Grading of Problem 1.

Problem 1 Let n be a nonzero natural number and $f:\mathbb{R}\to\mathbb{R}-\{0\}$ be a function such that $f(2014)=1-f(2013)$. Let $x_1, x_2, x_3, \ldots, x_n$ be ral numbers not equal to each other. If

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \ldots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \ldots & 1+f(x_n) \end{vmatrix} = 0, \qquad (1)$$

prove that f is not continuous.

Grading:

4 points: Calculate determinant:

$$\begin{vmatrix} 1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n) & f(x_2) & f(x_3) & \ldots & f(x_n) \\ 1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n) & 1+f(x_2) & f(x_3) & \ldots & f(x_n) \\ 1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n) & f(x_2) & 1+f(x_3) & \ldots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n) & f(x_2) & f(x_3) & \ldots & 1+f(x_n) \end{vmatrix} =$$

$$(1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)) \begin{vmatrix} 1 & f(x_2) & f(x_3) & \ldots & f(x_n) \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

The last determinant is equal to 1, therefore (from the equality given in the problem):

$$1+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)=0$$

2 points:

$$f(2014)+f(2013)+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)=0. \qquad (I)$$

4 points: Suppose that f is continuous and obtain a contradiction.

REMARK If it is proving that for some continuous function the equality (1) it is not true : **1 point**.

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Grading for Problem 2. Consider the sequence (x_n) given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 2.$$

Prove that the sequence $y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1}, \quad n \geq 1$ is convergent and find its limit.

Solution. It is clear that $x_n > 0 \ \forall n \geq 1$ and then

$$\sqrt{x_n^2 + 2x_n + 5} > \sqrt{(x_n + 1)^2} = x_n + 1, \qquad \forall\ n \geq 1.$$

Thus,

$$x_{n+1} > \frac{x_n + 1 + x_n + 1}{2} = x_n + 1 \qquad \forall n \geq 1.$$

It follows, by induction, that $x_n > x_1 + n = 2 + n \ \forall n \geq 1$ and then $x_n \to \infty$.
....2 points

Now, using the definition of x_{n+1} we get:

$$(2x_{n+1} - x_n - 1)^2 = x_n^2 + 2x_n + 5,$$

$$4x_{n+1}^2 - 4x_{n+1}(x_n + 1) + (x_n + 1)^2 = (x_n + 1)^2 + 4, \text{ and from here}$$

$$x_{n+1}^2 - 1 = x_{n+1}(x_n + 1) \Rightarrow \frac{x_{n+1}}{x_{n+1}^2 - 1} = \frac{1}{x_n + 1}.$$

....2 points

But one can write

$$\frac{x_{n+1}}{x_{n+1}^2 - 1} = \frac{1}{x_{n+1} + 1} + \frac{1}{x_{n+1}^2 - 1}$$

and then

$$\frac{1}{x_{n+1}^2 - 1} = \frac{1}{x_n + 1} - \frac{1}{x_{n+1} + 1} \qquad \forall n \geq 1.$$

....4 points

Finally,

$$y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1} = \frac{1}{3} + \sum_{k=2}^{n} \left(\frac{1}{x_{k-1} + 1} - \frac{1}{x_k + 1} \right) = \frac{1}{3} + \frac{1}{3} - \frac{1}{x_n + 1}.$$

Since $x_n \to \infty$ we get $y_n \to \frac{2}{3}$.
....2 points

Remark. The convergence of (y_n) can be proved noticing that $x_n > n + 2$ imply $\frac{1}{x_n^2 - 1} < \frac{1}{n^2}$ and thus

$$y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1} < \sum_{k=1}^{n} \frac{1}{k^2}.$$

(for this proof of convergence of y_n) ....3 points

Grading for Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $a \in \mathbb{C}^*$ such that $A - A^* = 2aI_n$, where $A^* = (\bar{A})^t$.

(a) Show that $|\det A| \geq |a|^n$

(b) Show that if $|\det A| = |a|^n$ then $A = aI_n$.

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a)

- proof for $a = i \cdot b,\ b \in \mathbb{R}$......2 points
- proof for $B^* = B$......1 point
- relation $\lambda_B \in \mathbb{R}$......2 points
- relation $\lambda_A = \lambda_B + a = \lambda_B + ib$......2 *points*
- Relation

  $|det(A)| = |\lambda_{1_A}| \cdot |\lambda_{2_A}| \cdot ... \cdot |\lambda_{n_A}| \geq |b|^n = |a|^n$......1*point*

b) proof of the relation......2 points

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Grading scheme for problem 4

a) $\rightarrow$ 3 points

- 1 point for applying a convergence theorem without arguing it.
- 3 points for a correct solution either by using a convergence theorem or by differentiating an integral with parameter.

b) $\rightarrow$ 7 points

- $$x_n = n^2 \int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x(x^2+1)} dx - n\frac{\pi}{2}$$
- 1 point:
$$\begin{aligned} x_n &= y_n - z_n, \text{ where} \\ y_n &= n^2 \int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x(x^2+1)} dx - n \int_0^n \frac{dx}{1+x^2} \\ z_n &= n \int_n^\infty \frac{dx}{1+x^2}. \end{aligned}$$
- 4 points:
$$\lim_{n\to\infty} y_n = \int_0^1 \frac{\operatorname{arctg} t - t}{t^3} dt.$$
- 1 point:
$$\lim_{n\to\infty} y_n = \frac{1}{2} - \frac{\pi}{4}.$$
- 1 point:
$$\lim_{n\to\infty} z_n = 1.$$

Hence
$$\lim_{n\to\infty} x_n = -\frac{1}{2} - \frac{\pi}{4}.$$

Note: Any other approach which can be completed to a right solution will obtain a corresponding amount of points.