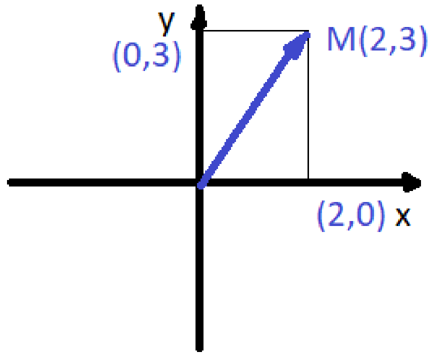
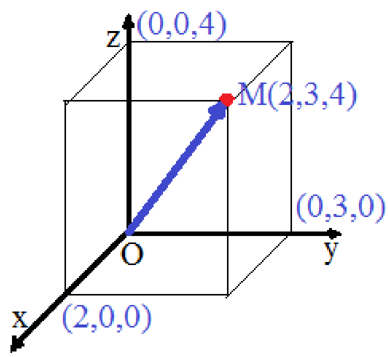
SEMINAR NR. 9, REZOLVĂRI
Algebră liniară şi Geometrie analitică

GEOMETRIE ANALITICĂ

Peste tot la disciplina Geometrie analitică se utilizează notaţiile (x, y) şi (x, y, z) în locul notaţiilor (x_1, x_2) şi (x_1, x_2, x_3) de la algebra liniară.





7. Algebra liniară a vectorilor liberi (din $\mathbb{V}_3$). Produse (scalar, vectorial, mixt)

Definiţie. Se numeşte *produsul scalar al perechii ordonate de vectori liberi* $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathbb{V}_3)^2$ numărul real definit prin

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \begin{cases} 0, & \text{dacă } \overrightarrow{u} = \overrightarrow{0} \text{ sau } \overrightarrow{v} = \overrightarrow{0} \\ \|\overrightarrow{u}\| \, \|\overrightarrow{v}\| \cos\left((\widehat{\overrightarrow{u}, \overrightarrow{v}}) \right), & \text{dacă } \overrightarrow{u} \neq \overrightarrow{0} \text{ şi } \overrightarrow{v} \neq \overrightarrow{0}. \end{cases}$$

Expresie analitică. Dacă $\overrightarrow{u} = x_1 \overrightarrow{i} + y_1 \overrightarrow{j} + z_1 \overrightarrow{k}$, $\overrightarrow{v} = x_2 \overrightarrow{i} + y_2 \overrightarrow{j} + z_2 \overrightarrow{k}$, atunci

$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2$,

$\|\overrightarrow{u}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Interpretare geometrică. a) $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0_{\mathbb{R}} \Leftrightarrow [\overrightarrow{u} = \overrightarrow{0}$ sau $\overrightarrow{v} = \overrightarrow{0}$ sau $\overrightarrow{u} \perp \overrightarrow{v}]$.

b) Dacă $\langle \overrightarrow{u}, \overrightarrow{v} \rangle \neq 0_{\mathbb{R}}$ atunci $(\widehat{\overrightarrow{u}, \overrightarrow{v}})$ are măsura în radiani, din $[0, \pi] \setminus \left\{ \frac{\pi}{2} \right\}$, dată de

$$\cos\left((\widehat{\overrightarrow{u}, \overrightarrow{v}}) \right) = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{u}\| \, \|\overrightarrow{v}\|}.$$

Proprietăţi. Cele din curs, în particular

$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \langle \overrightarrow{v}, \overrightarrow{u} \rangle$ (simetria).

Definiţie. Se numeşte *produsul vectorial al perechii ordonate de vectori liberi* $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathbb{V}_3)^2$ vectorul liber definit prin

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{cases} \overrightarrow{0}, & \text{dacă } \overrightarrow{u} = \overrightarrow{0} \text{ sau } \overrightarrow{v} = \overrightarrow{0} \text{ sau } \overrightarrow{u} \text{ şi } \overrightarrow{v} \text{ sunt coliniari} \\ \|\overrightarrow{u}\| \, \|\overrightarrow{v}\| \sin\left((\widehat{\overrightarrow{u}, \overrightarrow{v}}) \right) \overrightarrow{e}, & \text{dacă } \overrightarrow{u} \text{ şi } \overrightarrow{v} \text{ sunt necoliniari (deci şi } \overrightarrow{u} \neq \overrightarrow{0} \text{ şi } \overrightarrow{v} \neq \overrightarrow{0}). \end{cases}$$

unde $\overrightarrow{e}$ este un versor perpendicular şi pe $\overrightarrow{u}$ şi pe $\overrightarrow{v}$, astfel încât $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{e})$ să fie bază pozitiv orientată.

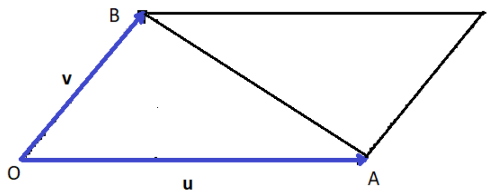
Expresie analitică. Dacă $\overrightarrow{u} = x_1 \overrightarrow{i} + y_1 \overrightarrow{j} + z_1 \overrightarrow{k}$, $\overrightarrow{v} = x_2 \overrightarrow{i} + y_2 \overrightarrow{j} + z_2 \overrightarrow{k}$, atunci

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \overrightarrow{k}.$$

Interpretare geometrică.

a) $\overrightarrow{u} \times \overrightarrow{v} = \overrightarrow{0} \Leftrightarrow [\overrightarrow{u} = \overrightarrow{0}$ sau $\overrightarrow{v} = \overrightarrow{0}$ sau $\overrightarrow{u}$ şi $\overrightarrow{v}$ sunt coliniari (paraleli ca vectori liberi sau pe aceeaşi dreaptă dacă se aleg reprezentanţi construiţi cu sursa în acelaşi punct)].
b) Dacă $\overrightarrow{u} \times \overrightarrow{v} \neq \overrightarrow{0}$ atunci $\|\overrightarrow{u} \times \overrightarrow{v}\|$ este aria paralelogramului construit pe vectorii $\overrightarrow{u}$ şi $\overrightarrow{v}$. Mai mult $\frac{1}{2}\|\overrightarrow{u} \times \overrightarrow{v}\|$ este aria triunghiului construit pe vectorii $\overrightarrow{u}$ şi $\overrightarrow{v}$.
Proprietăţi. Cele din curs, în particular

$\overrightarrow{u} \times \overrightarrow{v} = -\overrightarrow{v} \times \overrightarrow{u}$ (antisimetria).



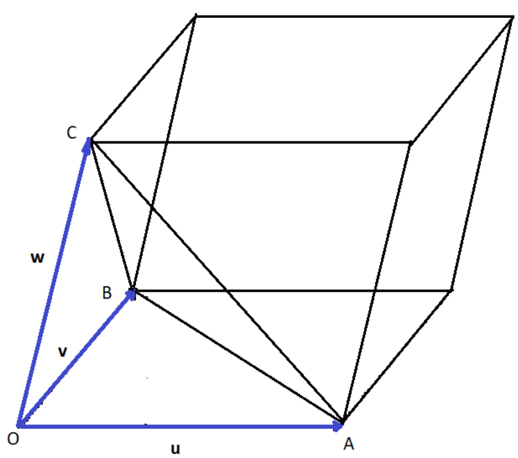
Definiţie. Se numeşte *produsul mixt al sistemului de vectori liberi* $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) \in (\mathbb{V}_3)^3$ numărul real definit prin

$\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle = \langle \overrightarrow{u}, \overrightarrow{v} \times \overrightarrow{w} \rangle$

Expresie analitică. Dacă $\overrightarrow{u} = x_1\overrightarrow{i} + y_1\overrightarrow{j} + z_1\overrightarrow{k}$, $\overrightarrow{v} = x_2\overrightarrow{i} + y_2\overrightarrow{j} + z_2\overrightarrow{k}$, $\overrightarrow{w} = x_3\overrightarrow{i} + y_3\overrightarrow{j} + z_3\overrightarrow{k}$, atunci

$$\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Interpretare geometrică.
a) $\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle = 0 \Leftrightarrow [\overrightarrow{u} = \overrightarrow{0}$ sau $\overrightarrow{v} = \overrightarrow{0}$ sau $\overrightarrow{w} = \overrightarrow{0}$ sau $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ sunt coplanari].
b) Dacă $\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle \neq 0$ atunci $|\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle|$ este volumul paralelipipedului construit pe vectorii $\overrightarrow{u}, \overrightarrow{v}$ şi $\overrightarrow{w}$. Mai mult $\frac{1}{6}|\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle|$ este volumul tetraedrului construit pe vectorii $\overrightarrow{u}, \overrightarrow{v}$ şi $\overrightarrow{w}$.



Proprietăţi. Cele din curs, în particular

$\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle = \langle\langle \overrightarrow{v}, \overrightarrow{w}, \overrightarrow{u} \rangle\rangle = \langle\langle \overrightarrow{w}, \overrightarrow{u}, \overrightarrow{v} \rangle\rangle$ (invarianţa la permutări circulare).

$\langle\langle \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \rangle\rangle = -\langle\langle \overrightarrow{u}, \overrightarrow{w}, \overrightarrow{v} \rangle\rangle = -\langle\langle \overrightarrow{w}, \overrightarrow{v}, \overrightarrow{u} \rangle\rangle = -\langle\langle \overrightarrow{v}, \overrightarrow{u}, \overrightarrow{w}, \rangle\rangle$ (comportarea la inversiuni)

Definiţie. Fie $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) \in (\mathbb{V}_3)^3$. Se numeşte *dublul produs vectorial al sistemului de vectori liberi* $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ vectorul liber definit prin $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w})$.
Teoremă de calcul. $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) = \langle \overrightarrow{u}, \overrightarrow{w} \rangle \overrightarrow{v} - \langle \overrightarrow{u}, \overrightarrow{v} \rangle \overrightarrow{w}$.

Exerciţiul 1. Se consideră vectorii $\overrightarrow{u} = 2\overrightarrow{i} - 5\overrightarrow{j} + 3\overrightarrow{k}$, $\overrightarrow{v} = \overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$. Se cere :
a) unghiul dintre vectorii $\overrightarrow{u}$ şi $\overrightarrow{v}$;
b) înalţimea corespunzătoare bazei $\overrightarrow{u}$ a paralelogramului construit pe vectorii $\overrightarrow{u}$ şi $\overrightarrow{v}$.
Rezolvare. a) Cum $\overrightarrow{u} \neq \overrightarrow{0}$ şi $\overrightarrow{v} \neq \overrightarrow{0} \Rightarrow$

$$\cos\left(\widehat{(\overrightarrow{u}, \overrightarrow{v})}\right) \overset{I.G.}{=} \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{u}\| \, \|\overrightarrow{v}\|} \overset{E.A.}{=} \frac{2\cdot 1 + (-5)\cdot 1 + 3\cdot(-1)}{\sqrt{2^2+(-5)^2+3^2}\sqrt{1^2+1^2+(-1)^2}} = \frac{-6}{\sqrt{38}\sqrt{3}}.$$

Cum $\frac{-6}{\sqrt{38}\sqrt{3}} \in \left]-1, 1\right[$ (chiar $\in \left]-1, 0\right[$) $\Rightarrow \alpha = \widehat{(\overrightarrow{u}, \overrightarrow{v})}$ este unica soluţie din $\left]0, \pi\right[$ (chiar din $\left]\frac{\pi}{2}, \pi\right[$) a ecuaţiei trigonometrice $\cos\alpha = \frac{-6}{\sqrt{38}\sqrt{3}}$.
b) Fie $\mathcal{A}$ aria paralelogramului construit pe vectorii $\overrightarrow{u}$ şi $\overrightarrow{v}$ şi h înalţimea acestui paralelogram corespunzătoare bazei $\overrightarrow{u}$. Atunci

$$\mathcal{A} = \|\overrightarrow{u} \times \overrightarrow{v}\| \text{ şi } \mathcal{A} = h\,\|\overrightarrow{u}\| \Rightarrow \boxed{h = \frac{\|\overrightarrow{u} \times \overrightarrow{v}\|}{\|\overrightarrow{u}\|}}.$$

Cum

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -5 & 3 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -5 & 3 \\ 1 & -1 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} 2 & -5 \\ 1 & 1 \end{vmatrix} \overrightarrow{k} =$$
$$= 2\overrightarrow{i} + 5\overrightarrow{j} + 7\overrightarrow{k} \Rightarrow$$
$$\|\overrightarrow{u} \times \overrightarrow{v}\| = \sqrt{2^2 + 5^2 + 7^2} = \sqrt{78},$$

se obţine $h = \frac{\sqrt{78}}{\sqrt{38}} = \frac{\sqrt{39}}{\sqrt{19}}$.

Exerciţiul 2. Se dau vectorii

$$\overrightarrow{a} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k},$$
$$\overrightarrow{b} = \overrightarrow{i} - \overrightarrow{j},$$
$$\overrightarrow{c} = -\overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k},$$

unde $\left(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right)$ este o bază ortonormată pozitiv orientată. Să se calculeze:
a) versorul vectorului $\overrightarrow{a}$;
b) $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle$, $\overrightarrow{b} \times \overrightarrow{c}$, $\left\langle\left\langle \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \right\rangle\right\rangle$, $\left\langle\left\langle \overrightarrow{b}, \overrightarrow{a}, \overrightarrow{c} \right\rangle\right\rangle$, $\overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c}\right)$;
c) aria paralelogramului construit cu vectorii $\overrightarrow{a}$ şi $\overrightarrow{b}$;
d) volumul tetraedrului construit pe vectorii $\overrightarrow{a}$, $\overrightarrow{b}$ şi $\overrightarrow{c}$;
e) înălţimile paralelogramului construit pe vectorii $\overrightarrow{a}$ şi $\overrightarrow{c}$;
f) înălţimile paralelipipedului construit pe vectorii $\overrightarrow{a}$, $\overrightarrow{b}$ şi $\overrightarrow{c}$;
$\circ$**g)** vectorul bisector al unghiului format de vectorii $\overrightarrow{a}$ şi $\overrightarrow{c}$.
Rezolvare.
a) $\boxed{\overrightarrow{a}_0 = \frac{1}{\|\overrightarrow{a}\|}\overrightarrow{a}} = \frac{1}{\sqrt{1^2+1^2+1^2}}\left(\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}\right) = \frac{1}{\sqrt{3}}\overrightarrow{i} + \frac{1}{\sqrt{3}}\overrightarrow{j} + \frac{1}{\sqrt{3}}\overrightarrow{k}$.
b) •<u>modul 1.</u> $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle \overset{E.A}{=} 1\cdot 1 + 1\cdot(-1) + 1\cdot 0 = 0.$
<u>modul 2.</u> $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = \left\langle \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}, \overrightarrow{i} - \overrightarrow{j} \right\rangle =$

$$= \left\langle \overrightarrow{i}, \overrightarrow{i} \right\rangle - \left\langle \overrightarrow{i}, \overrightarrow{j} \right\rangle + \left\langle \overrightarrow{j}, \overrightarrow{i} \right\rangle - \left\langle \overrightarrow{j}, \overrightarrow{j} \right\rangle + \left\langle \overrightarrow{k}, \overrightarrow{i} \right\rangle - \left\langle \overrightarrow{k}, \overrightarrow{j} \right\rangle \overset{\boxed{\left\langle \overrightarrow{i}, \overrightarrow{i} \right\rangle = 1^2} - \overrightarrow{i} \text{ versor}}{\underset{\boxed{\left\langle \overrightarrow{i}, \overrightarrow{j} \right\rangle = 0} \text{ ca vectori ortogonali}}{=}}$$

$$= 1 - 0 - 0 - 1 + 0 - 0 = 0.$$

$\left\langle \vec{a}, \vec{b} \right\rangle = 0 \Rightarrow \vec{a} \perp \vec{b}$.

•<u>modul 1.</u> $\vec{b} \times \vec{c} \overset{E.A}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = -2\vec{i} - 2\vec{j}$.

<u>modul 2.</u> $\vec{b} \times \vec{c} = \left(\vec{i} - \vec{j}\right) \times \left(-\vec{i} + \vec{j} + 2\vec{k}\right) =$

$$= \vec{i} \times \left(-\vec{i}\right) + \vec{i} \times \vec{j} + \vec{i} \times \left(2\vec{k}\right) + \left(-\vec{j}\right) \times \left(-\vec{i}\right) + \left(-\vec{j}\right) \times \vec{j} + \left(-\vec{j}\right) \times \left(2\vec{k}\right) =$$

$$= -\vec{i} \times \vec{i} + \vec{i} \times \vec{j} + 2\vec{i} \times \vec{k} + \vec{j} \times \vec{i} - \vec{j} \times \vec{j} - 2\vec{j} \times \vec{k} \underset{\boxed{\vec{i} \times \vec{j} = \vec{k},\ \vec{j} \times \vec{i} = -\vec{k}}\text{ plus permutări circ.}}{\overset{\boxed{\vec{i} \times \vec{i} = \vec{0}}\text{ – }\vec{i}\text{ coliniar cu }\vec{i}}{=}}$$

$$= -\vec{0} + \vec{k} - 2\vec{j} - \vec{k} - \vec{0} - 2\vec{i} = -2\vec{i} - 2\vec{j}.$$

$\vec{b} \times \vec{c} \neq \vec{0} \Rightarrow \vec{b}, \vec{c}$ nu sunt coliniari.

(Menţionăm că $\vec{b}$ şi $\vec{c}$ sunt coliniari$\Leftrightarrow$ au cordonatele în baza $\left(\vec{i}, \vec{j}, \vec{k}\right)$ proporţionale.

•<u>modul 1.</u> $\left\langle\left\langle \vec{a}, \vec{b}, \vec{c} \right\rangle\right\rangle \overset{E.A}{=} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = -4$.

<u>modul 2.</u> $\left\langle\left\langle \vec{a}, \vec{b}, \vec{c} \right\rangle\right\rangle = \left\langle \vec{a}, \vec{b} \times \vec{c} \right\rangle \overset{E.A}{=} 1 \cdot (-2) + 1 \cdot (-2) + 1 \cdot 0 = -4$.

$\left\langle\left\langle \vec{a}, \vec{b}, \vec{c} \right\rangle\right\rangle \neq 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ nu sunt coplanari.

•<u>modul 1.</u> $\left\langle\left\langle \vec{b}, \vec{a}, \vec{c} \right\rangle\right\rangle \overset{E.A}{=} \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{vmatrix} = 4$.

•<u>modul 1.</u> $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \overset{E.A}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -2 & -2 & 0 \end{vmatrix} = 2\vec{i} - 2\vec{j}$.

<u>modul 2.</u> $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \langle \vec{a}, \vec{c} \rangle \vec{b} - \left\langle \vec{a}, \vec{b} \right\rangle \vec{c} \overset{E.A}{=}$

$$= 2\left(\vec{i} - \vec{j}\right) - 0\left(-\vec{i} + \vec{j} + 2\vec{k}\right) = 2\vec{i} - 2\vec{j}$$

c) $\vec{a} \times \vec{b} \overset{E.A}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \vec{i} + \vec{j} - 2\vec{k}$.

Atunci $\mathcal{A} = \left\| \vec{a} \times \vec{b} \right\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$.

d) Volumul paralelipipedului construit pe vectorii $\vec{a}$, $\vec{b}$ şi $\vec{c}$ este

$\mathcal{V}_{paralelipiped} = \left| \left\langle\left\langle \vec{a}, \vec{b}, \vec{c} \right\rangle\right\rangle \right| = |-4| = 4$.

Atunci volumul tetraedrului construit pe vectorii $\vec{a}$, $\vec{b}$ şi $\vec{c}$ este

$\mathcal{V}_{tetraedru} = \frac{1}{6}\mathcal{V}_{paralelipiped} = \frac{2}{3}$.

e) $h_{\vec{a}} = \dfrac{\| \vec{a} \times \vec{c} \|}{\| \vec{a} \|}$ şi $h_{\vec{c}} = \dfrac{\| \vec{a} \times \vec{c} \|}{\| \vec{c} \|}$.

$\vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \vec{i} - 3\vec{j} + 2\vec{k}$.

$h_{\overrightarrow{a}} = \frac{\sqrt{1^2+(-3)^2+2^2}}{\sqrt{1^2+1^2+1^2}} = \sqrt{\frac{14}{3}}$ şi $h_{\overrightarrow{c}} = \frac{\sqrt{1^2+(-3)^2+2^2}}{\sqrt{(-1)^2+1^2+2^2}} = \sqrt{\frac{14}{6}}$.

f) Deoarece $\left\langle\left\langle \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \right\rangle\right\rangle \neq 0 \Rightarrow \overrightarrow{b} \times \overrightarrow{c} \neq \overrightarrow{0}$ şi $\overrightarrow{c} \times \overrightarrow{a} \neq \overrightarrow{0}$ şi $\overrightarrow{a} \times \overrightarrow{b} \neq \overrightarrow{0}$ ($\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ necoplanari$\Rightarrow$ $\overrightarrow{b}, \overrightarrow{c}$ necoliniari şi $\overrightarrow{c}, \overrightarrow{a}$ necoliniari şi $\overrightarrow{a}, \overrightarrow{b}$ necoliniari) şi

$$\boxed{h_{\overrightarrow{b},\overrightarrow{c}} = \frac{\left|\left\langle\left\langle \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \right\rangle\right\rangle\right|}{\left\| \overrightarrow{b} \times \overrightarrow{c} \right\|}}, \boxed{h_{\overrightarrow{c},\overrightarrow{a}} = \frac{\left|\left\langle\left\langle \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \right\rangle\right\rangle\right|}{\left\| \overrightarrow{c} \times \overrightarrow{a} \right\|}}, \boxed{h_{\overrightarrow{a},\overrightarrow{b}} = \frac{\left|\left\langle\left\langle \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \right\rangle\right\rangle\right|}{\left\| \overrightarrow{a} \times \overrightarrow{b} \right\|}}.$$

$$\overrightarrow{b} \times \overrightarrow{c} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = -2\overrightarrow{i} - 2\overrightarrow{j} + 0\overrightarrow{k}; \left\| \overrightarrow{b} \times \overrightarrow{c} \right\| = \sqrt{(-2)^2 + (-2)^2 + 0^2} = 2\sqrt{2}.$$

$$\overrightarrow{c} \times \overrightarrow{a} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\overrightarrow{i} + 3\overrightarrow{j} - 2\overrightarrow{k}; \| \overrightarrow{c} \times \overrightarrow{a} \| = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{14}.$$

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k}; \left\| \overrightarrow{a} \times \overrightarrow{b} \right\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

$h_{\overrightarrow{a}} = \frac{4}{2\sqrt{2}}, h_{\overrightarrow{b}} = \frac{4}{\sqrt{14}}, h_{\overrightarrow{c}} = \frac{4}{\sqrt{6}}$.

Exerciţiul 3. Să se determine al patrulea vârf al tetraedrului $ABCD$ cu $A(4, -2, 2)$, $B(3, 1, 1)$, $C(4, 2, 0)$ ştiind că $D \in Oz$ şi că volumul tetraedrului este egal cu 4. Să se determine lungimea înălţimii coborâte din D.

Rezolvare. •Se caută $D(x_D, y_D, z_D)$ a.î.

$$\begin{cases} D \in Oz \Leftrightarrow [x_D = 0, y_D = 0] \\ \mathcal{V} = 4 \neq 0 \Leftrightarrow \frac{1}{6}\left|\left\langle\left\langle \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD} \right\rangle\right\rangle\right| = 4. \end{cases}$$

$$\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}} = \left(3\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}\right) - \left(4\overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k}\right) = -\overrightarrow{i} + 3\overrightarrow{j} - \overrightarrow{k};$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(4\overrightarrow{i} + 2\overrightarrow{j} + 0\overrightarrow{k}\right) - \left(4\overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k}\right) = 4\overrightarrow{j} - 2\overrightarrow{k};$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(x_D\overrightarrow{i} + y_D\overrightarrow{j} + z_D\overrightarrow{k}\right) - \left(4\overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k}\right) =$$
$$= (x_D - 4)\overrightarrow{i} + (y_D + 2)\overrightarrow{j} + (z_D - 2)\overrightarrow{k}.$$

$$\left\langle\left\langle \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD} \right\rangle\right\rangle = \begin{vmatrix} -1 & 3 & -1 \\ 0 & 4 & -2 \\ x_D - 4 & y_D + 2 & z_D - 2 \end{vmatrix} = 12 - 2x_D - 2y_D - 4z_D.$$

Atunci caut $D(x_D, y_D, z_D)$ a.î.

$$\begin{cases} x_D = 0 \\ y_D = 0 \\ \frac{1}{6}|12 - 2x_D - 2y_D - 4z_D| = 4 \end{cases} \Leftrightarrow \begin{cases} x_D = 0 \\ y_D = 0 \\ |3 - z_D| = 6 \end{cases} \Leftrightarrow \begin{cases} x_D = 0 \\ y_D = 0 \\ z_D = -3 \end{cases} \text{sau} \begin{cases} x_D = 0 \\ y_D = 0 \\ z_D = 9 \end{cases}$$

S-au găsit $D_1(0, 0, -3)$ şi $D_2(0, 0, 9)$ două vărfuri cu proprietăţile cerute (deci două tetraedre).

•$h_{D_1} = \frac{\mathcal{V}}{\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{BC}\right\|}$ şi $h_{D_2} = \frac{\mathcal{V}}{\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{BC}\right\|}$.

(se precizează că $\mathcal{A}_{\text{triunghi}} = \frac{1}{2}\mathcal{A}_{\text{paralelogram}}$).

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 3 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -2\overrightarrow{i} - 2\overrightarrow{j} - 4\overrightarrow{k}.$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{BC}\right\| = \sqrt{(-2)^2+(-2)^2+(-4)^2} = 2\sqrt{6}.$$

Atunci $h_{D_1} = h_{D_2} = \frac{4}{\sqrt{6}}$.

Exerciţiul 4**.** Se dau punctele $A(-1,2,-2)$, $B(-2,5,1)$, $C(-1,6,0)$, $D(2,3,-6)$. Se cere:
a) să se verifice dacă patrulaterul $ABCD$ este plan;
b) să se calculeze aria figurilor determinate de aceste puncte.

Rezolvare. a) Patrulaterul $ABCD$ este plan $\Leftrightarrow$ vectorii $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ sunt coplanari $\Leftrightarrow$

$$\left|\left\langle\left\langle \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD} \right\rangle\right\rangle\right| = 0.$$

$$\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}} = \left(-2\,\overrightarrow{i} + 5\,\overrightarrow{j} + \overrightarrow{k}\right) - \left(-\,\overrightarrow{i} + 2\,\overrightarrow{j} - 2\,\overrightarrow{k}\right) = -\,\overrightarrow{i} + 3\,\overrightarrow{j} + 3\,\overrightarrow{k};$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(-\,\overrightarrow{i} + 6\,\overrightarrow{j} + 0\,\overrightarrow{k}\right) - \left(-2\,\overrightarrow{i} + 5\,\overrightarrow{j} + \overrightarrow{k}\right) = \overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k};$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \left(2\,\overrightarrow{i} + 3\,\overrightarrow{j} - 6\,\overrightarrow{k}\right) - \left(-\,\overrightarrow{i} + 6\,\overrightarrow{j} + 0\,\overrightarrow{k}\right) = 3\,\overrightarrow{i} - 3\,\overrightarrow{j} - 6\,\overrightarrow{k}.$$

$$\left\langle\left\langle \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD} \right\rangle\right\rangle = \begin{vmatrix} -2-(-1) & 5-2 & 1-(-2) \\ -1-(-2) & 6-5 & 0-1 \\ 2-(-1) & 3-6 & (-6)-0 \end{vmatrix} = 0.$$

b) Patrulaterul $ABCD$ este plan. Se notează (π) planul în care sunt situate punctele A, B, C, D. Nu se cunoaşte ordinea vârfurilor încât patrulaterul $ABCD$ să fie convex şi să i se poată determina aria. Rămâne doar să se calculeze ariile triunghiurilor determinate de cele 4 puncte.

$$\mathcal{A}_{\Delta ABC} = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{BC}\right\|; \mathcal{A}_{\Delta BCD} = \frac{1}{2}\left\|\overrightarrow{BC}\times\overrightarrow{CD}\right\|;$$

$$\mathcal{A}_{\Delta CDA} = \frac{1}{2}\left\|\overrightarrow{CD}\times\overrightarrow{DA}\right\|; \mathcal{A}_{\Delta DAB} = \frac{1}{2}\left\|\overrightarrow{DA}\times\overrightarrow{AB}\right\|.$$

$$\overrightarrow{AB}\times\overrightarrow{BC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 3 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -6\,\overrightarrow{i} + 2\,\overrightarrow{j} - 4\,\overrightarrow{k}; \left\|\overrightarrow{AB}\times\overrightarrow{BC}\right\| = 2\sqrt{14};$$

$$\overrightarrow{BC}\times\overrightarrow{CD} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & -1 \\ 3 & -3 & -6 \end{vmatrix} = -9\,\overrightarrow{i} + 3\,\overrightarrow{j} - 6\,\overrightarrow{k}; \left\|\overrightarrow{AB}\times\overrightarrow{BC}\right\| = 3\sqrt{14};$$

$$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = \left(-\,\overrightarrow{i} + 2\,\overrightarrow{j} - 2\,\overrightarrow{k}\right) - \left(2\,\overrightarrow{i} + 3\,\overrightarrow{j} - 6\,\overrightarrow{k}\right) = -3\,\overrightarrow{i} - \overrightarrow{j} + 4\,\overrightarrow{k}.$$

$$\overrightarrow{CD}\times\overrightarrow{DA} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & -3 & -6 \\ -3 & -1 & 4 \end{vmatrix} = -18\,\overrightarrow{i} + 6\,\overrightarrow{j} - 12\,\overrightarrow{k}; \left\|\overrightarrow{CD}\times\overrightarrow{DA}\right\| = 6\sqrt{14};$$

$$\overrightarrow{DA}\times\overrightarrow{AB} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -3 & -1 & 4 \\ -1 & 3 & 3 \end{vmatrix} = -15\,\overrightarrow{i} + 5\,\overrightarrow{j} - 10\,\overrightarrow{k}; \left\|\overrightarrow{CD}\times\overrightarrow{DA}\right\| = 5\sqrt{14}.$$

S-a obţinut că

$$\mathcal{A}_{\Delta ABC} = \tfrac{1}{2}2\sqrt{14}; \mathcal{A}_{\Delta BCD} = \tfrac{1}{2}3\sqrt{14}; \mathcal{A}_{\Delta CDA} = \tfrac{1}{2}6\sqrt{14}; \mathcal{A}_{\Delta DAB} = \tfrac{1}{2}5\sqrt{14}.$$

Mai mult, cum $2\sqrt{14} + 6\sqrt{14} = 3\sqrt{14} + 5\sqrt{14} \Rightarrow ABCD$ este patrulater convex în această ordine, şi deci $\mathcal{A}_{ABCD} = 4\sqrt{14}$.