APPROXIMATION AND CONVERGENCE THEOREMS FOR NONLINEAR SEMIGROUPS ASSOCIATED WITH SEMILINEAR EVOLUTION EQUATIONS

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Abstract An approximation theory for semilinear evolution equations is treated in terms of convergence theorems of nonlinear operator semigroups and three types of fundamental results are established. For a semilinear evolution problem in a general Banach space, a sequence of approximate evolution problems is formulated and so-called consistency and stability conditions for the approximate semilinear equations as well as the associated semigroups are introduced. Under these conditions, semilinear versions of the Lax equivalence theorem and Neveu-Trotter-Kato theorems are given. Also, in virtue of a characterization theorem of locally Lipschitzian semigroups, an approximation-solvability theorem is obtained.

1 Introduction

This paper is concerned with approximation theorems for nonlinear semigroups in a general Banach space $(X, |\cdot|)$ which provide mild solutions to semilinear problems of the form

(SP) $$(d/dt)\, u\,(t) = (A + B)\, u\,(t)\,, \quad t > 0; \qquad u\,(0) = x \in D.$$

Here A is assumed to be the generator of a (C_0)-semigroup $T = \{T\,(t)\,; t \geq 0\}$ and B a nonlinear operator from a subset D of X into X. We consider a semigroup $S = \{S(t); t \geq 0\}$ of nonlinear operators from D into itself which provides mild solutions to (SP) in the sense that given an initial-value $x \in D$ the function $u(t) \equiv S(t)x$ satisfies the integral equation

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds$$

for $t \geq 0$. This means that (SP) admits a global mild solution for each $x \in D$. In this paper we employ a lower semicontinuous functional $\varphi : X \to [0, \infty]$ to restrict the growth of the mild solutions and continuity of the operators B and $S(t)$. In what follows, we permit ourselves the common abbreviation, an l.s.c. functional, in referring a lower semicontinuous functional. It is then assumed that $D \subset D(\varphi) = \{x \in X; \varphi(x) < \infty\}$, and that B is continuous on each level set $D_\alpha = \{x \in X; \varphi(x) \le \alpha\}$, $\alpha \ge 0$. Moreover, the functional φ is assumed to restrict the growth of the mild solutions $u(\cdot)$ to (SP) in the sense that $\varphi(u(t))$ enjoys an exponential growth condition of the form

$$\varphi(u(t)) \le e^{at}(\varphi(x) + bt)$$

for $t \ge 0$ and some nonnegative constants a and b. The existence of a semigroup S as mentioned above means the global solvability of (SP).

In this paper we think of the following two cases. The first case is the case where we can formulate a system of approximate semilinear evolution problems

$$\text{(SP; n)} \qquad (d/dt)\, u_n(t) = (A_n + B_n)\, u_n(t), \quad t > 0; \qquad u_n(0) = x_n \in D_n$$

and assume that the semilinear problems (SP; n) are well-posed in the sense that for each n there exists one and only one semigroup $S_n = \{S_n(t); t \ge 0\}$ of nonlinear operators from D_n into itself which provides mild solutions to (SP; n). In this case it is natural to investigate general conditions under which the approximate semigroup S_n converges to the semigroup S associated with the original problem (SP). We here make an attempt to formulate appropriate consistency and stability conditions for $\{A_n + B_n\}$ and $\{S_n\}$ and establish semilinear versions of the Lax equivalence theorem and Neveu-Trotter-Kato theorem. The semilinear Lax equivalence theorem is new and the semilinear Neveu-Trotter-Kato theorem extends the convergence theorem due to Goldstein et al. [4]. The second case is the case in which it is not straightforward to specify the principal part of $A + B$ and treat (SP) through the direct application of the generation theorem. Such case may be considered for semilinear operators appearing for instance in Navier-Stokes equations.

In this case it may be natural to treat (SP) via an appropriate system of approximate evolution problems (SP;n) as well as the associated approximate semigroups S_n. For this approach we formulate appropriate consistency condition for $A_n + B_n$ and stability conditions for S_n and apply a characterization theorem for locally Lipschitzian semigroups to an approximation-solvability theorem.

This paper is organized as follows: Section 2 contains some basic results on mild solutions to semilinear problems which were established in [12] and a characterization theorem established in [1]. In Section 3, basic conditions (C), (S), (LQD), (RC) and (EC) are introduced and the main results are stated along with remarks. In Section 4, equicontinuity results for approximating operators are prepared and then the local uniformity of the subtangential condition is established. Section 5 contains our first main result. In Section 6, a semilinear version of Neveu-Trotter-Kato theorem for locally Lipschitzian semigroups is given under convexity conditions for D_n and φ_n. Finally, in Section 7, the approximation-solvability of (SP) is discussed and the proof of our third main result is obtained.

2 Semilinear evolution equations and semigroups

Let $(X, |\cdot|)$ be a real Banach space. The dual space of X is denoted by X^*. For $x \in X$ and $f \in X^*$, the value of f at x is denoted by $\langle x, f\rangle$. The duality mapping of X is the function $F : X \to 2^{X^*}$ defined by $Fx = \left\{ f \in X^*; \langle x, f \rangle = |x|^2 = \|f\|^2 \right\}$. Given a pair x, y in X, we define the upper and lower semiinner products $\langle y, x \rangle_s$, $\langle y, x \rangle_i$ by the supremum and infimum of the set $\{\langle y, f \rangle, f \in Fx\}$, respectively.

Let D be a subset of X and $\varphi : X \to [0, \infty]$ a l.s.c. functional on X such that $D \subset D(\varphi) = \{x \in X; \varphi(x) < \infty\}$. We denote by $D_\alpha = \{x \in D; \varphi(x) \leq \alpha\}$ a generic level set of D. A nonlinear operator $B : D \subset X \to X$ is said to be locally quasidissipative (respectively strongly locally quasidissipative) on $D(B)$ with respect to φ if for each $\alpha \geq 0$ there exists $\omega_\alpha \in \mathbb{R}$ such that

$$\langle Bx - By, x - y \rangle_i \leq \omega_\alpha |x - y|^2 \quad \text{for } x, y \in D_\alpha,$$

respectively

$$\langle Bx - By, x - y \rangle_s \leq \omega_\alpha |x - y|^2 \quad \text{for } x, y \in D_\alpha.$$

By a locally Lipschitzian semigroup on D with respect to φ is meant a one-parameter family $S = \{S(t); t \geq 0\}$ of (possibly nonlinear) operators from D into itself which satisfies the following three conditions below:

(S1) For $x \in D$ and $s, t \geq 0$, $S(t) S(s) x = S(t + s) x$, $S(0) x = x$.

(S2) For $x \in D$, $S(\cdot) x \in C([0, \infty); X)$.

(S3) For each $\alpha \geq 0$ and $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that

$$|S(t) x - S(t) y| \leq e^{\omega t} |x - y|$$

for $x, y \in D_\alpha$ and $t \in [0, \tau]$.

We consider the semilinear problem

(SP) $$u'(t) = (A + B) u(t), \quad t > 0; \qquad u(0) = x \in D,$$

and we assume the following hypotheses on A, B and D:

(A) $A : D(A) \subset X \to X$ generates a (C_0)-semigroup $T = \{T(t); t \geq 0\}$ on X such that $|T(t)x| \leq e^{\omega t}|x|$ for $x \in X$, $t \geq 0$ and some $\omega \in \mathbb{R}$.

(B) The level set D_α is closed for each $\alpha \geq 0$ and $B : D \subset X \to X$ is continuous on each D_α.

The semilinear problem (SP) may sometimes not have strong solutions and the variation of constants formula is employed to obtain solutions in a generalized sense. It is then said that a function $u(\cdot) \in C([0, \infty); X)$ is a mild solution to (SP) if $u(t) \in D$ for $t \geq 0$, $Bu(\cdot) \in C([0, \infty); X)$ and the integral equation

$$u(t) = T(t)x + \int_0^t T(t - s)Bu(s)ds$$

is satisfied for each $t \geq 0$.

In this paper we are concerned with the case in which (SP) is well-posed in the sense of semigroups. We say that a semigroup S is *associated with* (SP), if it provides mild solutions to (SP) in the sense that for each $x \in D$ the function $u(\cdot) = S(\cdot)x$ is a mild solution to (SP).

In this setting the following theorem, which was proved in [1], is valid.

Theorem 2.1. *Let $a, b \geq 0$ and suppose that* (A)*and* (B) *hold. Then the following statements are equivalent:*

(I) *There is a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D satisfying the following properties:*

(I.1) $S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds$ *for $t \geq 0$ and $x \in D$.*

(I.2) *For $\alpha > 0$ and $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that*

$$|S(t)x - S(t)y| \leq e^{\omega(\alpha,\tau)t}|x - y|$$

for $x, y \in D_\alpha$ and $t \in [0, \tau]$.

(I.3) $\varphi(S(t)x) \leq e^{at}(\varphi(x) + bt)$ *for $x \in D$ and $t \geq 0$.*

(II) *The semilinear operator $A+B$ satisfies the explicit subtangential condition and semilinear stability condition stated below:*

(II.1) *For $x \in D$ and $\varepsilon > 0$ there is $(h, x_h) \in (0, \epsilon] \times D$ such that*

$$(1/h)|T(h)x + hBx - x_h| \leq \varepsilon \quad \text{and} \quad \varphi(x_h) \leq e^{ah}(\varphi(x) + (b + \varepsilon)h).$$

(II.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbb{R}$ such that*

$$\varliminf_{h \downarrow 0}(1/h)[|T(h)(x - y) + h(Bx - By)| - |x - y|] \leq \omega_\alpha|x - y|$$

for $x, y \in D_\alpha$.

Moreover, if the subset D and the functional φ are assumed to be convex, then (I), (II) *and the following statements are equivalent:*

(III) *The semilinear operator $A+B$ satisfies the following density condition, quasidissipativity condition and range condition:*

(III.1) *The domain $D(A + B) = D(A) \cap D$ is dense in D.*

(III.2) *For α >0 there is $\omega_\alpha \in \mathbb{R}$ such that*

$$\langle (A + B)x - (A + B)y, x - y \rangle_i \leq \omega_\alpha|x - y|^2$$

for each $x, y \in D(A) \cap D_\alpha$.

(III.3) *For $\alpha > 0$ there is $\lambda_0 = \lambda_0(\alpha) \in (0, 1/a)$ such that for each $x \in D_\alpha$ and $\lambda \in (0, \lambda_0)$ there is $x_\lambda \in D(A) \cap D$ satisfying*

$$x_\lambda - \lambda(A + B)x_\lambda = x \quad \text{and} \quad \varphi(x_\lambda) \leq (1 - a\lambda)^{-1}(\varphi(x) + b\lambda).$$

(IV) *The semilinear operator $A+B$ satisfies the density condition, quasidissipativity condition and implicit subtangential condition which permits errors as stated below*

(IV.1) *$D(A) \cap D$ is dense in D.*

(IV.2) *For $\alpha > 0$ there is $\omega_\alpha \in \mathbb{R}$ such that*

$$\langle (A + B)x - (A + B)y, x - y \rangle_i \leq \omega_\alpha|x - y|^2$$

for $x, y \in D(A) \cap D_\alpha$*.*

(IV.3) *For* $\alpha > 0$ *and* $\varepsilon > 0$ *there exists* $\lambda_0 = \lambda_0(\alpha, \varepsilon)$ *such that for* $\lambda \in (0, \lambda_0)$ *and* $x \in D_\alpha$ *there exist* $x_\lambda \in D(A) \cap D$ *and* $z_\lambda \in X$ *satisfying* $|z_\lambda| < \varepsilon$*,*

$$x_\lambda - \lambda(A+B)x_\lambda = x + \lambda z_\lambda \text{ and } \varphi(x_\lambda) \leq (1-\lambda a)^{-1}(\varphi(x) + (b+\varepsilon)\lambda).$$

(V) *The semilinear operator* $A+B$ *satisfies the quasidissipativity condition and sequential implicit subtangential condition stated below:*

(V.1) *For each* $\alpha > 0$ *there is* $\omega_\alpha \in \mathbb{R}$ *such that*

$$\langle (A+B)x - (A+B)y, x-y \rangle_i \leq \omega_\alpha |x-y|^2$$

for $x, y \in D(A) \cap D_\alpha$*.*

(V.2) *For each* $x \in D$ *there exists a null sequence* $\{h_n\}$ *of positive numbers and a sequence* $\{x_n\}$ *in* $D(A) \cap D$ *such that*

(V.2a) $\lim_{n\to\infty} (1/h_n) \, |x_n - h_n(A+B)x_n - x| = 0$*,*

(V.2b) $\overline{\lim_{n\to\infty}} (1/h_n) \, [\varphi(x_n) - \varphi(x)] \leq a\varphi(x) + b$*,*

(V.2c) $\lim_{n\to\infty} |x_n - x| = 0$*.*

As mentioned in the Introduction, a semilinear operator $A + B$ is a nonlinear operator such that the linear part A plays an essential role in the characterization of mild solution to (SP). The following result shows the significance of the representation of a semilinear operator $A + B$. See [12].

Theorem 2.2. *Let* $S = \{S(t)\,; t \geq 0\}$ *be a nonlinear semigroup on* D *such that* $BS(\cdot)x \in C([0,\infty); X)$ *for each* $x \in D$*. The following statements are then equivalent*

(i) $S(t)x = T(t)x + \int_0^t T(t-s)\,BS(s)\,xds$ *for* $t \geq 0$ *and* $x \in D$*.*

(ii) $\lim_{h\downarrow 0} (1/h)\,[S(h)x - T(h)x] = Bx$ *for* $x \in D$*.*

(iii) $\lim_{h\downarrow 0} \langle (1/h)(S(h)x - x), x^* \rangle = \langle x, A^*x^* \rangle + \langle Bx, x^* \rangle$ *for* $x \in D$ *and* $x^* \in D(A^*)$*.*

(iv) $(d/dt)(\langle S(t)x, x^* \rangle) = \langle S(t)x, A^*x^* \rangle + \langle BS(t)x, x^* \rangle$ *for* $t \geq 0$*,* $x \in D$ *and* $x^* \in D(A^*)$*.*

(v) $\int_0^t S(s)\,xds \in D(A)$ *and* $S(t)x = x + A\int_0^t S(s)\,xds + \int_0^t BS(s)\,xds$ *for* $t \geq 0$ *and* $x \in D$*.*

In the above, condition (ii) states that $A + B$ is the semilinear infinitesimal generator of S; (iii) states that $A + B$ is the weak tangential operator to S; condition (iv) means that S provides weak solutions to (SP) in the sense of Ball; (v) describes that S yields integral solutions to (SP).

Condition (II.2) guarantees the uniqueness of mild solutions to (SP).

Theorem 2.3. *Suppose that condition* (II.2) *is satisfied. If* $u(\cdot)$ *and* $v(\cdot)$ *are locally* φ*-bounded mild solutions of* (SP)*, then*

$$|u(t) - v(t)| \leq e^{\omega(\alpha,\tau)} |u(0) - v(0)|$$

for $t \in [0, \tau]$ *with* $\varphi(u(t)), \varphi(v(t)) \leq \alpha$ *.*

3 Main results

We consider the semilinear problem (SP) under the basic hypotheses (A) and (B). Suppose that there exists a locally Lipschitzian nonlinear semigroup $S = \{S(t); t \geq 0\}$ satisfying the following conditions:

(3.1) For $t \geq 0$ and $x \in D, S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds.$

(3.2) For $t \geq 0$ and $x \in D,\ \varphi(S(t)x) \leq e^{at}(\varphi(x) + bt),$ where $a, b \geq 0$.

In other words, the locally Lipschitzian semigroup S provides mild solutions to (SP) and satisfies the growth condition (3.2). Note that Theorem 2.1 gives necessary and sufficient conditions for the existence of such semigroup.

We consider the approximate evolution problems

$$\text{(SP; n)} \qquad u_n'(t) = (A_n + B_n)u_n(t), \quad t > 0; \qquad u_n(0) = x_n \in D_n,$$

where $D_n \subset X$ and $\varphi_n : X \to [0, \infty]$ are proper l.s.c. functionals such that $D_n \subset D(\varphi_n) = \{x \in X; \varphi_n(x) < \infty\}$.

We assume that the operators A_n and B_n with domain D_n satisfy the basic hypotheses stated below:

(A_n) $A_n : D(A_n) \subset X \to X$ generates a (C_0)-semigroup $T_n = \{T_n(t); t \geq 0\}$ on X and there is $\omega_n \in \mathbb{R}$ such that $|T(t)x| \leq e^{\omega_n t}|x|$ for each $x \in X$ and $t \geq 0$.

(B_n) $D_{n,\alpha} = \{x \in D_n, \varphi_n(x) \leq \alpha\}$ is closed in X and $B_n : D_n \to X$ is nonlinear and continuous from $D_{n,\alpha}$ into X.

Suppose that for each n there exists a locally Lipschitzian semigroup $S_n = \{S_n(t); t \geq 0\}$ on D_n such that

$$S_n(t)x_n = T_n(t)x_n + \int_0^t T_n(t-s)B_nS_n(s)x_nds \tag{3.3}$$

and

$$\varphi_n(S_n(t)x_n) \leq e^{at}(\varphi_n(x_n) + bt) \tag{3.4}$$

for each $t \geq 0$ and $x_n \in D_n$.

The operators A_n and B_n with domains D_n are supposed to be the approximate operators to the operator A and the operator B with domain D. In order to assure this, we impose the so-called *consistency condition* for A_n, B_n and D_n. In what follows we say that $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-*bounded* sequence if $x_n \in D_n$ for each $n \geq 1$ and $\sup_{n\geq 1} \varphi_n(x_n) < \infty$.

Consistency Condition

(C) The following conditions are satisfied:

(C1) For $x \in X$ and $\tau > 0$, $T_n(t)x \to T(t)x$ as $n \to \infty$, uniformly with respect to $t \in [0, \tau]$.

(C2) For all $\alpha > 0$ there is $\beta = \beta(\alpha) > 0$ such that for each $x \in D_\alpha$ there is $\{x_n\}$ with $x_n \in D_{n,\beta}$ and $x_n \to x$ as $n \to \infty$.

(C3) If $x \in D$, $x_n \in D_n$, $\overline{\lim}_{n\to\infty} \varphi_n(x_n) < \infty$, and $x_n \to x$ as $n \to \infty$, then $B_n x_n \to Bx$ in X as $n \to \infty$.

Condition (C1) is understood to be a consistency condition for A_n's in the sense that the convergence of their resolvents $(I - \lambda A_n)^{-1}$ to $(I - \lambda A)^{-1}$ is derived by taking the Laplace transforms of T_n's. Condition (C2) states that each level set D_α is approximated by elements of the level sets $D_{n,\beta}$ such that β is independent of n and is chosen so that $\beta > \alpha$ in general. Condition (C3) may be interpreted as a $\{\varphi_n\}$-bounded continuous convergence of B_n to B.

Remark 3.1. If $D_n \equiv D$ is independent of n, (C) becomes much simpler since it is not necessary to choose sequences $\{x_n\}_{n\geq 1}$. However, if $\varepsilon A + B$ is regarded as a singular perturbation of B then we necessitate treating the case in which $D_n \subset D$; if $A + B$ is treated through discrete approximations, then we have the condition that $D_n \supset D$. Therefore, it is important to assume that D_n does depend upon n and formulate conditions (C2) and (C3).

In addition to conditions (C1) through (C3), we employ the following condition:

(EC) For $x \in D$ and for a $\{\varphi_n\}$-bounded sequence $\{x_n\}_{n\geq 1}$ with $x_n \to x$ as $n \to \infty$,

$$\sup_{n\geq 1} |S_n(t)x_n - x_n| \to 0 \text{ as } t \to 0.$$

This condition states that the family $\{S_n(\cdot)x_n\}$ is equicontinuous at $t = 0$ from the right.

The above condition may be called an *equicontinuity condition* for S_n's. It should be noted that (EC) implies (S) via the uniform boundedness principle provided that A_n, B_n and $S_n(t)$ are all linear.

It is also necessary to impose uniformity for the local Lipschitz continuity of the approximate semigroups S_n, which prevent blow-up situations in their convergence. We then impose the following stability condition:

Stability Condition

(S) There is a separately nondecreasing function $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$|S_n(t)x_n - S_n(t)y_n| \leq e^{\omega(\alpha,t)t} |x_n - y_n|$$

for $t \geq 0$, $\alpha \geq 0$, x_n, $y_n \in D_\alpha$ and $n = 0, 1, 2, \cdots$, where S_0 is understood to be the limit semigroup S.

Under the above assumptions we obtain our first main result (Theorem 5.1).

Theorem 1. *Let $\{S_n\}_{n\geq 0}$ be a sequence of locally Lipschitzian semigroups satisfying* (3.3) *and* (3.4). *Suppose that the consistency condition* (C) *and stability condition* (S) *are satisfied. Then* (EC) *holds if and only if the statement* (I) *below is valid.*

(I) *If* $x \in D$, $\{x_n\}_{n\geq 1}$ *is a* $\{\varphi_n\}$*-bounded sequence and* $x_n \to x$ *as* $n \to \infty$, *then*

$$S_n(t)\, x_n \to S_0(t)\, x \text{ as } n \to \infty \text{ for } t \geq 0,$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$.

The above result may be interpreted as a semilinear version of the Lax equivalence theorem. In fact, if B is linear, then by the uniform boundedness principle condition (EC) implies (S). This means that under the consistency condition the convergence of $\{S_n\}$ is equivalent to the uniform boundedness of $\{S_n\}$.

In order to formulate a semilinear version of Neveu-Trotter-Kato theorem, we employ the following two conditions listed below. Here $A_0 \equiv A$, $B_0 \equiv B$, $D_0 \equiv D$, $\varphi_0 \equiv \varphi$ and $S_0 \equiv S$.

The first condition (LQD) means that the family $\{A_n + B_n\}_{n\geq 0}$ is uniformly quasidissipative in a local sense:

(LQD) For each $n \geq 0$ and $\alpha > 0$ there are $\omega_{n,\alpha} \in \mathbb{R}$ with $\sup_{n\geq 0} \omega_{n,\alpha} < \infty$ such that

$$\langle (A_n + B_n)\, x_n - (A_n + B_n)\, y_n, x_n - y_n \rangle_i \leq \omega_{n,\alpha} |x_n - y_n|^2$$

for each $x_n, y_n \in D_{n,\alpha}$.

The second condition (RC) states that the domain $D(A_n) \cap D_n$ of $A_n + B_n$ is dense in D_n and the range of $I - \lambda(A_n + B_n)$ is sufficiently large for each n.

(RC) For $n = 0,\ 1,\ 2 \cdots,$ $D(A_n) \cap D_n$ is dense in D_n; for $\alpha > 0$ and there is $\lambda_{0,n} = \lambda_{0,n}(\alpha) \in (0, 1/a)$ such that to $\lambda \in (0, \lambda_{0,n})$ and $x_n \in D_{n,\alpha}$ there corresponds $x_n^\lambda \in D(A_n) \cap D_n$ satisfying

$$x_n^\lambda - \lambda (A_n + B_n)\, x_n^\lambda = x_n \text{ and } \varphi_n\left(x_n^\lambda\right) \leq (1 - \lambda a)^{-1} (\varphi_n(x_n) + b\lambda)\,.$$

From our characterization theorem, Theorem 2.1, it follows that both (LQD) and (RC) hold if and only if there exists a sequence $\{S_n\}_{n\geq 0}$ of locally Lipschitzian semigroups satisfying (S), (3.3) and (3.4) provided that D_n and φ_n are all convex. In view of this fact, our second main result (Theorem 6.1) may be stated as follows:

Theorem 2. *Let* $\{S_n\}_{n\geq 0}$ *be a sequence of locally Lipschitzian semigroups satisfying* (3.3) *and* (3.4). *Asssume that conditions* (C) *and* (S) *hold, and that* D_n *and* φ_n *are convex for* $n = 0,\ 1,\ 2, \cdots$. *Then the following three statements are equivalent:*

(I) *If* $x_0 \in D_0$, $\{x_n\}_{n\geq 1}$ *is a* $\{\varphi_n\}$*-bounded sequence,* $x_n \to x_0$ *as* $n \to \infty$, *then*

$$S_n(t)\, x_n \to S_0(t)\, x_0 \text{ as } n \to \infty \text{ for } t \geq 0$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$.

(II) *For each* $\alpha > 0$ *there is* $\beta = \beta(\alpha) > 0$ *such that to* $x_0 \in D(A_0) \cap D_{0,\alpha}$ *there corresponds a sequence* $\{x_n\}_{n\geq 1}$ *such that*

$$x_n \in D(A_n) \cap D_{n,\beta}, x_n \to x \text{ and } (A_n + B_n)\, x_n \to (A_0 + B_0)\, x_0 \text{ as } n \to \infty.$$

(III) *The following statements are valid:*

(III.1) *For each* $\alpha > 0$ *there is* $\lambda_1 = \lambda_1(\alpha) \in (0, 1/a)$ *such that if* $\lambda \in (0, \lambda_1)$, $x_n \in D_{n,\alpha}$ *for* $n \geq 1$, $x_0 \in D_{0,\alpha}$, *and* $x_n \to x_0$ *as* $n \to \infty$, *then there exist* $x_n^\lambda \in D(A_n) \cap D_n$ *and* $x_0^\lambda \in D(A_0) \cap D_0$ *satisfying*

$$x_n^\lambda - \lambda (A_n + B_n) x_n^\lambda = x_n \ , \ \varphi_n \left(x_n^\lambda\right) \leq (1 - \lambda a)^{-1} (\varphi_n (x_n) + b\lambda),$$
$$x_0^\lambda - \lambda (A_0 + B_0) x_0^\lambda = x_0 \ , \ \varphi_0 \left(x_0^\lambda\right) \leq (1 - \lambda a)^{-1} (\varphi (x_0) + b\lambda),$$

and $x_n^\lambda \to x_0^\lambda$ *as* $n \to \infty$.

(III.2) *If* $\varepsilon > 0$, $x_0 \in D_0$, $\{x_n\}_{n\geq 1}$ *is* $\{\varphi_n\}$*-bounded and* $x_n \to x_0$ *as* $n \to \infty$, *then there are a* $\{\varphi_n\}$*-bounded sequence* $\{z_n\}_{n\geq 1}$ *and* $z_0 \in D_0 \cap D(A_0)$ *such that*

$$z_n \in D(A_n) \cap D_n \ , \ z_n \to z_0 \text{ as } n \to \infty, \text{ and } \sup_{n\geq 0} |z_n - x_n| < \varepsilon.$$

It should be noted that the convexity assumptions for the domains D_n and the functionals φ_n are essential for the verification of the implications (I) ⇒ (II) and (I) ⇒ (III).

There are many cases in which it is difficult to treat (SP) unless we apply appropriate regularization or approximation procedures. In such situations it is important to construct the solutions of (SP) using the approximate solutions of (SP; n). Our third main result (Theorem 7.1) is called the approximation-solvability theorem.

Theorem 3. *Let* D *be a closed subset of* X,

$$\widetilde{D} = \{x \in X, x \text{ is a limit of some } \{x_n\} \text{ with } x_n \in D_n \text{ for } n \geq 1\}$$

and $\Phi : X \to [0, \infty]$ *a functional defined by*

$$\Phi(x) = \begin{cases} \inf\{\overline{\lim\limits_{n\to\infty}} \varphi_n(x_n); x_n \in D_n; x_n \to x \text{ as } n \to \infty\} & \text{for } x \in \widetilde{D} \\ \infty & \text{otherwise.} \end{cases}$$

Suppose that (C1), (C3), (EC), (S) *for* $n \geq 1$ *and the following condition are satisfied:*

(C4) *The following conditions are valid:*

(C4.a) *For each* $x \in D$ *there is a sequence* $\{x_n\}$ *such that* $x_n \in D_n$, $\overline{\lim\limits_{n\to\infty}} \varphi_n(x_n) < \infty$ *and* $x_n \to x$ *as* $n \to \infty$.

(C4.b) *There is* $\beta \geq 0$ *such that* $D_{n,\beta} \neq \emptyset$ *for each* $n \geq 1$.

(C4.c) *If* $x_n \in D_n$, $\overline{\lim\limits_{n\to\infty}} |x_n| < \infty$ *and* $\overline{\lim\limits_{n\to\infty}} \varphi_n(x_n) < \infty$, *then* $\underline{\lim\limits_{n\to\infty}}\, d(x_n, D_\alpha) = 0$ *for each* $\alpha > \overline{\lim\limits_{n\to\infty}} \varphi_n(x_n)$.

Then there exists a locally Lipschitzian semigroup $S = \{S(t); t \geq 0\}$ *satisfying*

(i) $S(t)x = T(t)x + \int_0^t T(t-s) BS(s) x ds$ *for* $t \geq 0$ *and* $x \in D$;

(ii) $\Phi(S(t)x) \leq e^{at} (\Phi(x) + bt)$ *for* $t \geq 0$ *and* $x \in D$.

Moreover, if $x \in D$, $x_n \in D_n$, $\overline{\lim\limits_{n\to\infty}} \varphi_n(x_n) < \infty$ *and* $x_n \to x$ *as* $n \to \infty$, *then*

$$S_n(t) x_n \to S(t) x \quad \text{as } n \to \infty,$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$.

4 Equicontinuity of approximating operators

This section is concerned with the uniformity on compact sets of the subtangential condition (II.1) in Theorem 2.1. To this end, we first discuss equicontinuity of the family of nonlinear operators $\{B_n\}_{n\geq 1}$ on level sets and demonstrate that condition (EC) is equivalent to the equicontinuity as well as uniform boundedness of $\{B_nS_n\}_{n\geq 1}$.

Lemma 4.1. *Under condition* (A), (A$_n$) *and* (C1)*, we have:*

(i) *For each $\tau > 0$ there exists $M_\tau < \infty$ such that*

$$\sup_{n\geq 1} |T_n(t)| \leq M_\tau \text{ for } t \in [0,\tau]. \tag{4.1}$$

(ii) *If $\{x_n\}$ is convergent in X , then*

$$\sup_{n\geq 1} |T_n(t)x_n - x_n| \to 0 \quad \text{as } t \downarrow 0. \tag{4.2}$$

(iii) *$T_n(t)x_n \to T(t)x$ uniformly on bounded sets of $[0,\infty)$ for each $x \in X$ and each $\{x_n\}$ convergent to x.*

The first statement follows from the uniform boundedness principle applied to the family of operators $T_n(t)$. The third statement (iii) is obvious from (i) and (C1). It is easily seen that (iii) implies (ii). As well-known, (4.1) implies $\|T_n(t)\| \leq Me^{\omega t}$ for all $n \geq 1$ and some M, $\omega \in \mathbb{R}$.

Under conditions (B$_n$) and (C3), it is shown that the family $\{B_n\}$ of nonlinear operators is equicontinuous in the following sense.

Lemma 4.2. *Suppose that conditions* (B$_n$) *and* (C3) *hold. Let $\varepsilon > 0$, $\alpha > 0$, $x \in D$ and let $x_n \in D_{n,\alpha}$ be such that $x_n \to x$ in X as $n \to \infty$. Then there is a number $r = r(\varepsilon, \alpha, \{x_n\}, x) > 0$ such that*

$$\sup_{n\geq 1} |B_nx_n - B_ny_n| \leq \varepsilon \tag{4.3}$$

for any sequence $\{y_n\}$ satisfying $y_n \in D_{n,\alpha}$ and $\sup_{n\geq 1} |y_n - x_n| \leq r$.

Proof. It suffices to show that if $\left\{\{y_n^{(k)}\}_{n\geq 1}\right\}_{k\geq 1}$ is a sequence of sequences such that $y_n^k \in D_{n,\alpha}$ and $\sup_{n\geq 1}\left|y_n^{(k)} - x_n\right| \to 0$ as $k \to \infty$, then $\sup_{n\geq 1}\left|B_ny_n^{(k)} - B_nx_n\right| \to 0$ as $k \to \infty$. Suppose to the contrary that the above statement does not hold. Then there exists a sequence $\left\{\{y_n^{(k)}\}_{n\geq 1}\right\}_{k\geq 1}$, a number $\varepsilon_0 > 0$ and a divergent subsequence $\{k_l\}_{l\geq 1}$ such that $y_n^{(k)} \in D_{n,\alpha}$, $\sup_{n\geq 1}\left|y_n^{(k)} - x_n\right| \to 0$ as $k \to \infty$ and $\sup_{n\geq 1}\left|B_ny_n^{(k_l)} - B_nx_n\right| \geq \varepsilon_0$ for $l \geq 1$. Then for each l there is $n_l \geq 1$ such that

$$\left|B_{n_l}y_{n_l}^{(k_l)} - B_{n_l}x_{n_l}\right| \geq \varepsilon_0/2. \tag{4.4}$$

For the subsequence $\{n_l\}_{l\geq 1}$ so chosen, we consider the following two cases:

Case 1. If $\{n_l\}$ is bounded, then it contains a constant subsequence for which we also write $\{n_l\}$. Since $\left|y_{n_l}^{(k_l)} - x_{n_l}\right| \le \sup_{n\ge1}\left|y_n^{(k_l)} - x_n\right| \to 0$ as $l \to \infty$, (4.4) contradicts the continuity of B_{n_l} on level sets.

Case 2. If $\{n_l\}$ is unbounded, then it contains a divergent subsequence, which we also denote by $\{n_l\}$. Since $\left|y_{n_l}^{(k_l)} - x\right| \le \sup_{n\ge1}\left|y_n^{(k_l)} - x_n\right| + |x_{n_l} - x| \to 0$ as $l \to \infty$, and $y_{n_l}^{(k_l)} \in D_{n_l,\alpha}$ we use (C3) to conclude that $B_{n_l} y_{n_l}^{(k_l)} \to Bx$ as $l \to \infty$. But $B_{n_l} x_{n_l}$ also converges to Bx, which contradicts (4.4). This completes the proof.

□

Remark 4.1. As a sufficient condition for (EC), we may assume that the family of nonlinear operators $\{B_n\}$ is equi-Lipschitz on level sets, namely, for $n \ge 1$ and $\alpha \ge 0$ there exist $\omega_{n,\alpha} \in \mathbb{R}$ such that

$$|B_n x_n - B_n y_n| \le \omega_{n,\alpha}\,|x_n - y_n|$$

for $x_n, y_n \in D_{n,\alpha}$, and $\sup_{n\ge1}\omega_{n,\alpha} < \infty$ for each $\alpha > 0$.

In fact, let $S_n = \{S_n(t); t \ge 0\}$, $n \in \mathbb{N}$, be a sequence of locally Lipschitzian semigroups satisfying (3.3) and (3.4). Let $\alpha > 0$, $\delta > 0$, $\beta > e^{a\delta}(\alpha + b\delta)$, $x \in D$, $x_n \in D_{n,\alpha}$ and $x_n \to x$. Then, by Lemma 4.1, $|T_n(s)| \le M_\delta$, for $n \ge 1$, $s \in [0,\delta]$ and some M_δ. Since $S_n(s)x_n \in D_{n,\beta}$ for $s \in [0,\delta]$ and $n \ge 1$, we obtain

$$\begin{aligned}|S_n(t)x_n - x_n| &\le |T_n(t)x_n - x_n| + \int_0^t |T_n(t-s)\left[B_n S_n(s)x_n - B_n x_n\right]| \,ds \\ &\quad + \int_0^t |T_n(t-s) B_n x_n| \,ds \\ &\le |T_n(t)x_n - x_n| + M_\delta \int_0^t |B_n S_n(s)x_n - B_n x_n|\,ds + M_\delta t\,|B_n x_n|\end{aligned}$$

for $t \in [0,\delta]$. Using the equi-Lipschitz continuity of B_n, we have

$$|S_n(t)x_n - x_n| \le \sup_{n\ge1}|T_n(t)x_n - x_n| + tM_\delta \sup_{n\ge1}|B_n x_n| + M_\delta\Big(\sup_{n\ge1}\omega_{n,\beta}\Big)\int_0^t |S_n(t)x_n - x_n|\,ds.$$

Applying Gronwall's inequality, we get

$$|S_n(t)x_n - x_n| \le c(\delta)\, e^{M_\delta \omega_\beta t},$$

where

$$c(\delta) = \sup_{t\in[0,\delta]}\left[\sup_{n\ge1}|T_n(t)x_n - x_n|\right] + \delta M_\delta \sup_{n\ge1}|B_n x_n| \text{ and } \omega_\beta = \sup_{n\ge1}\omega_{n,\beta}.$$

By Lemma 4.1, $c(\delta) \to 0$ as $\delta \to 0$ and so it is shown that (EC) holds.

The advantage of imposing the equi-Lipschitz continuity condition on $\{B_n\}$ is that by means of appropriate renorming we may employ (C_0)-semigroups $\{T_n\}$ such that $|T_n(t)| \le Me^{wt}$ for $t \ge 0$ and some $M > 1$. See [13].

We now show that (EC) is related to the equicontinuity as well as uniform boundedness of $\{B_n S_n(\cdot)x_n\}$.

Lemma 4.3. *Let $\{S_n\}_{n\geq 1}$ be a sequence of locally Lipschitzian semigroups satisfying* (3.3) *and* (3.4)*. Assume that conditions* (C1) *and* (C3) *hold. Then condition* (EC) *is equivalent to any one of the following conditions:*

(I) *If $x \in D$ and $\{x_n\}$ is a φ_n-bounded sequence, $x_n \to x$ as $n \to \infty$, then*

$$\sup_{n\geq 1} |B_n S_n(t) x_n - B_n x_n| \to 0 \quad \text{as } t \downarrow 0.$$

(II) *If $x \in D$ and $\{x_n\}$ is a φ_n-bounded sequence, $x_n \to x$ as $n \to \infty$, then there exist $M > 0$ and $\delta > 0$ such that*

$$\sup_{n\geq 1} |B_n S_n(t) x_n| \leq M$$

for $t \in [0, \delta]$.

Proof. (EC) $\Rightarrow$ (I) : Let $\varepsilon > 0, x \in D$ and let $\{x_n\}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \to x$ as $n \to \infty$. Let $\alpha = \sup_{n\geq 1} \varphi_n(x_n)$, $h_0 > 0$ and $\beta > e^{ah_0}(\alpha + bh_0)$. Then $S_n(t) x_n \in D_{n,\beta}$ for $t \in [0, h_0]$. Let $r = r(\varepsilon, \beta, \{x_n\}, x)$ be a number given by Lemma 4.2 and choose any $h_r \in [0, h_0]$ so that

$$\sup_{n\geq 1} |S_n(t) x_n - x_n| \leq r \quad \text{for } t \in [0, h_r].$$

Then $\sup_{n\geq 1} |B_n S_n(t) x_n - B_n x_n| \leq \varepsilon$ for $t \in [0, h_r]$ by Lemma 4.2. This shows that statement (I) is valid.

(I) $\Rightarrow$ (II) : Let $\varepsilon > 0$ and $\delta > 0$ be such that $\sup_{n\geq 1} |B_n S_n(t) x_n - B_n x_n| < \varepsilon$ for each $t \in [0, \delta]$. One then obtains

$$\sup_{n\geq 1} |B_n S_n(t) x_n| \leq \varepsilon + \sup_{n\geq 1} |B_n x_n|$$

for $t \in [0, \delta]$. Since $B_n x_n \to Bx$ by (C3), $\sup_{n\geq 1} |B_n S_n(t) x_n|$ is bounded on $[0, \delta]$.

(II) $\Rightarrow$ (EC) : Let $M > 0$ and $\delta > 0$ be the numbers given in (II) and let $t \in (0, \delta)$. Using Lemma 4.1, we have

$$|S_n(t) x_n - x_n| \leq |T_n(t) x_n - x_n| + M M_\delta t,$$

and therefore

$$\sup_{n\geq 1} |S_n(t) x_n - x_n| \leq \sup_{n\geq 1} |T_n(t) x_n - x_n| + M M_\delta t.$$

Letting here $t \to 0$, we obtain condition (EC). □

Applying Lemmas 4.1 and 4.2, we obtain the following key result which represents the local uniformity of the subtangential condition.

Theorem 4.1. *Let $S_n = \{S_n(t); n \geq 1\}$, $n \in \mathbb{N}$, be a sequence of locally Lipschitzian semigroups satisfying* (3.3) *and* (3.4). *Suppose that conditions* (C) *and* (EC) *hold. Let $x \in D$, $\{x_n\}$ be a $\{\varphi_n\}$-bounded sequence, and let $x_n \to x$ as $n \to \infty$. Then*

$$\lim_{h\downarrow 0}\left[\sup_{n\geq 1}(1/h)\left|T_n(h)\,x_n + hB_nx_n - S_n(h)\,x_n\right|\right] = 0. \tag{4.5}$$

Proof. Let $\varepsilon > 0, \delta_0 > 0$ and $\beta > e^{a\delta_0}\left(\sup_{n\geq 1}\varphi_n(x_n) + b\delta_0\right)$. Then

$$\varphi_n(S_n(t)x_n) \leq e^{at}(\varphi_n(x_n) + bt) < \beta$$

for $t \in [0, \delta_0]$. Let M_{δ_0} be a positive number given by Lemma 4.1, (i). Also, we may choose with the aid of Lemma 4.2 a positive number $r = r(\varepsilon/2M_{\delta_0}, \beta, \{x_n\}, x)$ such that $y_n \in D_{n,\beta}$ and $\sup_{n\geq 1}|y_n - x_n| < r$ imply $\sup_{n\geq 1}|B_ny_n - B_nx_n| < \varepsilon/2M_{\delta_0}$. Since $B_nx_n \to Bx$ by (C3), it follows from Lemma 4.1 and (EC) that there exists $h_0 \in (0, \delta_0)$ such that

$$\sup_{n\geq 1}|T_n(t)\,B_nx_n - B_nx_n| < \varepsilon/2 \quad \text{and} \sup_{n\geq 1}|S_n(t)\,x_n - x_n| < r$$

for $t \in [0, h_0]$. Hence we have

$$|T_n(t)\,B_ny_n - B_nx_n| \leq M_{\delta_0}|B_nx_n - B_ny_n| + \sup_{t\in[0,h]}\left(\sup_{n\geq 1}|T_n(t)\,B_nx_n - B_nx_n|\right) < \varepsilon \tag{4.6}$$

for $y_n \in D_{n,\beta}$, $\sup_{n\geq 1}|y_n - x_n| < r$ and $t \in [0, h_0]$. Therefore, we obtain the estimate

$$(1/t)\,|T_n(t)x_n + tB_nx_n - S_n(t)x_n| \leq (1/t)\int_0^t |T_n(t-s)B_nS_n(s)x_n - B_nx_n|\,ds < \varepsilon$$

by (4.6). This concludes that (4.5) holds. □

5 Convergence theorem

This section is devoted to the proof of Theorem 1 which is stated as follows.

Theorem 5.1. *Let $\{S_n\}_{n\geq 0}$ be a sequence of locally Lipschitzian semigropus satisfying* (3.3) *and* (3.4). *Suppose that conditions* (C) *and* (S) *are satisfied. Then condition* (EC) *is equivalent to the statement* (I) *below:*

(I) *If $x \in D$, $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-bounded sequence and $x_n \to x$ as $n \to \infty$, then*

$$S_n(t)\,x_n \to S_0(t)\,x \text{ as } n \to \infty \text{ for } t \geq 0,$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$.

Proof. (I) $\Rightarrow$ (EC) : We use Kisyński's sequence space. See also [2], [3], [8], [9], [10]. Let

$$\mathscr{X} = c(X) = \left\{ \mathscr{x} = \{x_n\}_{n\geq 0}\,; x_n \in X, x_n \to x_0 \text{ as } n \to \infty \right\}$$

be the space of convergent sequences in X, with norm $|\{x_n\}_{n\geq 0}| = \sup_{n\geq 0} |x_n|$. Let $\{x_n\}_{n\geq 1}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \to x_0$ as $n \to \infty$, for some x_0 in D.

For $t \geq 0$, we define an $\mathscr{X}$-valued function by

$$V(t) = \{S_n(t)x_n\}_{n\geq 0}\,.$$

This is well-defined in $\mathscr{X}$ under condition (I). For each $N \geq 1$ and each $t \geq 0$ we define

$$V_N(t) = \{v_n(t)\}_{n\geq 0}\,, \quad v_n(t) = \begin{cases} S_n(t)x_n & \text{for } 0 \leq n \leq N-1, \\ S_0(t)x_0 & \text{for } n \geq N. \end{cases}$$

It is easily seen that, for each $N \geq 1$, $V_N(\cdot)$ is continuous over $[0, \infty)$ in $\mathscr{X}$ and

$$|V_N(t) - V(t)| = \sup_{n\geq N} |S_n(t)x_n - S_0(t)x_0|\,.$$

From statement (I), for each $\varepsilon > 0$ and $\tau > 0$ there is $N = N_{\varepsilon,\tau}$ such that $n \geq N_{\varepsilon,\tau}$ implies

$$|S_n(t)x_n - S_0(t)x_0| < \varepsilon \text{ for each } t \in [0, \tau].$$

Hence $N \geq N_{\varepsilon,\tau}$ implies $\sup_{t\in[0,\tau]} |V_N(t) - V(t)| \leq \varepsilon$, and consequently $V(\cdot)$ is continuous in $\mathscr{X}$. Since the sequence $\{x_n\}_{n\geq 1}$ was arbitrary, one obtains that condition (EC) is satisfied.

We next demonstrate that (EC) implies (I). Assume that (EC) holds. Let $x \in D$ and $\{x_n\}$ be a $\{\varphi_n\}$-bounded sequence converging to x. Let $\tau > 0$ and $\alpha > 0$ be a number such that $\alpha > e^{a\tau}(\varphi(x) + b\tau)$, which implies $S(t)x \in D_\alpha$ for $t \in [0, \tau]$. Let $\beta = \beta(\alpha)$ be a number given by (C2). Without loss of generality we may assume that $\beta > e^{a\tau}\left(\sup_{n\geq 1} \varphi_n(x_n) + b\tau\right)$, which implies $S_n(t) \in D_{n,\beta}$ for $t \in [0, \tau]$ and $n \geq 1$.

Step 1 Let $\varepsilon > 0$. In this step we construct a finite sequence $\{t_k\}_{k=0}^N$ in $[0, \infty)$ and a finite sequence $\left\{\{y_n^{(k)}\}_{n\geq 1}\right\}_{k=0}^N$ of sequences in X satisfying the following conditions:

(i) $t_0 = 0$, $t_N = \tau$, $y_n^{(0)} = x_n$ for $n \geq 1$;

(ii) $0 < t_{k+1} - t_k < \varepsilon$;

(iii) $y_n^{(k)} \in D_{n,\beta}$, $y_n^{(k)} \to S(t_k)x$ as $n \to \infty$;

(iv) $\overline{\lim_{n\to\infty}} \left| T_n(t_{k+1} - t_k)y_n^{(k)} + (t_{k+1} - t_k)B_n y_n^{(k)} - S_n(t_{k+1} - t_k)y_n^{(k)} \right| \leq (t_{k+1} - t_k)\varepsilon$
and

$$|T(t_{k+1} - t_k)S(t_k)x + (t_{k+1} - t_k)BS(t_k)x - S(t_{k+1} - t_k)S(t_k)x| \leq (t_{k+1} - t_k)\varepsilon.$$

First we set $t_0 = 0$ and $\{y_n^{(0)}\}_{n\geq 1} = \{x_n\}_{n\geq 1}$. Here (i) and (iii) are valid for $k = 0$. Let $k \geq 0$ and assume that $\{t_j\}_{j=0}^{k}$ and $\{\{y_n^{(j)}\}\}_{j=0}^{k}$ have been constructed in such a way that (ii), (iii) and (iv) are satisfied.

If $t_k < \tau$, we define

$$\hat{h}_k = \sup\{h \in (0,\varepsilon] \cap (0, \tau - t_k];\ (5.2) \text{ and } (5.3) \text{ hold}\}, \tag{5.1}$$

where

$$\overline{\lim_{n\to\infty}} \left|T_n(h)\, y_n^{(k)} + hB_n y_n^{(k)} - S_n(h)\, y_n^{(k)}\right| \leq h\varepsilon, \tag{5.2}$$

and

$$|T(h)\, S(t_k)\, x - hBS(t_k)\, x - S(h)\, S(t_k)\, x| \leq h\varepsilon. \tag{5.3}$$

By Theorems 2.2 and 4.1, it is seen that $\hat{h}_k > 0$. We then choose an appropriate number $h_k \in [\hat{h}_k/2, \hat{h}_k]$ so that (5.2) and (5.3) hold for $h = h_k$. We put $t_{k+1} = t_k + h_k\ (\leq \tau)$ and apply (C2) to find a new sequence $\{y_n^{(k+1)}\}_{n\geq 1}$ such that $y_n^{(k+1)} \in D_{n,\beta}$ and $y_n^{(k+1)} \to S(t_{k+1})\, x$. Hence (iii) holds for t_{k+1}. It is also seen from the definition of $\hat{h}_k$ that (ii) and (iv) hold for t_{k+1}. One now continue constructing the numbers t_j and sequences $\{y_n^{(j)}\}$ so far as $t_j < \tau$.

Next, we show that τ is attained in finite steps. Suppose to the contrary that $t_k < \tau$ for all $k \geq 0$. Then there exists $s = \lim_{k\to\infty} t_k \leq \tau$, and so $S(s)\, x \in D_\alpha$. By (C2) a sequence $\{z_n\}$ can be found such that $z_n \in D_{n,\beta}$ and $z_n \to S(s)\, x$ as $n \to \infty$. From Theorems 2.2 and 4.1 we infer that there exists $h \in (0, \varepsilon]$ such that

$$\sup_{n\geq 1} (1/h)\, |T_n(h)\, z_n + hB_n z_n - S_n(h)\, z_n| < \varepsilon/3 \tag{5.4}$$

and

$$(1/h)\, |S(h)\, S(s)\, x - hBS(s)\, x - T(h)\, S(s)\, x| < \varepsilon/3. \tag{5.5}$$

Our aim here is to show that (5.4) or (5.5) is violated. Since the series $\sum_{n=1}^{\infty} h_n$ is summable, there must exist $N \geq 1$ such that $\hat{h}_k < h$ for all $k \geq N$. This implies that for each $k \geq N$ we have either

$$\overline{\lim_{n\to\infty}} \left|T_n(h)\, y_n^{(k)} + hB_n y_n^{(k)} - S_n(h)\, y_n^{(k)}\right| > h\varepsilon,$$

or

$$|T(h)\, S(t_k)\, x + hBS(t_k)\, x - S(h)\, S(t_k)\, x| > h\varepsilon.$$

Hence, it would be concluded that either

$$\overline{\lim_{n\to\infty}} \left|T_n(h)\, y_n^{(k)} + hB_n y_n^{(k)} - S_n(h)\, y_n^{(k)}\right| > h\varepsilon$$

for infinitely many $k \geq N$, or

$$|T(h)\, S(t_k)\, x + hBS(t_k)\, x - S(h)\, S(t_k)\, x| > h\varepsilon$$

for infinitely many $k \geq N$. In the first case, there is a subsequence $\{k_l\}_{l\geq 1}$ such that $k_l \geq N$, $k_l \to \infty$ and

$$\overline{\lim_{n\to\infty}} \left|T_n(h)\, y_n^{(k_l)} + hB_n y_n^{(k_l)} - S_n(h)\, y_n^{(k_l)}\right| > h\varepsilon$$

for $l \geq 1$. Then there is a subsequence $n_l \geq 1$ such that $n_l > n_{l-1}$,

$$\left|T_{n_l}(h)\, y_{n_l}^{(k_l)} + hB_{n_l} y_{n_l}^{(k_l)} - S_{n_l}(h)\, y_{n_l}^{(k_l)}\right| > h\varepsilon/2 \quad \text{for } l \geq 1, \tag{5.6}$$

and

$$\left|y_{n_l}^{(k_l)} - S(t_{k_l})\, x\right| \leq 1/k_l. \tag{5.7}$$

Since $S(t_{k_l})\, x \to S(s)\, x$ and $z_{n_l} \to S(s)\, x$ as $k_l \to \infty$, we deduce from (5.7) that

$$\left|z_{n_l} - y_{n_l}^{(k_l)}\right| \to 0 \text{ as } l \to \infty. \tag{5.8}$$

We here observe the inequality

$$\begin{aligned} |T_{n_l}(h)\, z_{n_l} + hB_{n_l} z_{n_l} - S_{n_l}(h)\, z_{n_l}| &\geq \left|T_{n_l}(h)\, y_{n_l}^{(k_l)} + hB_{n_l} y_{n_l}^{(k_l)} - S_{n_l}(h)\, y_{n_l}^{(k_l)}\right| \\ &-h\left|B_{n_l} y_{n_l}^{(k_l)} - B_{n_l} z_{n_l}\right| - \left|T_{n_l}(h)\, y_{n_l}^{(k_l)} - T_{n_l}(h)\, z_{n_l}\right| - \left|S_{n_l}(h)\, y_{n_l}^{(k_l)} - S_{n_l}(h)\, z_{n_l}\right| \end{aligned}$$

and denote the second, third and fourth terms in the right-hand side by J_1, J_2 and J_3, respectively. We have

$$J_2 = \left|T_{n_l}(h)\, y_{n_l}^{(k_l)} - T_{n_l}(h)\, z_{n_l}\right| \leq M_h \left|y_{n_l}^{(k_l)} - z_{n_l}\right| \to 0$$

as $l \to \infty$, by (5.8), where M_h is a constant given by Lemma 4.1 (i). One also has

$$J_1 = h\left|B_{n_l} y_{n_l} - B_{n_l} z_{n_l}\right| \to 0$$

as $l \to \infty$, since both $B_{n_l} y_{n_l}$ and $B_{n_l} z_{n_l}$ tends to $BS(s)\, x$ as $l \to \infty$. Moreover,

$$J_3 = \left|S_{n_l}(h)\, y_{n_l}^{(k_l)} - S_{n_l}(h)\, z_{n_l}\right| \leq e^{\omega(\beta,h)h} \left|y_{n_l}^{(k_l)} - z_{n_l}\right| \to 0$$

as $k_l \to \infty$, by (5.8).

Hence, for l sufficiently large, (5.6) implies $|T_{n_l}(h)\, z_{n_l} + hB_{n_l} z_{n_l} - S_{n_l}(h) z_{n_l}| > h\varepsilon/3$, which contradicts (5.4).

In the second case, there is a subsequence $\{k_l\}_{l\geq 1}$, such that $k_l \geq N$ for $l \geq 1$, $k_l \to \infty$ as $l \to \infty$, and

$$|T(h)\, S(t_{k_l})\, x + hBS(t_{k_l})\, x - S(h)\, S(t_{k_l})\, x| > h\varepsilon$$

for $l \geq 1$. Letting $l \to \infty$ we obtain

$$|T(h)\, S(s)\, x + hBS(s)\, x - S(h)\, S(s)\, x| \geq h\varepsilon,$$

which contradicts (5.5).

Thus it is concluded that $t_N = \tau$ for some $N \geq 1$, and that Step 1 is complete.

STEP 2 In this step we demonstrate that S_n converges to S.

Let $\{t_k\}_{k=0}^{N}$ and $\{y_n^{(k)}\}_{n\geq 1}$, be sequences constructed in Step 1. Let $t \in (t_k, t_{k+1}]$ for some k with $0 \leq n \leq N-1$. Then

$$|S_n(t)\,x_n - S(t)\,x| \leq |S_n(t)\,x_n - S_n(t_{k+1})\,x_n| + |S_n(t_{k+1})\,x_n - S(t_{k+1})\,x| + |S(t_{k+1})\,x - S(t)\,x|\,. \tag{5.9}$$

Using the stability condition (S), one can show that

$$|S_n(t)\,x_n - S_n(t_{k+1})\,x_n| \leq e^{\omega(\beta,\tau)\tau}\,|S_n(t_{k+1}-t)\,x_n - x_n| \tag{5.10}$$

for $n \geq 1$, and

$$|S(t_{k+1})\,x - S(t)\,x| \leq e^{\omega(\alpha,\tau)\tau}\,|S(t_{k+1}-t)\,x - x|\,. \tag{5.11}$$

It now remains to estimate the second term on the right-hand side of (5.9). Using condition (S), one obtains

$$|S_n(t_{k+1})\,x_n - S(t_{k+1})\,x| \leq e^{\omega(\beta,\tau)(t_{k+1}-t_k)}\left(|S_n(t_k)\,x_n - S(t_k)\,x| + \left|S(t_k)\,x - y_n^{(k)}\right|\right) + \left|S_n(t_{k+1}-t_k)\,y_n^{(k)} - S(t_{k+1})\,x\right|. \tag{5.12}$$

We also have

$$\begin{aligned}
&\left|S_n(t_{k+1}-t_k)\,y_n^{(k)} - S(t_{k+1})\,x\right| \\
&\quad\leq \left|S_n(t_{k+1}-t_k)\,y_n^{(k)} - T_n(t_{k+1}-t_k)\,y_n^{(k)} - (t_{k+1}-t_k)\,B_n y_n^{(k)}\right| \\
&\qquad + \left|T_n(t_{k+1}-t_k)\,y_n^{(k)} - T(t_{k+1}-t_k)\,S(t_k)\,x\right| + (t_{k+1}-t_k)\left|B_n y_n^{(k)} - BS(t_k)\,x\right| \\
&\qquad + |T(t_{k+1}-t_k)\,S(t_k)\,x + (t_{k+1}-t_k)\,BS(t_k)\,x - S(t_{k+1})\,x|\,.
\end{aligned}$$

Taking the limit superior in (5.12) gives

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\,|S_n(t_{k+1})\,x_n - S(t_{k+1})\,x| &\leq e^{\omega(\beta,\tau)(t_{k+1}-t_k)}\,\overline{\lim_{n\to\infty}}\,|S_n(t_k)\,x_n - S(t_k)\,x| + 2\varepsilon\,(t_{k+1}-t_k) \\
&\quad + \overline{\lim_{n\to\infty}}\left|T_n(t_{k+1}-t_k)\,y_n^{(k)} - T(t_{k+1}-t_k)\,S(t_k)\,x\right|.
\end{aligned}$$

Since $\overline{\lim_{n\to\infty}}\left|T_n(t_{k+1}-t_k)\,y_n^{(k)} - T(t_{k+1}-t_k)\,S(t_k)\,x\right| = 0$ by Lemma 4.1, it follows that

$$\overline{\lim_{n\to\infty}}\,|S_n(t_{k+1})\,x_n - S(t_{k+1})\,x| \leq e^{\omega(\beta,\tau)(t_{k+1}-t_k)}\,\overline{\lim_{n\to\infty}}\,|S_n(t_k)\,x_n - S(t_k)\,x| + 2\varepsilon\,(t_{k+1}-t_k)\,.$$

Denoting $\overline{\lim_{n\to\infty}}\,|S_n(t_k)\,x_n - S(t_k)\,x|$ by L_k, we obtain

$$L_{k+1} \leq e^{\omega(\beta,\tau)(t_{k+1}-t_k)}L_k + 2\varepsilon\,(t_{k+1}-t_k)\ ,\ L_0 = 0.$$

This recurrent inequality implies

$$\overline{\lim_{n\to\infty}}\,|S_n(t_{k+1})\,x_n - S(t_{k+1})\,x| \leq 2\varepsilon\tau e^{\omega(\beta,\varepsilon)\tau}. \tag{5.13}$$

Passing to limit superior as $n \to \infty$ in (5.9) and applying (5.10), (5.11) and (5.13), one obtains

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \left|S_n(t)\,x_n - S(t)\,x\right| \\
&\quad \le e^{\omega(\beta,\tau)\tau}\left[\overline{\lim_{n\to\infty}}\left|S_n(t_{k+1}-t)\,x_n - x_n\right| + 2\varepsilon\tau\right] + e^{\omega(\alpha,\tau)\tau}\left|S(t_{k+1}-t)\,x - x\right| \\
&\quad \le e^{\omega(\beta,\tau)\tau}\left[\sup_{h\in[0,\varepsilon]}\left(\sup_{n\ge 1}\left|S_n(h)\,x_n - x_n\right|\right) + 2\varepsilon\tau + \sup_{h\in[0,\varepsilon]}\left|S(h)\,x - x\right|\right].
\end{aligned}$$

Since

$$\sup_{h\in[0,\varepsilon]}\left(\sup_{n\ge 1}\left|S_n(h)\,x_n - x_n\right|\right) \to 0 \quad \text{as } \varepsilon \downarrow 0$$

by (EC) and

$$\sup_{h\in[0,\varepsilon]}\left|S(h)\,x - x\right| \to 0 \quad \text{as } \varepsilon \downarrow 0$$

we conclude that

$$\overline{\lim_{n\to\infty}}\left|S_n(t)\,x_n - S(t)\,x\right| = 0 \quad \text{uniformly on } [0,\tau].$$

This means that $S_n(\cdot)\,x_n$ converges to $S(\cdot)\,x$ uniformly on $[0,\tau]$. Thus the proof is complete. □

6 Semilinear Neveu-Trotter-Kato theorem

In this section we discuss a semilinear version of the Neveu-Trotter-Kato theorem under the assumption that D_n and φ_n, $n = 0, 1, 2, \cdots$, are convex. Let $\{S_n\}_{n\ge 0}$ be a sequence of locally Lipschitzian semigroups satisfying (3.3) and (3.4). Our aim is to give the proof of a semilinear Neveu-Trotter-Kato theorem under the stability condition (S). As mentioned in Section 3, condition (S) is equivalent to the combination of (LQD) and (RC) stated as below:

(LQD) For $n \ge 0$ and $\alpha > 0$ there exist $\omega_{n,\alpha} \in \mathbb{R}$ such that $\sup_{n\ge 0}\omega_{n,\alpha} < \infty$ and

$$\left\langle (A_n + B_n)\,x_n - (A_n + B_n)\,y_n, x_n - y_n\right\rangle_i \le \omega_{n,\alpha}\left|x_n - y_n\right|^2 \quad \text{for each } x_n, y_n \in D_{n,\alpha}.$$

(RC) For $n = 0,\ 1,\ 2\cdots$, $D(A_n) \cap D_n$ is dense in D_n; for $\alpha > 0$ and there is $\lambda_{0,n} = \lambda_{0,n}(\alpha) \in (0, 1/a)$ such that to $\lambda \in (0, \lambda_{0,n})$ and $x_n \in D_{n,\alpha}$ there corresponds $x_n^\lambda \in D(A_n) \cap D_n$ satisfying

$$x_n^\lambda - \lambda(A_n + B_n)\,x_n^\lambda = x_n \text{ and } \varphi_n\left(x_n^\lambda\right) \le (1 - \lambda a)^{-1}\left(\varphi_n(x_n) + b\lambda\right).$$

Remark 6.1. As shown in [11, Theorem 3.1], it is seen that $\lambda_{0,n}$ can be chosen independently of n. More precisely, for each $\alpha > 0$ we may take the constant

$$\lambda_{0,n}(\alpha) = \lambda_0(\alpha) = \min\left\{\left(\max\{\sup_{n\ge 0}\omega_{n,\alpha}, 0\}\right)^{-1}, \left(a(\alpha+2) + (b+1)\right)^{-1}\right\}.$$

We now state our second main result

Theorem 6.1. *Let $\{S_n\}_{n\geq 0}$ be a sequence of locally Lipschitzian semigroups satisfying* (3.3) *and* (3.4). *Asssume that conditions* (C) *and* (S) *hold, and that D_n and φ_n are convex for $n = 0,\ 1,\ 2, \cdots$. Then the following statements are equivalent.*

(I) *If $x_0 \in D_0$, $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-bounded sequence, $x_n \to x_0$ as $n \to \infty$, then*

$$S_n\,(t)\,x_n \to S_0\,(t)\,x_0 \text{ as } n \to \infty \text{ for } t \geq 0$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$.

(II) *For each $\alpha > 0$ there is $\beta = \beta(\alpha) > 0$ such that to $x_0 \in D\,(A_0) \cap D_{0,\alpha}$ there corresponds a sequence $\{x_n\}_{n\geq 1}$ such that*

$$x_n \in D\,(A_n) \cap D_{n,\beta}, x_n \to x \text{ and } (A_n + B_n)\,x_n \to (A_0 + B_0)\,x_0 \text{ as } n \to \infty.$$

(III) *The following statements are valid:*

(III.1) *For each $\alpha > 0$ there is $\lambda_1 = \lambda_1\,(\alpha) \in (0, 1/a)$ such that if $\lambda \in (0, \lambda_1)$, $x_n \in D_{n,\alpha}$ for $n \geq 1$, $x_0 \in D_{0,\alpha}$, and $x_n \to x_0$ as $n \to \infty$, then there exist $x_n^\lambda \in D\,(A_n) \cap D_n$ and $x_0^\lambda \in D\,(A_0) \cap D_0$ satisfying*

$$x_n^\lambda - \lambda\,(A_n + B_n)\,x_n^\lambda = x_n\ ,\ \varphi_n\left(x_n^\lambda\right) \leq (1 - \lambda a)^{-1}\,(\varphi_n\,(x_n) + b\lambda)\,,$$
$$x_0^\lambda - \lambda\,(A_0 + B_0)\,x_0^\lambda = x_0\ ,\ \varphi_0\left(x_0^\lambda\right) \leq (1 - \lambda a)^{-1}\,(\varphi\,(x_0) + b\lambda)\,,$$

and $x_n^\lambda \to x_0^\lambda$ as $n \to \infty$.

(III.2) *If $\varepsilon > 0$, $x_0 \in D_0$, $\{x_n\}_{n\geq 1}$ is $\{\varphi_n\}$-bounded and $x_n \to x_0$ as $n \to \infty$, then there are a $\{\varphi_n\}$-bounded sequence $\{z_n\}_{n\geq 1}$ and $z_0 \in D_0 \cap D(A_0)$ such that*

$$z_n \in D\,(A_n) \cap D_n\ ,\ z_n \to z_0 \text{ as } n \to \infty, \text{ and } \sup_{n\geq 0} |z_n - x_n| < \varepsilon.$$

Proof. (I) $\Rightarrow$ (II) : Suppose that (I) holds. Let $\alpha > 0$ and $x_0 \in D\,(A_0) \cap D_{0,\alpha}$. By (C2), one finds a number $\gamma > 0$ and a sequence $\{x_n\}_{n\geq 1}$ such that $x_n \in D_{n,\gamma}$ for $n \geq 0$ and $x_n \to x_0$ as $n \to \infty$. Set $x_n^h{=}(1/h) \int_0^h S_n\,(t)\,x_n dt$, for $n \geq 0$ and $h > 0$.

We see from Theorem 2.2 that $x_n^h \in D\,(A_n) \cap D_n$ and

$$A_n x_n^h + (1/h) \int_0^h B_n S_n\,(t)\,x_n\,dt = (1/h)\,(S_n\,(h)\,x_n - x_n) \text{ for } n \geq 0.$$

Let $\beta > e^{ah_0}\,(\gamma + b)$. Since

$$\varphi_n\left(x_n^h\right) \leq (1/h) \int_0^h \varphi_n\,(S_n\,(t)\,x_n)\,dt \leq e^{ah}\,(\varphi_n\,(x_n) + bh) < \beta$$

for $n \geq 1$, it follows that $x_n^h \in D_{n,\beta}$ for $n \geq 1$ and $h \in (0, 1]$.

Let $\varepsilon > 0$. We have

$$\begin{aligned}&\left|(A_n + B_n)\, x_n^h - (1/h)\,(S_n(h)\,x_n - x_n)\right| \\ &\quad\le \left|B_n x_n^h - (1/h)\int_0^h B_n S_n(t)\, x_n dt\right| \\ &\quad\le (1/h)\int_0^h \left|B_n S_n(t)\,x_n - B_n x_n\right| dt + \left|B_n x_n^h - B_n x_n\right|.\end{aligned}$$

From condition (I) and Lemma 4.3 it follows that there is $\delta \in (0,1]$ such that

$$\left|(A_n + B_n)\, x_n^h - (1/h)\,(S_n(h)\,x_n - x_n)\right| < \varepsilon \quad \text{for } h \in (0,\delta] \text{ and } n \ge 1. \tag{6.1}$$

Since $x_0^h \to x_0$ as $h \to 0$ and $x_0 \in D(A_0) \cap D_0$, we have

$$\left|x_0^h - x_0\right| \le (1/h)\int_0^h \left|S_0(t)\,x_0 - x_0\right| dt < \varepsilon \tag{6.2}$$

and, by Theorem 2.2 (ii),

$$\left|(A_0 + B_0)\,x_0 - (1/h)\,(S_0(h)\,x_0 - x_0)\right| < \varepsilon, \tag{6.3}$$

for $h \in \left[0, \hat{\delta}\right]$ and some $\hat{\delta}$. We here take $\hat{\delta}$ to be smaller than δ. Then $\left|x_n^h - x_0\right| < \left|x_n^h - x_n\right| + |x_n - x_0|$, and so (6.2) implies that $\overline{\lim_{n\to\infty}} \left|x_n^h - x_0\right| \le \varepsilon$,

$$\left|(A_n + B_n)\,x_n^h - (A_0 + B_0)\,x_0\right| \le 2\varepsilon + \left|(1/h)\,(S_n(h)\,x_n - x_n) - (1/h)\,(S_0(h)\,x_0 - x_0)\right|.$$

which implies that $\overline{\lim_{n\to\infty}} \left|(A_n + B_n)\,x_n^h - (A_0 + B_0)\,x_0\right| \le 2\varepsilon$. From this we infer that there exists a sequence $\{y_n\}_{n\ge1}$ such that

$$y_n \in D_{n,\beta} \text{ for } n \ge 1\,,\; y_n \to x_0 \text{ and } (A_n + B_n)\,y_n \to (A_0 + B_0)\,x_0 \text{ as } n \to \infty.$$

Thus (II) follows.

(II)$\Rightarrow$(III) : 1. We first derive (III.1). Let $\alpha > 0$, $x_n \in D_{n,\alpha}$ for $n \ge 0$, and let $x_n \to x_0$ as $n \to \infty$. Let $\lambda_0(\alpha)$ be the number specified in Remark 6.1 and β a number given for $\gamma = (1 - \lambda_0(\alpha))^{-1}(\alpha + b\lambda_0(\alpha))$ (instead of α) by (II). Then it follows from (RC) that for $n \ge 0$ and $\lambda \in (0, \lambda_0(\alpha))$ there exists an $x_n^\lambda \in D_n$ such that

$$x_n^\lambda - \lambda\,(A_n + B_n)\,x_n^\lambda = x_n \text{ and } \varphi_n\left(x_n^\lambda\right) \le (1 - \lambda a)^{-1}(\varphi_n(x_n) + b\lambda) \le \gamma.$$

Now, for each $\lambda \in (0, \lambda_0(\alpha))$, (II) asserts the existence of a sequence $\left\{y_n^\lambda\right\}_{n\ge1}$ such that

$$y_n^\lambda \in D(A_n) \cap D_{n,\beta}\,,\; y_n^\lambda \to x_0^\lambda \text{ and } (A_n + B_n)\,y_n^\lambda \to (A_0 + B_0)\,x_0^\lambda \text{ as } n \to \infty.$$

At this point we necessitate assuming that $\beta > \gamma$ and choosing a number $\lambda_1(\alpha)$ so that $\lambda_1(\alpha) < \min\left\{\lambda_0(\alpha)\;,\; \left(\sup_{n\ge1} \omega_{n,\beta}\right)\right\}$. Let $\lambda \in (0, \lambda_1(\alpha))$ and $z_n^\lambda = y_n^\lambda - \lambda\,(A_n + B_n)\,y_n^\lambda$ for $n \ge 1$. Then, by (LQD), we have

$$\left|z_n^\lambda - x_n\right| = \left|y_n^\lambda - \lambda\,(A_n + B_n)\,y_n^\lambda - x_n^\lambda + \lambda\,(A_n + B_n)\,x_n^\lambda\right| \tag{6.4}$$

$$\geq (1-\lambda \sup_{n\geq 1} \omega_{n,\beta}) \left| x_n^\lambda - y_n^\lambda \right| \qquad \text{for } n \geq 1.$$

Since $\left| z_n^\lambda - x_n \right| \leq \left| z_n^\lambda - x_0 \right| + |x_0 - x_n|$, and

$$\left| z_n^\lambda - x_0 \right| = \left| y_n^\lambda - \lambda \left(A_n + B_n \right) y_n^\lambda - x_0^\lambda + \lambda \left(A_0 + B_0 \right) x_0^\lambda \right| \to 0 \quad \text{as } n \to \infty,$$

it follows that $\left| y_n^\lambda - x_n^\lambda \right| \to 0$ as $n \to \infty$. Since $y_n^\lambda \to x_0^\lambda$, it is concluded that $x_n^\lambda \to x_0^\lambda$ as $n \to \infty$.

We next show (III.2). Let $\varepsilon > 0$. Let $\{x_n\}_{n\geq 0}$ be a $\{\varphi_n\}$-bounded sequence and assume that $x_n \to x_0$ as $n \to \infty$. By (RC) one finds $x_0^\varepsilon \in D\left(A_0\right) \cap D_0$ such that $|x_0^\varepsilon - x_0| < \varepsilon/3$. Let $\alpha = \varphi\left(x_0^\varepsilon\right)$. Then there is $\beta > 0$ and a sequence $\{y_n^\varepsilon\}_{n\geq 1}$, $y_n^\varepsilon \in D\left(A_n\right) \cap D_{n,\beta}$ such that

$$y_n^\varepsilon \to x_0^\varepsilon \text{ and } \left(A_n + B_n \right) y_n^\varepsilon \to \left(A_0 + B_0 \right) x_0^\varepsilon. \tag{6.5}$$

Let $N_\varepsilon > 0$ such that

$$|x_n - x_0| < \varepsilon/3 \text{ and } |y_n^\varepsilon - x_0^\varepsilon| < \varepsilon/3 \ \text{ for } n \geq N_\varepsilon. \tag{6.6}$$

For $1 \leq n \leq N_\varepsilon - 1$, (RC) guarantees the existence of $w_n^\varepsilon \in D\left(A_n\right) \cap D_n$ such that $|w_n^\varepsilon - x_n| < \varepsilon$. We then define $\{z_n\}_{n\geq 0}$ by

$$z_n = \begin{cases} x_0^\varepsilon & \text{if } n = 0 \\ w_n^\varepsilon & \text{if } 1 \leq n \leq N_\varepsilon - 1 \\ y_n^\varepsilon & \text{if } n \geq N_\varepsilon. \end{cases}$$

For $n \geq N_\varepsilon$, (6.6) implies

$$|y_n^\varepsilon - x_n| \leq |x_n - x_0| + |x_0 - x_0^\varepsilon| + |y_n^\varepsilon - x_0^\varepsilon| < \varepsilon.$$

Combining the above-mentioned we conclude that $|z_n - x_n| < \varepsilon$ for all $n \geq 0$. Finally, by (6.5) we see that $z_n \to z_0$ as $n \to \infty$, and that (III.2) is obtained.

(III) $\Rightarrow$ (I) : We again employ the Kisyński sequence space defined in Section 5. We first define a linear operator $\mathscr{A} : D\left(\mathscr{A}\right) \subset \mathscr{X} \to \mathscr{X}$, a nonlinear operator $\mathscr{B} : \mathscr{D} \subset \mathscr{X} \to \mathscr{X}$ and a functional $: \mathscr{X} \to \mathscr{X}$ by the following

$$D\left(\mathscr{A}\right) = \left\{ \mathscr{x} = \{x_n\}_{n\geq 0} \in \mathscr{X};\ x_n \in D\left(A_n\right) \text{ for each } n \geq 0,\ A_n x_n \to A_0 x_0 \text{ as } n \to \infty \right\},$$

$$\mathscr{A}\left(\{x_n\}_{n\geq 0}\right) = \{A_n x_n\}_{n\geq 0},$$

$$\mathscr{D} = \left\{ \mathscr{x} = \{x_n\}_{n\geq 0} \in \mathscr{X};\ x_n \in D_n \text{ for } n \geq 0 \text{ and } \sup_{n\geq 0} \varphi_n\left(x_n\right) < \infty \right\};$$

$$\mathscr{B}\left(\{x_n\}_{n\geq 0}\right) = \{B_n x_n\}_{n\geq 0};$$

$$\left(\{x_n\}_{n\geq 1}\right) = \begin{cases} \sup_{n\geq 0} \varphi_n(x_n) & \text{if } x \in \mathscr{D} \\ \infty & \text{otherwise.} \end{cases}$$

We then define level sets with respect to by

$$\mathscr{D}_\alpha = \left\{ \{x_n\}_{n\geq 1};\ \left(\{x_n\}_{n\geq 1}\right) \leq \alpha \right\}, \quad \alpha \geq 0.$$

It is easily seen that the lower semicontinuity of each φ_n implies the lower semicontinuity of . Also, $\mathscr{x} = \{x_n\}_{n\geq 0} \in \mathscr{D}_\alpha$ if and only if $x_n \in D_{n,\alpha}$ for each $n \geq 0$. Hence each $\mathscr{D}_\alpha$ is closed in $\mathscr{X}$. Moreover, we observe that $\{x_n\}_{n\geq 0}$ is a $\{\varphi_n\}$-bounded sequence and $x_n \to x_0$ as $n \to \infty$ if and only if $\{x_n\}_{n\geq 0}$ belongs to some $\mathscr{D}_\gamma$.

Now, $\mathscr{B}$ is well-defined by (C3). The Neveu-Trotter-Kato theorem and Lemma 4.1 together imply that $\mathscr{A}$ generates a (C_0)-semigroup $\mathscr{T} = \{\mathscr{T}(t)\,; t \geq 0\}$ on $\mathscr{X}$ given by

$$\mathscr{T}(t)\left(\{x_n\}_{n\geq 0}\right) = \lim_{n\to\infty} \left(\mathscr{I} - (t/m)\,\mathscr{A}\right)^{-m} \{x_n\}_{n\geq 0} = \{T_n(t)\,x_n\}_{n\geq 0}.$$

From (LQD) we obtain

$$(1 - \lambda\omega_\alpha)|x_n - y_n| \leq |(I - \lambda(A_n + B_n))x_n - (I - \lambda(A_n + B_n))y_n|$$

for $n \geq 0$, x_n, $y_n \in D_{n,\alpha}$, $\lambda > 0$ and $\alpha \geq 0$, where $\omega_\alpha = \sup_{n\geq 0} \omega_{n,\alpha}$. Hence

$$(1 - \lambda\omega_\alpha)|\mathscr{x} - \mathscr{y}| \leq |(\mathscr{I} - \lambda(\mathscr{A} + \mathscr{B}))\mathscr{x} - (\mathscr{I} - \lambda(\mathscr{A} + \mathscr{B}))\mathscr{y}|$$

for $\mathscr{x}$, $\mathscr{y}$ in $\mathscr{D}_\alpha$, $\lambda \in (0, \omega_\alpha)$ and $\alpha \geq 0$. From this, we see that $\mathscr{A} + \mathscr{B}$ is quasidissipative on $\mathscr{D}_\alpha$, $\alpha \geq 0$, namely

$$\langle (\mathscr{A} + \mathscr{B})\,\mathscr{x} - (\mathscr{A} + \mathscr{B})\,\mathscr{y}, \mathscr{x} - \mathscr{y} \rangle_i \leq \left(\sup_{n\geq 0} \omega_{n,\alpha}\right) |\mathscr{x} - \mathscr{y}|^2$$

for $\mathscr{x}$, $\mathscr{y} \in \mathscr{D}_\alpha$, $\alpha \geq 0$, where $\langle \cdot, \cdot \rangle_i$ stands for the lower semiinnner product in $\mathscr{X}$. It follows from Lemma 4.2 that $\mathscr{B}$ is continuous on each $\mathscr{D}_\alpha$, $\alpha > 0$.

Let $\alpha > 0$ and $\mathscr{x} = \{x_n\}_{n\geq 0} \in \mathscr{D}_\alpha$. Then $x_n \in D_{n,\alpha}$ for $n \geq 0$ and $x_n \to x_0$ as $n \to \infty$. By (III) there exists $\lambda_0 = \lambda_0(\alpha) > 0$ such that for each $\lambda \in (0, \lambda_0)$ and each $n \geq 0$ there exists $x_n^\lambda \in D_n \cap D(A_n)$ such that

$$x_n^\lambda - \lambda\,(A_n + B_n)\,x_n^\lambda = x_n,\;\; \varphi_n\left(x_n^\lambda\right) \leq (1 - a\lambda)^{-1}\left(\varphi_n(x_n) + b\lambda\right) \text{ for } n \geq 0 \text{ and } x_n^\lambda \to x_0^\lambda,$$

which means that

$$\mathscr{x}^\lambda - \lambda\,(\mathscr{A} + \mathscr{B})\,\mathscr{x}^\lambda = \mathscr{x} \text{ and also } \left(\mathscr{x}^\lambda\right) \leq (1 - a\lambda)^{-1}\left(\,(\mathscr{x}) + b\lambda\right) \text{ in } \mathscr{X}.$$

Now (III.2) implies that $D(\mathscr{A}) \cap \mathscr{D}$ is dense in $\mathscr{D}$. Applying Theorem 2.1 to the Kisyński space $\mathscr{X}$, one obtains a locally Lipschitzian semigroup $\mathscr{S}$ satisfying

$$\mathscr{S}(t)\left(\{x_n\}_{n\geq 0}\right) = \mathscr{T}(t)\left(\{x_n\}_{n\geq 0}\right) + \int_0^t \mathscr{T}(t-s)\,\mathscr{B}\mathscr{S}(s)\left(\{x_n\}_{n\geq 0}\right) ds$$

and

$$\left(\mathscr{S}(t)\,\{x_n\}_{n\geq 0}\right) \leq e^{at}\left(\,\left(\{x_n\}_{n\geq 0}\right) + bt\right).$$

Here the uniqueness theorem, Theorem 2.3, asserts that $\mathscr{S}(t)\left(\{x_n\}_{n\geq 0}\right) = \{S_n(t)x_n\}_{n\geq 0}$ for $t \geq 0$ and $\{x_n\}_{n\geq 0} \in \mathscr{D}$. Therefore, we may write

$$\int_0^t \mathscr{T}(t-s)\,\mathscr{B}\mathscr{S}(s)\left(\{x_n\}_{n\geq 0}\right) ds = \left\{\int_0^t T_n(t-s)\,B_n S_n(s)\,x_n ds\right\}_{n\geq 0}.$$

The continuity of the semigroup $\mathscr{S}$ implies

$$\left|\mathscr{S}(t)\left(\{x_n\}_{n\geq 0}\right) - \{x_n\}_{n\geq 0}\right| \to 0 \quad \text{as } t \downarrow 0 \text{ for } \mathscr{x} = \{x_n\}_{n\geq 0} \in \mathscr{D}.$$

This is actually nothing but condition (EC). This implies in turn, from Theorem 5.1, the required uniform convergence. □

Remark 6.2. The second statement corresponds to the convergence in the sense of graphs on a core of $D(A_0)$ in the Neveu-Trotter-Kato theorem, while the third one corresponds to the convergence in the sense of resolvents. See [7] for details and [6] for a discussion on the consistency and stability condition in the linear case, and on error estimates for smooth initial data. See also [5] for a recent application of Neveu-Trotter-Kato theorem to an age-structured population dynamics model.

Remark 6.3. If the family $\{B_n\}_{n\geq 0}$ is uniformly Lipschitz in the sense of Remark 4.1, then $\mathscr{B}$ becomes a locally Lipschitz operator and (III.2) is no longer necessary. See [13] for details.

7 Approximation solvability theorem

This section corresponds to the case in which it is not straightforward to verify the hypotheses of the known generation theorems for the semilinear problem

(SP) $$u'(t) = (A+B)\,u(t), \quad t>0; \qquad u(0) = x \in D.$$

Here A is assumed to be the generator of a (C_0)-semigroup $T = \{T(t)\,; t \geq 0\}$, D is a closed subset of X and $B : D \to X$ is a nonlinear operator.

In this case one can try to obtain the semigroup S as an uniform limit of the approximate semigroups S_n, using suitable approximations for the operators A and B.

We consider again the approximate semilinear problems

(SP; n) $$u_n'(t) = (A_n + B_n)\,u_n(t), \quad t>0; \qquad u_n(0) = x_n \in D_n,$$

with A_n and B_n satisfying respectively the hypotheses (A_n) and (B_n) given in Section 2. Also, to each (SP; n) one associates a proper l.s.c. functional such that $D \subset D(\varphi_n)$. We assume that for each n there exists a nonlinear semigroup $S_n = \{S_n(t)\,; t \geq 0\}$ on D_n satisfying (3.3) and (3.4).

Define a set $\widetilde{D}$ in X by

(7.1) $$\widetilde{D} = \{x \in X,\ x \text{ is a limit of some } \{x_n\} \text{ with } x_n \in D_n \text{ for } n \geq 1\},$$

and a functional $\Phi : X \to [0, \infty]$ such that

(7.2) $$\Phi(x) = \begin{cases} \inf\left\{\overline{\lim_{n\to\infty}}\,\varphi_n(x_n)\,;\ x_n \in D_n \text{ for } n \geq 1,\ x_n \to x \text{ as } 4n \to \infty\right\} & \text{for } x \in \widetilde{D} \\ \infty & \text{otherwise.} \end{cases}$$

Suppose now that the following condition is satisfied:

(C4) The following statements hold:

(C4.a) For each $x \in D$ there is a sequence $\{x_n\}$ such that $x_n \in D_n$, $\overline{\lim_{n\to\infty}} \varphi_n(x_n) < \infty$ and $x_n \to x$ in X as $n \to \infty$.

(C4.b) There is $\beta \geq 0$ such that $D_{n,\beta} \neq \emptyset$ for each $n \geq 1$.

(C4.c) If $x_n \in D_n$, $\overline{\lim_{n\to\infty}} |x_n| < \infty$ and $\overline{\lim_{n\to\infty}} \varphi_n(x_n) < \infty$, then $\underline{\lim_{n\to\infty}} d(x_n, D_\alpha) = 0$, for each $\alpha > \overline{\lim_{n\to\infty}} \varphi_n(x_n)$.

together with (C1) and (C3). Note that, by (C4.a), $D \subset \widetilde{D}$ and $\Phi(x) < \infty$ for each $x \in D$.

We intend to use the newly-defined functional Φ as a growth function to establish the well-posedness of (SP). To this goal, we necessitate establishing some properties of Φ and $\widetilde{D}$.

Lemma 7.1. *The set $\widetilde{D}$ is closed in X.*

Proof. Let $\widetilde{x}_n \in \widetilde{D}$, $\widetilde{x}_n \to \widetilde{x} \in X$. We may use the definition of $\widetilde{D}$ to obtain the sequences $\{\{x_n^m\}_m\}_n$ such that $x_n^m \in D_m$ for each $m, n \geq 1$ and $x_n^m \to \widetilde{x}_n$ as $m \to \infty$.

Then for each $i \geq 1$ we can choose N_i such that

$$|x_i^m - \widetilde{x}_i| \leq 1/i \text{ for } m \geq N_i, \text{and also } N_i \geq N_{i-1} \text{ for } i \geq 2. \tag{7.3}$$

Define now $\{y_n\}$ by $y_n = \begin{cases} x_1^n & \text{for } 1 \leq n \leq N_2 - 1 \\ x_i^n & \text{for } i \geq 2 \text{ and } N_i \leq n \leq N_{i+1} - 1. \end{cases}$

Let $\varepsilon > 0$ and $N_\varepsilon \geq 1$ great enough, so that

$$|\widetilde{x}_n - \widetilde{x}| < \varepsilon/2 \text{ for } n \geq N_\varepsilon \text{ and also } 1/N_\varepsilon < \varepsilon/2. \tag{7.4}$$

Then $|y_n - \widetilde{x}| \leq |y_n - \widetilde{x}_N| + |\widetilde{x}_N - x|$ for some $N \geq N_\varepsilon$ corresponding to y_n by the defining procedure (that is, $y_n = x_N^n$ for that N), and from (7.3) and (7.4) we obtain that $|y_n - \widetilde{x}| < \varepsilon$ for $n \geq N_N$. This shows that $y_n \to x$ and so $\widetilde{x} \in \widetilde{D}$ and $\widetilde{D}$ is closed. □

We also observe that the infimum in (7.2) is actually a minimum, as seen from the following lemma.

Lemma 7.2. *For each $x \in \widetilde{D}$ there is a sequence $\{x_n\}$, $x_n \in D_n$, $x_n \to x$ as $n \to \infty$ such that $\Phi(x) = \overline{\lim_{n\to\infty}} \varphi_n(x_n)$.*

Proof. Let $x \in \widetilde{D}$ and $\varepsilon > 0$.

If $\Phi(x) < \infty$, then for each $k > 0$ there is a sequence $\left\{x_n^k\right\}_n$ such that $x_n^k \in D_n$ for $n \geq 1$, $x_n^k \to x$ as $n \to \infty$ and $\overline{\lim_{n\downarrow\infty}} \varphi_n\left(x_n^k\right) < \Phi(x) + 1/(2k)$. Then for each $k \geq 1$ we can choose N_k such that

$$\varphi_n\left(x_n^k\right) < \Phi(x) + 1/k\,,\ \left|x_n^k - x\right| < 1/k \text{ for all } n > N_k \text{ and } N_k \geq N_{k-1} \text{ for } k \geq 2. \tag{7.5}$$

As in the proof of Lemma 7.1, we define a diagonal sequence $\{y_n\}$ by

$$y_n = \begin{cases} x_n^1 & \text{for } 1 \leq n \leq N_2 - 1 \\ x_n^i & \text{for } i \geq 2 \text{ and } N_i \leq n \leq N_{i+1} - 1. \end{cases}$$

Let $m \geq 1$ such that $1/m < \varepsilon$. Then

$$|y_n - x| < \varepsilon \text{ for } n \geq N_m. \tag{7.6}$$

We also see that $\overline{\lim_{n\to\infty}} \varphi_n(y_n) \leq \Phi(x)$ by (7.5) and, since $\overline{\lim_{n\downarrow\infty}} \varphi_n(y_n) \geq \Phi(x)$ by the definition of Φ, we obtain $\overline{\lim_{n\to\infty}} \varphi_n(x_n) = \Phi(x)$. Combining this with (7.6) we see that $\{y_n\}$ is the required sequence.

If $\Phi(x) = \infty$, then

$$\inf\left\{ \overline{\lim_{n\to\infty}} \varphi_n(x_n), x_n \in D_n, x_n \to x \text{ as } n \to \infty \right\} = \infty,$$

and so $\overline{\lim_{n\to\infty}} \varphi_n(x_n) = \infty$ for each $\{x_n\}$ such that $x_n \in D_n$ for all n and $x_n \to x$, so Lemma 7.2 is proved. □

The existence of such minimizing sequence will play a central role in the proof of the continuity of B on level sets, as it will be seen in what follows.

Lemma 7.3. Φ *is proper l.s.c.* .

Proof. Set $x \in X$ and let $\{x_n\}$ be a sequence which converges to x as $n \to \infty$.

If $x \notin \widetilde{D}$ then $x_n \notin \widetilde{D}$ for n greater than some N; otherwise $x \in \overline{\widetilde{D}} = \widetilde{D}$. Then $\Phi(x_n) = \infty$ for $n \geq N$, and so $\Phi(x) = \underline{\lim}_{n\to\infty} \Phi(x_n) = \infty$.

If $x \in \widetilde{D}$ and $\underline{\lim}_{n\to\infty} \Phi(x_n) < \infty$, then there is a subsequence $\{n_k\}$ such that $\Phi(x_{n_k}) \to \underline{\lim}_{n\to\infty} \Phi(x_n)$ as $k \to \infty$. Using the definition of Φ, for each $k \geq 1$ we can choose sequences $\left\{x_{n_k,m}^k\right\}_m$ such that

$$x_{n_k,m}^k \in D_m \text{ for each } m \geq 1 \text{ , } x_{n_k,m}^k \to x_{n_k} \text{ as } m \to \infty$$

and

$$\overline{\lim_{m\to\infty}} \varphi_m\left(x_{n_k,m}^k\right) < \Phi(x_{n_k}) + 1/2k.$$

Then, as in the proof of Lemma 7.2, for each $k \geq 1$ we can choose $N_k \geq 1$ such that

$$\varphi_m\left(x_{n_k,m}^k\right) < \Phi(x_{n_k}) + 1/k \text{ , } \left|x_{n_k}^k - x_{n_k}\right| < 1/k \text{ for } m \geq N_k \text{ and } N_k \geq N_{k-1} \text{ for } k \geq 2. \tag{7.7}$$

We now define a sequence $\{y_m\}$ by

$$y_m = \begin{cases} x_{n_1,m}^1 & \text{for } 1 \leq m \leq N_2 - 1 \\ x_{n_i,m}^i & \text{for } i \leq 2 \text{ and } N_i \leq m \leq N_{i+1} - 1. \end{cases}$$

Let $\varepsilon > 0$ and $k_\varepsilon \geq 1$ such that

$$\Phi\left(x_{n_k}\right) - \varliminf_{n\to\infty} \Phi\left(x_n\right) < \varepsilon/2\,,\ \left|x_{n_k} - x\right| < \varepsilon/2 \text{ and } 1/k < \varepsilon/2 \text{ for } k \geq k_\varepsilon. \tag{7.8}$$

Then

$$\varphi_m\left(y_m\right) - \varliminf_{n\to\infty} \Phi\left(x_n\right) = \left(\varphi_m\left(y_m\right) - \Phi\left(x_{n_k}\right)\right) + \left(\Phi\left(x_{n_k}\right) - \varliminf_{n\to\infty} \Phi\left(x_n\right)\right) < \varepsilon$$

and $|y_m - x| < \varepsilon$ for $m \geq N_{k_\varepsilon}$, by (7.7) and (7.8). Hence $y_n \to x$ as $m \to \infty$ and

$$\varlimsup_{n\to\infty} \varphi_n\left(y_n\right) \leq \varliminf_{n\to\infty} \Phi\left(x_n\right). \tag{7.9}$$

From (7.9), using again the definition of Φ, we get $\Phi(x) \leq \varliminf_{n\to\infty} \Phi(x_n)$, and so Φ is l.s.c. We have already seen that $D \subset \widetilde{D}$ and Φ is proper by (C4.a), which completes the proof. □

Next we prove that B is continuous on the level sets of D with respect to Φ.

Lemma 7.4. *B is continuous on D_α for each $\alpha \geq 0$.*

Proof. Set $\alpha > 0$, $x \in D_\alpha$ and $\delta > 0$. By Lemma 7.2, there is a sequence $\{x_n\}$, $x_n \in D_n$, such that $x_n \to x$ as $n \to \infty$ and $\varlimsup_{n\to\infty} \varphi_n(x_n) \leq \alpha$. Let $\varepsilon > 0$ and denote $\gamma = \sup_{n\geq 1} \varphi_n(x_n) < \infty$.

Let $\overline{\gamma} = \max\{\alpha + \delta, \gamma\}$ and let $r = r(\varepsilon, \overline{\gamma}, \{x_n\}, x)$ be the constant given by Lemma 4.2. We will prove that each $y \in D_\alpha$ with $|y - x| \leq r/2$ satisfies $|By - Bx| \leq \varepsilon$.

Let $y \in D_\alpha$ with $|y - x| \leq r/2$. From Lemma 7.2 one obtains a sequence $\{y_n\}$, $y_n \in D_n$, $y_n \to y$ as $n \to \infty$ and $\varlimsup_{n\to\infty} \varphi_n(y_n) = \Phi(y)$.

Let $N \geq 1$ such that

$$|x_n - x| < r/4\,,\ \varphi_n(y_n) \leq \alpha + \delta \text{ and } |y_n - y| < r/4 \text{ for each } n \geq N. \tag{7.10}$$

We define a sequence $\widetilde{y}_n$ by

$$\widetilde{y}_n = \begin{cases} x_n & \text{for } n < N \\ y_n & \text{for } n \geq N. \end{cases}$$

One can see that $\widetilde{y}_n \in D_{n,\overline{\gamma}}$ for $n \geq 1$ and, since

$$|x_n - \widetilde{y}_n| = \begin{cases} 0 & \text{for } n < N \\ |x_n - y_n| & \text{for } n \geq N, \end{cases}$$

the inequalities in (7.10) imply that $\sup_{n\geq 1} |x_n - \widetilde{y}_n| \leq r$.

Then, by Lemma 4.2, $\sup_{n\geq 1} |B_n x_n - B_n \widetilde{y}_n| \leq \varepsilon$. Using (C3) one obtains that $|By - Bx| \leq \varepsilon$, which finishes the proof of Lemma 7.4. □

We now employ condition (C4.b) to show that (C2) is verified.

Set $\alpha > 0$, $\delta > 0$ and $x \in D_\alpha$. By Lemma 7.2, there is a sequence $\{x_n\}$, $x_n \in D_n$, $x_n \to x$ as $n \to \infty$ and $\overline{\lim_{n\to\infty}} \varphi_n(x_n) = \Phi(x)$. This implies that

$$\varphi_n(x_n) < \Phi(x) + \delta, \text{ for } n \geq N_{\delta,x} \text{ great enough.}$$

We define the sequence $\{\widetilde{x}_n\}$ by

$$\widetilde{x}_n = \begin{cases} x_n^\beta & \text{for } 0 \leq n \leq N_{\delta,x} - 1 \\ x_n & \text{for } n \geq N_{\delta,x}, \end{cases}$$

where the x_n^β are arbitrary elements in $D_{n,\beta}$ given by (C4.b).

Let $\gamma = \max(\alpha + \delta, \beta)$. Then $\widetilde{x}_n \in D_{n,\gamma}$ for each $n \geq 1$ and $\widetilde{x}_n \to x$ as $n \to \infty$, so condition (C2) is satisfied. We remark that if D is closed then $D_\alpha = D \cap \{x \in X; \Phi(x) \leq \alpha\}$ is also closed. If the closedness of the level sets can be obtained by other methods, then the closedness of D is unnecessary. We also observe that up to now we have used only (C1), (C2)′, (C3) and (C4.b).

Applying Theorems 2.1 and 5.1 we obtain the following generation theorem.

Theorem 7.1. *Let Φ be the functional defined by* (7.2)*. Suppose that conditions* (C1)*,* (C3)*,* (C4)*,* (EC) *and* (S) *are satisfied.*

Then there exists a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D, satisfying

$$S(t)x = T(t)x + \int_0^t T(t-s) BS(s) x ds, \tag{7.11}$$

$$\Phi(S(t)x) \leq e^{at}(\Phi(x) + bt) \text{ for } t \geq 0 \text{ and } x \in D. \tag{7.12}$$

Moreover, if $x \in D$ and $\{x_n\}$ is a $\{\varphi_n\}$-bounded sequence with $x_n \to x$ as $n \to \infty$, then $S_n(t)x_n \to S(t)x$ and the convergence is uniform on bounded subintervals of $[0, \infty)$*.*

Proof. Set $\alpha > 0$ and $\varepsilon > 0$. Let x be an arbitrary element of D_α. By Lemma 7.2, there is a sequence $\{x_n\}$, $x_n \in D_n$, $x_n \to x$ as $n \to \infty$ with $\overline{\lim_{n\to\infty}} \varphi_n(x_n) = \Phi(x)$.

Applying Theorem 4.1 we get

$$\lim_{h\downarrow 0} \left[\sup_{n\geq 1} (1/h) |T_n(h)x_n + hB_n x_n - S_n(h)x_n| \right] = 0, \tag{7.13}$$

and hence we can choose $h \in (0, \varepsilon]$ such that

$$|T_n(h)x_n + hB_n x_n - S_n(h)x_n| < h\varepsilon/3 \text{ for each } n \geq 1 \tag{7.14}$$

and also, by (EC),

$$\sup_{n\geq 1} |S_n(h)x_n| \leq M \text{ for some } M < \infty. \tag{7.15}$$

For this h, we obtain also

(7.16)
$$(1/h)\left|T_n(h)x_n + hB_nx_n - T(h)x - hBx\right| \le (1/h)\left|T(h)x - T_n(h)x_n\right| + \left|B_nx_n - Bx\right| \le \varepsilon/3$$

for $n \ge N_1$. Using the inequality

$$\overline{\lim_{n\to\infty}}\,\varphi_n(S_n(h)x_n) \le e^{ah}(\Phi(x) + bh)$$

and (7.15), one gets from (C4.c) that $\lim_{n\to\infty} d(S_n(h)x_n, D_\delta) = 0$, for each $\delta > e^{ah}(\Phi(x) + bh)$.

Let $\gamma = e^{ah}(\Phi(x) + (b+\varepsilon)h)$. One can find some n and $x_h \in D_\gamma$ such that

$$|S_n(h)x_n - x_h| \le h\varepsilon/3 \text{ for each } k \ge 1. \tag{7.17}$$

This yields

$$\begin{aligned}(1/h)\left|T(h)x + hBx - x_h\right| &\le (1/h)\left|T(h)x + hBx - T_n(h)x_n - hB_nx_n\right| \\ &\quad + (1/h)\left|S_n(h)x_n - x_h\right| + (1/h)\left|T_n(h)x_n + hB_nx_n - S_n(h)x_n\right| \\ &< \varepsilon\end{aligned}$$

by (7.14), (7.16) and (7.17), and so the subtangential condition (II.2) in Theorem 2.1 holds. We now verify the semilinear stability condition.

Let $\alpha > 0$, $\beta > \alpha$ and $x, y \in D_\alpha$. Then, by Lemma 7.2, there are sequences $\{x_n\}$, $\{y_n\}$, $x_n \in D_n$, $y_n \in D_n$ for all n, $x_n \to x$, $y_n \to y$ as $n \to \infty$ and $\overline{\lim_{n\to\infty}}\,\varphi_n(x_n) = \Phi(x)$, $\overline{\lim_{n\to\infty}}\,\varphi_n(y_n) = \Phi(y)$. Thus $x_n \in D_{n,\beta}$ and $y_n \in D_{n,\beta}$ for $n \ge N$ great enough. Then

$$\begin{aligned}&\underline{\lim}_{h\downarrow 0}(1/h)\left[\left|T(h)(x-y) + h(Bx - By)\right| - |x-y|\right] \\ &\qquad = \underline{\lim}_{h\downarrow 0}(1/h)\left[\lim_{n\to\infty}\left(\left|T_n(h)(x_n - y_n) + h(B_nx_n - B_ny_n)\right| - |x_n - y_n|\right)\right]\end{aligned}$$

and from (S) and (7.13) one gets

$$\begin{aligned}\underline{\lim}_{h\downarrow 0}(1/h)\left[\left|T(h)(x-y) + h(Bx - By)\right| - |x-y|\right] &\le \underline{\lim}_{h\downarrow 0}(1/h)\left[e^{\omega_1(\beta,h)h} - 1\right]|x-y| \\ &= \underline{\lim}_{h\downarrow 0}\omega_1(\beta,h)\,|x-y|.\end{aligned}$$

Since $\beta > \alpha$ was arbitrary, the semilinear stability condition is also proved. We also observe that the hypotheses of Theorem 5.1 are verified, which finishes the proof. □

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