SEMINAR NR. 4, REZOLVĂRI
Matematici Speciale, AIA

FUNCŢII COMPLEXE

1. Mulţimea numerelor complexe

1.1. Definiţii. Exemple. Structura algebrică a mulţimii $\mathbb{C}$

Noţiunile **de număr complex, număr complex sub formă algebrică, număr complex sub formă trigonometrică,** precum şi **operaţii, proprietăţi-** A se vedea Curs.

Exerciţiul 1. Să se determine conjugatele următoarelor numere complexe:
a) $z = 1 + \frac{1}{e}\,\mathrm{j}; \overline{z} = 1 - \frac{1}{e}\,\mathrm{j}$.**b)** $z = \sqrt{3} - 2\,\mathrm{j}; \overline{z} = \sqrt{3} + 2\,\mathrm{j}$.
c) $z = \frac{7}{3}\,\mathrm{j} - 5 = -5 + \frac{7}{3}\,\mathrm{j}; \overline{z} = -5 - \frac{7}{3}\,\mathrm{j}$.**d)** $z = 5 = 5 + 0\,\mathrm{j}; \overline{z} = 5 - 0\,\mathrm{j} = 5$.
e) $z = 4\,\mathrm{j} = 0 + 4\,\mathrm{j}; \overline{z} = 0 - 4\,\mathrm{j}$.**f)** $z = 0 = 0 + 0\,\mathrm{j}; \overline{z} = 0 - 0\,\mathrm{j} = 0$.

Exerciţiul 2. Să se calculeze:
a) $(2 + 3\,\mathrm{j}) + (2 - \mathrm{j})(3 + 2\,\mathrm{j}) = 2 + 3\,\mathrm{j} + 6 + 4\,\mathrm{j} - 3\,\mathrm{j} - 2\,\mathrm{j}^2 = 10 + 4\,\mathrm{j}$;
b) $\left(\sqrt{2} + 3\,\mathrm{j}\right)\left(3 - \sqrt{2}\,\mathrm{j}\right) = 3\sqrt{2} - 2\,\mathrm{j} + 9\,\mathrm{j} - 3\sqrt{2}\,\mathrm{j}^2 = 6\sqrt{2} + 7\,\mathrm{j}$;
c) ${}^{1+2\,\mathrm{j})}\dfrac{4}{1 - 2\,\mathrm{j}} = \dfrac{4\,(1 + 2\,\mathrm{j})}{(1 - 2\,\mathrm{j})\,(1 + 2\,\mathrm{j})} = \dfrac{4 + 8\,\mathrm{j}}{1 - 4\,\mathrm{j}^2} = \dfrac{4 + 8\,\mathrm{j}}{5} = \dfrac{4}{5} + \dfrac{8}{5}\,\mathrm{j}$;
d) ${}^{1+\mathrm{j}\sqrt{3})}\dfrac{1 + \mathrm{j}\sqrt{3}}{1 - \mathrm{j}\sqrt{3}} = \dfrac{\left(1 + \mathrm{j}\sqrt{3}\right)^2}{\left(1 - \mathrm{j}\sqrt{3}\right)\left(1 + \mathrm{j}\sqrt{3}\right)} = \dfrac{1 + 3\,\mathrm{j}^2 + 2\,\mathrm{j}\sqrt{3}}{1 - 3\,\mathrm{j}^2} = \dfrac{-2 + 2\,\mathrm{j}\sqrt{3}}{4} = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}\,\mathrm{j}$;

Exerciţiul 3. Să se calculeze:
a) $E_1 = \mathrm{j}^6 + \mathrm{j}^{16} + \mathrm{j}^{26} + \mathrm{j}^{36} + \mathrm{j}^{46}$
Rezolvare: <u>modul 1.</u> $E_1 = \left(\mathrm{j}^2\right)^3 + \left(\mathrm{j}^2\right)^8 + \left(\mathrm{j}^2\right)^{13} + \left(\mathrm{j}^2\right)^{18} + \left(\mathrm{j}^2\right)^{23} =$
$= (-1)^3 + (-1)^8 + (-1)^{13} + (-1)^{18} + (-1)^{23} = -1 + 1 - 1 + 1 - 1 = -1.$
<u>modul 2.</u> $E_1 = \mathrm{j}^{4\cdot1+2} + \mathrm{j}^{4\cdot4+0} + \mathrm{j}^{4\cdot6+2} + \mathrm{j}^{4\cdot9+0} + \mathrm{j}^{4\cdot11+2} =$
$= \left(\mathrm{j}^4\right)^1 \cdot \mathrm{j}^2 + \left(\mathrm{j}^4\right)^4 + \left(\mathrm{j}^4\right)^6 \cdot \mathrm{j}^2 + \left(\mathrm{j}^4\right)^9 + \left(\mathrm{j}^4\right)^{11} \cdot \mathrm{j}^2 = -1 + 1 - 1 + 1 - 1 = -1.$
b) $E_2 = \mathrm{j}^1 + \mathrm{j}^2 + ... + \mathrm{j}^n, n \in \mathbb{N}, n \geq 4.$
Rezolvare: $E_2 = \mathrm{j}^1 + \mathrm{j}^2 + ... + \mathrm{j}^n = \mathrm{j} \cdot \dfrac{\mathrm{j}^n - 1}{\mathrm{j} - 1}.$

<u>cazul 1.</u> $n = 4k, k \in \mathbb{N}^* \Rightarrow \mathrm{j}^n = \left(\mathrm{j}^4\right)^k = 1 \Rightarrow E_2 = \mathrm{j} \cdot \dfrac{1 - 1}{\mathrm{j} - 1} = 0.$

<u>cazul 2.</u> $n = 4k + 1, k \in \mathbb{N}^* \Rightarrow \mathrm{j}^n = \left(\mathrm{j}^4\right)^k \cdot \mathrm{j} = \mathrm{j} \Rightarrow E_2 = \mathrm{j} \cdot \dfrac{\mathrm{j} - 1}{\mathrm{j} - 1} = \mathrm{j}.$

<u>cazul 3.</u> $n = 4k + 2, k \in \mathbb{N}^* \Rightarrow \mathrm{j}^n = \left(\mathrm{j}^4\right)^k \cdot \mathrm{j}^2 = -1 \Rightarrow E_2 = \mathrm{j} \cdot \dfrac{-1 - 1}{\mathrm{j} - 1} = \mathrm{j} \cdot {}^{1+\mathrm{j})}\dfrac{2}{1 - \mathrm{j}} = -1 + \mathrm{j}.$

<u>cazul 4.</u> $n = 4k + 3, k \in \mathbb{N}^* \Rightarrow \mathrm{j}^n = \left(\mathrm{j}^4\right)^k \cdot \mathrm{j}^3 = -\mathrm{j} \Rightarrow E_2 = \mathrm{j} \cdot \dfrac{-\mathrm{j} - 1}{\mathrm{j} - 1} = \mathrm{j} \cdot {}^{1+\mathrm{j})}\dfrac{\mathrm{j} + 1}{1 - \mathrm{j}} = -1.$

c) $E_3 = \mathrm{j}^1 \cdot \mathrm{j}^2 \cdot ... \cdot \mathrm{j}^{100}$.
Rezolvare: $E_3 = \mathrm{j}^{\frac{100\cdot101}{2}} = \mathrm{j}^{5050} = \mathrm{j}^{1262\cdot4+2} = -1.$
d) $E_4 = \mathrm{j}^n + \mathrm{j}^{n+1} + \mathrm{j}^{n+2} + \mathrm{j}^{n+3}, n \in \mathbb{N}.$
Rezolvare: $E_4 = \mathrm{j}^n\left(1 + \mathrm{j} + \mathrm{j}^2 + \mathrm{j}^3\right) = 0.$

Exerciţiul 4. Să se determine $x, y \in \mathbb{R}$ din ecuaţiile:

a) $(*)\,(5x+3\,\mathrm{j}\,y)+(2y-\mathrm{j}\,x)=3-\mathrm{j};$

Rezolvare: $(*) \Leftrightarrow (5x+2y)+\mathrm{j}\,(3y-x)=3-\mathrm{j} \Leftrightarrow \begin{cases} 5x+2y=3 \\ -x+3y=-1 \end{cases} \Leftrightarrow \begin{cases} x=\frac{11}{17} \\ y=\frac{-2}{17} \end{cases}.$

b) $(*)\,\left(-3y+\frac{1}{2}\,\mathrm{j}\,x\right)-(-8x+5\,\mathrm{j}\,y)=-2+12\,\mathrm{j};$

Rezolvare: $(*) \Leftrightarrow (-3y+8x)+\mathrm{j}\left(\frac{1}{2}x-5y\right)=-2+12\,\mathrm{j} \Leftrightarrow \begin{cases} 8x-3y=-2 \\ \frac{1}{2}x-5y=12 \end{cases} \Leftrightarrow \begin{cases} x=\frac{-92}{77} \\ y=\frac{-194}{77} \end{cases}.$

c) $(*)\,\dfrac{x-2}{1-\mathrm{j}}+\dfrac{y-3}{1+\mathrm{j}}=1-3\,\mathrm{j};$

Rezolvare: $(*) \Leftrightarrow \dfrac{^{1+\mathrm{j})}x-2}{1-\mathrm{j}}+\dfrac{^{1-\mathrm{j})}y-3}{1+\mathrm{j}}=1-3\,\mathrm{j} \Leftrightarrow \dfrac{x-2+\mathrm{j}\,(x-2)}{2}+\dfrac{y-3-\mathrm{j}\,(y-3)}{2}=1-3\,\mathrm{j} \Leftrightarrow$

$\Leftrightarrow (x-2+y-3)+\mathrm{j}\,(x-2-y+3)=2-6\,\mathrm{j} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x+y-5=2 \\ x-y+1=-6 \end{cases} \Leftrightarrow \begin{cases} x+y=7 \\ x-y=-7 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=7 \end{cases}.$

d) $(*)\,(\mathrm{j}\,x-y)^2=6-8\,\mathrm{j}+(x+\mathrm{j}\,y)^2;$

Rezolvare: $(*) \Leftrightarrow -x^2+y^2-2\,\mathrm{j}\,xy=6-8\,\mathrm{j}+x^2-y^2+2\,\mathrm{j}\,xy \Leftrightarrow$

$\Leftrightarrow \begin{cases} -x^2+y^2=6+x^2-y^2 \\ -2xy=-8+2xy \end{cases} \Leftrightarrow \begin{cases} x^2-y^2=-3 \\ xy=2 \end{cases}$

$y=\frac{2}{x} \Rightarrow x^2-\frac{4}{x^2}=-3 \Rightarrow x^4+3x^2-4=0.$

$x^2=1 \overset{x,y\in\mathbb{R}}{\Rightarrow} \begin{cases} x=1 \\ y=2 \end{cases}$ sau $\begin{cases} x=-1 \\ y=-2 \end{cases}$

$x^2=-4$ nu are soluţii $x, y \in \mathbb{R}$.

e) $(*)\,\dfrac{\mathrm{j}}{x}+\dfrac{\mathrm{j}}{y}+\dfrac{1}{a}=\dfrac{1}{x}-\dfrac{1}{y}+\dfrac{b\,\mathrm{j}}{y}, a, b \in \mathbb{R}, a \neq 0, b \neq 2;$

Rezolvare: $(*) \Leftrightarrow \begin{cases} \dfrac{1}{a}=\dfrac{1}{x}-\dfrac{1}{y} \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{b}{y} \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{a} \\ \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{b}{y}=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \dfrac{1}{x}=\dfrac{1}{y}+\dfrac{1}{a} \\ \dfrac{2}{y}-\dfrac{b}{y}=\dfrac{1}{a} \end{cases} \Leftrightarrow \begin{cases} x=\dfrac{ya}{y+a} \\ y=a\,(2-b) \end{cases} \Leftrightarrow \begin{cases} x=\dfrac{a\,(2-b)}{1-b} \\ y=a\,(2-b) \end{cases}.$

Exerciţiul 5. Fie $m \in \mathbb{R}$. Să se simplifice în $\mathbb{C}$ fracţiile

a) $\dfrac{15x^2-8mx+m^2}{12x^2-mx-m^2}=\dfrac{15\left(x-\frac{m}{3}\right)\left(x-\frac{m}{5}\right)}{12\left(x-\frac{m}{3}\right)\left(x+\frac{m}{4}\right)}=\dfrac{5\left(x-\frac{m}{5}\right)}{4\left(x+\frac{m}{4}\right)}=\dfrac{5x-m}{4x+m}.$

b) $\dfrac{x^2+3\,\mathrm{j}\,x-2}{x^2+\mathrm{j}\,x+2}=\dfrac{1\,(x+2\,\mathrm{j})\,(x+\mathrm{j})}{1\,(x+2\,\mathrm{j})\,(x-\mathrm{j})}=\dfrac{x+\mathrm{j}}{x-\mathrm{j}}.$

Exerciţiul 6. Să se scrie sub formă trigonometrică numerele

a) $z=1+\mathrm{j}$;**b)** $z=-2+2\,\mathrm{j}$;**c)** $z=-\sqrt{3}-\mathrm{j}$;**d)** $z=1-\mathrm{j}\,\sqrt{3}$

e) $z=2$;**f)** $z=-\frac{3}{2}$;**g)** $z=2\,\mathrm{j}$;**h)** $z=-2\,\mathrm{j}$

Rezolvare. A se vedea Curs

i) $z=\cos a-\mathrm{j}\sin a, a \in \mathbb{R}$.

Rezolvare. $z=1\,(\cos(-a)+\mathrm{j}\sin(-a))$, cu $r=1$ şi $t=-a \in \mathbb{R}$.

Sau $r=|z|=\sqrt{(\cos a)^2+(-\sin a)^2}=1$

$$\arg z = t^* = \begin{cases} \operatorname{arctg}\left(\dfrac{-\sin a}{\cos a}\right) + \widetilde{k}\pi, \text{ unde } \begin{cases} \widetilde{k} = 0 \text{ dacă } \cos a > 0 \text{ şi } -\sin a > 0 \\ \widetilde{k} = 1 \text{ dacă } \cos a < 0 \text{ şi } (-\sin a > 0 \text{ sau } -\sin a < 0) \\ \widetilde{k} = 2 \text{ dacă } \cos a > 0 \text{ şi } -\sin a < 0 \end{cases} \\ 0, \text{ dacă } -\sin a = 0, \cos a > 0 \\ \pi, \text{ dacă } -\sin a = 0, \cos a < 0 \\ \frac{\pi}{2}, \text{ dacă } \cos a = 0, -\sin a > 0 \\ \frac{3\pi}{2}, \text{ dacă } \cos a = 0, -\sin a < 0 \end{cases}$$

j) $z = \sin a + \mathrm{j}\,(1 + \cos a)\,, a \in \mathbb{R}$.

Rezolvare. $z = 2\sin\frac{a}{2}\cos\frac{a}{2} + \mathrm{j}\cdot 2\cos^2\frac{a}{2} = 2\cos\frac{a}{2}\left(\sin\frac{a}{2} + \mathrm{j}\cos\frac{a}{2}\right) = 2\cos\frac{a}{2}\left(\cos\frac{\pi-a}{2} + \mathrm{j}\sin\frac{\pi-a}{2}\right)$

$$= \begin{cases} 2\cos\frac{a}{2}\left(\cos\frac{\pi-a}{2} + \mathrm{j}\sin\frac{\pi-a}{2}\right), \text{ dacă } 2\cos\frac{a}{2} > 0 \\ -2\cos\frac{a}{2}\left(\cos\frac{3\pi-a}{2} + \mathrm{j}\sin\frac{3\pi-a}{2}\right), \text{ dacă } -2\cos\frac{a}{2} > 0 \end{cases},$$

cu $r = \begin{cases} 2\cos\frac{a}{2}, \text{ dacă } 2\cos\frac{a}{2} > 0 \\ -2\cos\frac{a}{2}, \text{ dacă } -2\cos\frac{a}{2} > 0 \end{cases}$ şi $t^* = \begin{cases} \frac{\pi-a}{2}, \text{ dacă } 2\cos\frac{a}{2} > 0 \\ \frac{3\pi-a}{2}, \text{ dacă } -2\cos\frac{a}{2} > 0 \end{cases}$

Sau $r = |z| = \sqrt{(\sin a)^2 + (1+\cos a)^2} = \sqrt{2 + 2\cos a} = \sqrt{4\cos^2\frac{a}{2}} = 2\left|\cos\frac{a}{2}\right|$.

$$\arg z = t^* = \begin{cases} \operatorname{arctg}\left(\dfrac{1+\cos a}{\sin a}\right) + \widetilde{k}\pi, \text{ unde } \begin{cases} \widetilde{k} = 0 \text{ dacă } \sin a > 0 \text{ şi } 1+\cos a > 0 \\ \widetilde{k} = 1 \text{ dacă } \sin a < 0 \text{ şi } (1+\cos a > 0 \text{ sau } 1+\cos a < 0) \\ \widetilde{k} = 2 \text{ dacă } \sin a > 0 \text{ şi } 1+\cos a < 0 \end{cases} \\ 0, \text{ dacă } 1+\cos a = 0, \sin a > 0 \\ \pi, \text{ dacă } 1+\cos a = 0, \sin a < 0 \\ \frac{\pi}{2}, \text{ dacă } \sin a = 0, 1+\cos a > 0 \\ \frac{3\pi}{2}, \text{ dacă } \sin a = 0, 1+\cos a < 0 \end{cases}$$

Exerciţiul 7. Să se calculeze

a) $E_1 = \left(-1 + \mathrm{j}\sqrt{3}\right)^{10}$;**b)** $E_2 = \dfrac{(1+\mathrm{j})^{2013}}{(1-\mathrm{j})^{2000}}$.

Rezolvare. a) A se vedea Curs

b) <u>modul 1.</u> (algebric-mai ales dacă se cere forma algebrică a E_2)

O variantă de calcul algebric este:

$z_1 = 1 + \mathrm{j}$

$z_1^2 = (1+\mathrm{j})^2 = 2\,\mathrm{j}$

$z_1^4 = (2\,\mathrm{j})^2 = -2^2$

$z_1^{2013} = z_1^{503\cdot 4+1} = \left(z_1^4\right)^{503}\cdot z = \left(-2^2\right)^{503}\cdot(1+\mathrm{j}) = -2^{1006}\cdot(1+\mathrm{j}) = -2^{1006} - 2^{1006}\,\mathrm{j}$.

$z_2 = 1 - \mathrm{j}$

$z_2^2 = (1-\mathrm{j})^2 = -2\,\mathrm{j}$

$z_2^4 = (-2\,\mathrm{j})^2 = -2^2$

$z_2^{2000} = z_2^{500\cdot 4+0} = \left(z_2^4\right)^{500} = \left(-2^2\right)^{500} = 2^{1000}$

$E_2 = \dfrac{-2^{1006}\cdot(1+\mathrm{j})}{2^{1000}\cdot 1} = -2^6\cdot(1+\mathrm{j}) = -2^6 - 2^6\,\mathrm{j}$.

<u>modul 2.</u> (trigonometric-mai ales dacă se cere forma trigonometrică a E_2)

Analog cu exerciţiul 6, se reprezintă trigonometric

$z_1 = 1 + \mathrm{j} = \sqrt{2}\left(\cos\dfrac{\pi}{4} + \mathrm{j}\sin\dfrac{\pi}{4}\right)$; $z_2 = 1 - \mathrm{j} = \sqrt{2}\left(\cos\dfrac{7\pi}{4} + \mathrm{j}\sin\dfrac{7\pi}{4}\right) \Rightarrow$

$$E_2 = \frac{(z_1)^{2013}}{(z_2)^{2000}} = \frac{\left(\sqrt{2}\right)^{2013}\left(\cos\dfrac{2013\pi}{4} + \mathrm{j}\sin\dfrac{2013\pi}{4}\right)}{\left(\sqrt{2}\right)^{2000}\left(\cos\dfrac{2000\cdot 7\pi}{4} + \mathrm{j}\sin\dfrac{2000\cdot 7\pi}{4}\right)} =$$

$$= \frac{\left(\sqrt{2}\right)^{2013}\left(\cos\left(503\pi + \dfrac{\pi}{4}\right) + \mathrm{j}\sin\left(502\pi + \dfrac{5\pi}{4}\right)\right)}{\left(\sqrt{2}\right)^{2000}\left(\cos\left(3500\pi\right) + \mathrm{j}\sin\left(3500\pi\right)\right)} =$$

$$= \frac{\left(\sqrt{2}\right)^{2013}\left(\cos\dfrac{5\pi}{4} + \mathrm{j}\sin\dfrac{5\pi}{4}\right)}{\left(\sqrt{2}\right)^{2000}\left(\cos\left(0\pi\right) + \mathrm{j}\sin\left(0\pi\right)\right)} = \left(\sqrt{2}\right)^{13}\left(\cos\left(\frac{5\pi}{4} - 0\pi\right) + \mathrm{j}\sin\left(\frac{5\pi}{4} - 0\pi\right)\right) =$$

$$= \left(\sqrt{2}\right)^{13}\left(\cos\frac{5\pi}{4} + \mathrm{j}\sin\frac{5\pi}{4}\right).$$

Exerciţiul 8. Fie $\forall n \in \mathbb{N}^*$. Să se calculeze $\mathrm{Re}\left(s\left(z\right)\right)$ pentru

$s\left(z\right) = 1 + z + ... + z^n$, unde $z = \dfrac{1+\mathrm{j}}{1-\mathrm{j}}$.

Rezolvare. $s\left(z\right) = 1 + z + ... + z^n = 1 \cdot \dfrac{z^{n+1} - 1}{z - 1}, z \neq 1.$

$$z = {}^{1+\mathrm{j})}\frac{1+\mathrm{j}}{1-\mathrm{j}} = \frac{1 + 2\,\mathrm{j} + \mathrm{j}^2}{1 - \mathrm{j}^2} = \mathrm{j} = 1\left(\cos\tfrac{\pi}{2} + \mathrm{j}\sin\tfrac{\pi}{2}\right).$$

$$s\left(\tfrac{1+\mathrm{j}}{1-\mathrm{j}}\right) = 1 \cdot \frac{\left(\tfrac{1+\mathrm{j}}{1-\mathrm{j}}\right)^{n+1} - 1}{z - 1} = \frac{1^{n+1}\left(\cos\tfrac{\pi}{2} + \mathrm{j}\sin\tfrac{\pi}{2}\right)^{n+1} - 1}{1\left(\cos\tfrac{\pi}{2} + \mathrm{j}\sin\tfrac{\pi}{2}\right) - 1} =$$

$$= \frac{\cos\tfrac{(n+1)\pi}{2} + \mathrm{j}\sin\tfrac{(n+1)\pi}{2} - 1}{\cos\tfrac{\pi}{2} + \mathrm{j}\sin\tfrac{\pi}{2} - 1} = \frac{2\sin^2\tfrac{(n+1)\pi}{2\cdot 2} - 2\,\mathrm{j}\sin\tfrac{(n+1)\pi}{2\cdot 2}\cos\tfrac{(n+1)\pi}{2\cdot 2}}{2\sin^2\tfrac{\pi}{2\cdot 2} - 2\,\mathrm{j}\sin\tfrac{\pi}{2\cdot 2}\cos\tfrac{\pi}{2\cdot 2}} =$$

$$= \frac{{}^{\mathrm{j})}2\sin\tfrac{(n+1)\pi}{4}\left(\sin\tfrac{(n+1)\pi}{4} - \mathrm{j}\cos\tfrac{(n+1)\pi}{4}\right)}{2\sin\tfrac{\pi}{4}\left(\sin\tfrac{\pi}{4} - \mathrm{j}\cos\tfrac{\pi}{4}\right)} = \frac{\sin\tfrac{(n+1)\pi}{4}\left(\cos\tfrac{(n+1)\pi}{4} + \mathrm{j}\sin\tfrac{(n+1)\pi}{4}\right)}{\sin\tfrac{\pi}{4}\left(\cos\tfrac{\pi}{4} + \mathrm{j}\sin\tfrac{\pi}{4}\right)} =$$

$$= \frac{\sin\tfrac{(n+1)\pi}{4}}{\sin\tfrac{\pi}{4}}\left(\cos\left(\tfrac{(n+1)\pi}{4} - \tfrac{\pi}{4}\right) + \mathrm{j}\sin\left(\tfrac{(n+1)\pi}{4} - \tfrac{\pi}{4}\right)\right) = \frac{\sin\tfrac{(n+1)\pi}{4}}{\sin\tfrac{\pi}{4}}\left(\cos\tfrac{n\pi}{4} + \mathrm{j}\sin\tfrac{n\pi}{4}\right).$$

În consecinţă, $\mathrm{Re}\left(s\left(\tfrac{1+\mathrm{j}}{1-\mathrm{j}}\right)\right) = \dfrac{\sin\tfrac{(n+1)\pi}{4}}{\sin\tfrac{\pi}{4}} \cdot \cos\tfrac{n\pi}{4}.$

Exerciţiul 9. Folosind numere sub formă trigonometrică şi formula lui Moivre, să se arate că, $\forall n \in \mathbb{N}^*$ şi $\forall x \in \mathbb{R}, x \neq 2k\pi, k \in \mathbb{Z}$, are loc

$$\cos x + \cos 2x + ... + \cos nx = \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\cos\left(\tfrac{(n+1)x}{2}\right); \sin x + \sin 2x + ... + \sin nx = \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\sin\left(\tfrac{(n+1)x}{2}\right).$$

Rezolvare. Fie $n \in \mathbb{N}^*$. Se notează

$$A_n = \cos x + \cos 2x + ... + \cos nx = \sum_{k=1}^{n}\cos kx; B_n = \sin x + \sin 2x + ... + \sin nx = \sum_{k=1}^{n}\sin kx.$$

$$A_n + \mathrm{j}\,B_n = \sum_{k=1}^{n}\cos kx + \mathrm{j}\sum_{k=1}^{n}\sin kx = \sum_{k=1}^{n}\left(\cos kx + \mathrm{j}\sin kx\right) =$$

$$= \sum_{k=1}^{n}\left(\cos x + \mathrm{j}\sin x\right)^k = \left(\cos x + \mathrm{j}\sin x\right)\frac{\left(\cos x + \mathrm{j}\sin x\right)^n - 1}{\left(\cos x + \mathrm{j}\sin x\right) - 1} =$$

$$= (\cos x + \mathrm{j}\sin x)\,\frac{(\cos nx + \mathrm{j}\sin nx) - 1}{(\cos x + \mathrm{j}\sin x) - 1} = (\cos x + \mathrm{j}\sin x)\,\frac{{}^{\mathrm{j})}2\sin^2\frac{nx}{2} + 2\,\mathrm{j}\sin\frac{nx}{2}\cos\frac{nx}{2}}{2\sin^2\frac{x}{2} + 2\,\mathrm{j}\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= (\cos x + \mathrm{j}\sin x)\,\frac{\sin\frac{nx}{2}\left(\cos\frac{nx}{2} + \mathrm{j}\sin\frac{nx}{2}\right)}{\sin\frac{x}{2}\left(\cos\frac{x}{2} + \mathrm{j}\sin\frac{x}{2}\right)} = \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\left(\cos\left(x + \frac{nx}{2} - \frac{x}{2}\right) + \mathrm{j}\sin\left(x + \frac{nx}{2} - \frac{x}{2}\right)\right).$$

Deci $A_n = \dfrac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\cos\left(\frac{(n+1)x}{2}\right)$ şi $B_n = \dfrac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\sin\left(\frac{(n+1)x}{2}\right)$.

Definiţia 3. Fie $z \in \mathbb{C}^*$ şi $n \in \mathbb{N}^*$. Se numeşte *rădăcină de ordinul* n *a* z orice număr complex Z care verifică ecuaţia

$Z^n = z.$

Propoziţia 6. Fie $z = r(\cos t^* + \mathrm{j}\sin t^*) \in \mathbb{C}^*$, cu $r = |z|, t^* = \arg z \in [0, 2\pi[$(dacă se lucrează cu tăietura Ox_+; altfel $[0+\alpha, 2\pi+\alpha[, \alpha \in \mathbb{R}$). Numărul complex nenul z are exact n rădăcini complexe de ordin n, şi anume

$$\boxed{(\sqrt[n]{z})_{\mathbb{C}} = \left\{Z_k = \sqrt[n]{r}\left(\cos\frac{t^* + 2k\pi}{n} + \mathrm{j}\sin\frac{t^* + 2k\pi}{n}\right); k \in \{0, 1, ..., n-1\}\right\}.}$$

Exerciţiul 10. Să se rezolve în $\mathbb{C}$ ecuaţiile

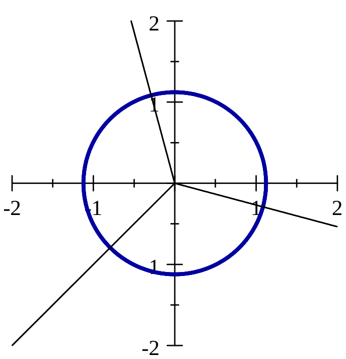
a) $(2+\mathrm{j})\,Z^3 - 3 + \mathrm{j} = 0;$

Rezolvare.$(*) \Leftrightarrow Z^3 = \dfrac{{}^{2-\mathrm{j})}3 - \mathrm{j}}{2+\mathrm{j}} \Leftrightarrow Z^3 = \dfrac{6 - 1 - 5\,\mathrm{j}}{5} \Leftrightarrow Z^3 = 1 - \mathrm{j}\,.$

Se scrie sub formă trigonometrică $1 - \mathrm{j} = \sqrt{2}\left(\cos\frac{7\pi}{4} + \mathrm{j}\sin\frac{7\pi}{4}\right)$. Atunci

$$Z^3 = \sqrt{2}\left(\cos\tfrac{7\pi}{4} + \mathrm{j}\sin\tfrac{7\pi}{4}\right) \Rightarrow$$
$$Z \in \left(\sqrt[3]{1-\mathrm{j}}\right)_{\mathbb{C}} = \left\{Z_k = \sqrt[3]{\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4} + 2k\pi}{3} + \mathrm{j}\sin\frac{\frac{7\pi}{4} + 2k\pi}{3}\right); k \in \{0, 1, 2\}\right\}$$
$$Z_0 = \sqrt[6]{2}\left(\cos\frac{\frac{7\pi}{4} + 0}{3} + \mathrm{j}\sin\frac{\frac{7\pi}{4} + 0}{3}\right) = \sqrt[6]{2}\left(\cos\tfrac{7\pi}{12} + \mathrm{j}\sin\tfrac{7\pi}{12}\right).$$
$$Z_1 = \sqrt[6]{2}\left(\cos\frac{\frac{7\pi}{4} + 2\pi}{3} + \mathrm{j}\sin\frac{\frac{7\pi}{4} + 2\pi}{3}\right) = \sqrt[6]{2}\left(\cos\tfrac{5\pi}{4} + \mathrm{j}\sin\tfrac{5\pi}{4}\right).$$
$$Z_2 = \sqrt[6]{2}\left(\cos\frac{\frac{7\pi}{4} + 4\pi}{3} + \mathrm{j}\sin\frac{\frac{7\pi}{4} + 4\pi}{3}\right) = \sqrt[6]{2}\left(\cos\tfrac{23\pi}{12} + \mathrm{j}\sin\tfrac{23\pi}{12}\right).$$

Dacă se reprezintă soluţiile ecuaţiei, Z_0, Z_1, Z_2, în planul complex, ele sunt vârfurile unui triunghi echilateral încris într-un cerc cu centrul în origine şi de rază $\sqrt[3]{\sqrt{2}}$.



b) $\left(\sqrt{3} - \mathrm{j}\right)Z^4 - 4\,\mathrm{j} = 0;$

Rezolvare.$(*) \Leftrightarrow Z^4 = \dfrac{\sqrt{3}+\mathrm{j})4\,\mathrm{j}}{\sqrt{3}-\mathrm{j}} \Leftrightarrow Z^4 = \dfrac{4\,\mathrm{j}\left(\sqrt{3}+\mathrm{j}\right)}{3+1} \Leftrightarrow Z^4 = -1+\mathrm{j}\,\sqrt{3}.$

Se scrie sub formă trigonometrică $-1+\mathrm{j}\,\sqrt{3} = 2\left(\cos\frac{2\pi}{3}+\mathrm{j}\sin\frac{2\pi}{3}\right)$. Atunci

$$Z^4 = 2\left(\cos\tfrac{2\pi}{3}+\mathrm{j}\sin\tfrac{2\pi}{3}\right) \Rightarrow$$

$$Z \in \left(\sqrt[4]{-1+\mathrm{j}\,\sqrt{3}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+2k\pi}{4}+\mathrm{j}\sin\frac{\frac{2\pi}{3}+2k\pi}{4}\right); k\in\{0,1,2,3\}\right\}$$

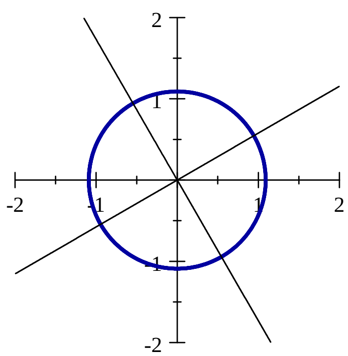
$$Z_0 = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+0}{4}+\mathrm{j}\sin\frac{\frac{2\pi}{3}+0}{4}\right) = \sqrt[4]{2}\left(\cos\tfrac{\pi}{6}+\mathrm{j}\sin\tfrac{\pi}{6}\right).$$

$$Z_1 = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{4}+\mathrm{j}\sin\frac{\frac{2\pi}{3}+2\pi}{4}\right) = \sqrt[4]{2}\left(\cos\tfrac{2\pi}{3}+\mathrm{j}\sin\tfrac{2\pi}{3}\right).$$

$$Z_2 = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+4\pi}{4}+\mathrm{j}\sin\frac{\frac{2\pi}{3}+4\pi}{4}\right) = \sqrt[4]{2}\left(\cos\tfrac{7\pi}{6}+\mathrm{j}\sin\tfrac{7\pi}{6}\right).$$

$$Z_3 = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+6\pi}{4}+\mathrm{j}\sin\frac{\frac{2\pi}{3}+6\pi}{4}\right) = \sqrt[4]{2}\left(\cos\tfrac{5\pi}{3}+\mathrm{j}\sin\tfrac{5\pi}{3}\right).$$

Dacă se reprezintă soluţiile ecuaţiei, Z_0, Z_1, Z_2, Z_3, în planul complex, ele sunt vârfurile unui pătrat înscris într-un cerc cu centrul în origine şi de rază $\sqrt[4]{2}$.



c) $5\,(1-3\,\mathrm{j})\,Z^5+4\,(2-\mathrm{j})=0;$

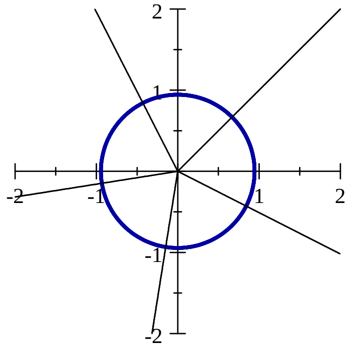
Rezolvare. $(*) \Leftrightarrow Z^5 = \dfrac{1+3\,\mathrm{j})\,(-4\,(2-\mathrm{j}))}{5\,(1-3\,\mathrm{j})} \Leftrightarrow Z^5 = \dfrac{4\,\mathrm{j}\left(\sqrt{3}+\mathrm{j}\right)}{3+1} \Leftrightarrow Z^4 = -\frac{2}{5}-\frac{2}{5}\,\mathrm{j}.$

Se scrie sub formă trigonometrică $-\frac{2}{5}-\frac{2}{5}\,\mathrm{j} = \frac{2\sqrt{2}}{5}\left(\cos\frac{5\pi}{4}+\mathrm{j}\sin\frac{5\pi}{4}\right)$.Atunci

$$Z^5 = \tfrac{2\sqrt{2}}{5}\left(\cos\tfrac{5\pi}{4}+\mathrm{j}\sin\tfrac{5\pi}{4}\right) \Rightarrow$$

$$Z \in \left(\sqrt[5]{-\tfrac{2}{5}-\tfrac{2}{5}\,\mathrm{j}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[5]{\tfrac{2\sqrt{2}}{5}}\left(\cos\frac{\frac{5\pi}{4}+2k\pi}{5}+\mathrm{j}\sin\frac{\frac{5\pi}{4}+2k\pi}{5}\right); k\in\{0,1,2,3,4\}\right\}$$

Dacă se reprezintă soluţiile în planul complex, ele sunt vârfurile unui pentagon regulat încris într-un cerc cu centrul în origine şi de rază $\sqrt[5]{\frac{2\sqrt{2}}{5}}$.



d) $Z^6 - 9Z^3 + 8 = 0;$
Rezolvare. $w^2 - 9w + 8 = 0; \Delta = 81 - 4 \cdot 1 \cdot 8 = 49 \Rightarrow w_1 = 1$ şi $w_2 = 8.$
$(*_1)$ $Z^3 = 1 + 0\,\mathrm{j}$. Se scrie sub formă trigonometrică $1 + 0\,\mathrm{j} = 1\,(\cos 0 + \mathrm{j}\sin 0)$.
Atunci $Z^3 = 1\,(\cos 0 + \mathrm{j}\sin 0) \Rightarrow$

$$Z \in \left(\sqrt[3]{1+0\,\mathrm{j}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[3]{1}\left(\cos\frac{0+2k\pi}{3} + \mathrm{j}\sin\frac{0+2k\pi}{3}\right); k \in \{0,1,2\}\right\}$$

$$Z_0 = \sqrt[3]{1}\left(\cos\frac{0+0}{3} + \mathrm{j}\sin\frac{0+0}{3}\right) = \cos 0 + \mathrm{j}\sin 0.$$

$$Z_1 = \sqrt[3]{1}\left(\cos\frac{0+2\pi}{3} + \mathrm{j}\sin\frac{0+2\pi}{3}\right) = \cos\tfrac{2\pi}{3} + \mathrm{j}\sin\tfrac{2\pi}{3}.$$

$$Z_2 = \sqrt[3]{1}\left(\cos\frac{0+4\pi}{3} + \mathrm{j}\sin\frac{0+4\pi}{3}\right) = \cos\tfrac{4\pi}{3} + \mathrm{j}\sin\tfrac{4\pi}{3}.$$

$(*_2)$ $Z^3 = 8 + 0\,\mathrm{j}$. Se scrie sub formă trigonometrică $8 + 0\,\mathrm{j} = 8\,(\cos 0 + \mathrm{j}\sin 0)$.
Atunci $Z^3 = 8\,(\cos 0 + \mathrm{j}\sin 0) \Rightarrow$

$$Z \in \left(\sqrt[3]{8+0\,\mathrm{j}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[3]{8}\left(\cos\frac{0+2k\pi}{3} + \mathrm{j}\sin\frac{0+2k\pi}{3}\right); k \in \{0,1,2\}\right\}$$

$$Z_0 = \sqrt[3]{8}\left(\cos\frac{0+0}{3} + \mathrm{j}\sin\frac{0+0}{3}\right) = 2\,(\cos 0 + \mathrm{j}\sin 0).$$

$$Z_1 = \sqrt[3]{8}\left(\cos\frac{0+2\pi}{3} + \mathrm{j}\sin\frac{0+2\pi}{3}\right) = 2\left(\cos\tfrac{2\pi}{3} + \mathrm{j}\sin\tfrac{2\pi}{3}\right).$$

$$Z_2 = \sqrt[3]{8}\left(\cos\frac{0+4\pi}{3} + \mathrm{j}\sin\frac{0+4\pi}{3}\right) = 2\left(\cos\tfrac{4\pi}{3} + \mathrm{j}\sin\tfrac{4\pi}{3}\right).$$

Dacă se reprezintă soluţiile în planul complex, ele sunt vârfurile a două triunghiuri echilaterale, unul încris într-un cerc cu centrul în origine şi de rază 1, celălalt într-un cerc cu centrul în origine şi de rază 2.

e) $Z^8 + (1 - \mathrm{j})\,Z^4 - \mathrm{j} = 0;$
Rezolvare. $w^2 + (1-\mathrm{j})\,w - \mathrm{j} = 0; \Delta = (1-\mathrm{j})^2 - 4 \cdot 1 \cdot (-\mathrm{j}) = 1 + 2\,\mathrm{j} + \mathrm{j}^2 = (1+\mathrm{j})^2$

$$w_1 = \frac{-(1-\mathrm{j}) - (1+\mathrm{j})}{2 \cdot 1} = -1 \text{ şi } w_2 = \frac{-(1-\mathrm{j}) + (1+\mathrm{j})}{2 \cdot 1} = \mathrm{j}.$$

$(*_1)$ $Z^4 = -1$. Se scrie sub formă trigonometrică $-1 + 0 \cdot \mathrm{j} = 1\,(\cos\pi + \mathrm{j}\sin\pi)$.
Atunci $Z^4 = 1\,(\cos\pi + \mathrm{j}\sin\pi) \Rightarrow$

$$Z \in \left(\sqrt[4]{-1+0\cdot\mathrm{j}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[4]{1}\left(\cos\frac{\pi+2k\pi}{4} + \mathrm{j}\sin\frac{\pi+2k\pi}{4}\right); k \in \{0,1,2,3\}\right\}$$

$$Z_0 = \sqrt[4]{1}\left(\cos\frac{\pi+0}{4} + \mathrm{j}\sin\frac{\pi+0}{4}\right) = \cos\tfrac{\pi}{4} + \mathrm{j}\sin\tfrac{\pi}{4}.$$

$$Z_1 = \sqrt[4]{1}\left(\cos\frac{\pi+2\pi}{4} + \mathrm{j}\sin\frac{\pi+2\pi}{4}\right) = \cos\tfrac{3\pi}{4} + \mathrm{j}\sin\tfrac{3\pi}{4}.$$
$$Z_2 = \sqrt[4]{1}\left(\cos\frac{\pi+4\pi}{4} + \mathrm{j}\sin\frac{\pi+4\pi}{4}\right) = \cos\tfrac{5\pi}{4} + \mathrm{j}\sin\tfrac{5\pi}{4}.$$
$$Z_3 = \sqrt[4]{1}\left(\cos\frac{\pi+6\pi}{4} + \mathrm{j}\sin\frac{\pi+6\pi}{4}\right) = \cos\tfrac{7\pi}{4} + \mathrm{j}\sin\tfrac{7\pi}{4}.$$

$(*_2)$ $Z^4 = \mathrm{j}$. Se scrie sub formă trigonometrică $0 + 1\cdot\mathrm{j} = 1\left(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}\right)$.

Atunci $Z^4 = 1\left(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}\right) \Rightarrow$

$$Z \in \left(\sqrt[4]{-1+0\cdot\mathrm{j}}\right)_{\mathbb{C}} = \left\{ Z_k = \sqrt[4]{1}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{4} + \mathrm{j}\sin\frac{\frac{\pi}{2}+2k\pi}{4}\right); k \in \{0,1,2,3\}\right\}$$
$$Z_0 = \sqrt[4]{1}\left(\cos\frac{\frac{\pi}{2}+0}{4} + \mathrm{j}\sin\frac{\frac{\pi}{2}+0}{4}\right) = \cos\tfrac{\pi}{8} + \mathrm{j}\sin\tfrac{\pi}{8}.$$
$$Z_1 = \sqrt[4]{1}\left(\cos\frac{\frac{\pi}{2}+2\pi}{4} + \mathrm{j}\sin\frac{\frac{\pi}{2}+2\pi}{4}\right) = \cos\tfrac{5\pi}{8} + \mathrm{j}\sin\tfrac{5\pi}{8}.$$
$$Z_2 = \sqrt[4]{1}\left(\cos\frac{\frac{\pi}{2}+4\pi}{4} + \mathrm{j}\sin\frac{\frac{\pi}{2}+4\pi}{4}\right) = \cos\tfrac{9\pi}{8} + \mathrm{j}\sin\tfrac{9\pi}{8}.$$
$$Z_3 = \sqrt[4]{1}\left(\cos\frac{\frac{\pi}{2}+6\pi}{4} + \mathrm{j}\sin\frac{\frac{\pi}{2}+6\pi}{4}\right) = \cos\tfrac{13\pi}{8} + \mathrm{j}\sin\tfrac{13\pi}{8}.$$

Dacă se reprezintă soluţiile în planul complex, ele sunt vârfurile a două pătrate, ambele încrise într-un cerc cu centrul în origine şi de rază $\sqrt[4]{1}$.

1.2$^{\bigcirc}$. **Structura topologică a mulţimii** $\mathbb{C}$-NU

1.3$^{\bigcirc}$. **Mulţimea numerelor complexe extinsă** $\overline{\mathbb{C}}$-NU

1.4$^{\bigcirc}$. Şiruri de numere complexe-...; **1.5**$^{\bigcirc}$. Serii de numere complexe...

2. Funcţii complexe de o variabilă reală $f : A \subseteq \mathbb{R} \to \mathbb{C}$- A se vedea Curs