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PERMANENCE, PERIODICITY AND STABILITY FOR AN IMPULSIVELY PERTURBED SINGLE SPECIES MODEL

BY

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Abstract. A single species model which is subject to periodic impulsive perturbations is investigated from the viewpoint of finding sufficient conditions for permanence, for the existence of periodic solutions, and for their global asymptotic stability. First, an auxiliary equation, whose solutions are continuous functions but which incorporates the effects of impulsive perturbations, is constructed, the relationship between its solutions and the solutions of the initial system being investigated. The permanence of the system is then established via a comparison argument, while the existence and global asymptotic stability of periodic solutions makes use, in addition to comparison estimations, of Brouwer's fixed point theorem.

Key words: single species model, impulsive perturbations, periodic solutions, Brouwer's fixed point theorem.

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1. Introduction

A rather simple, yet useful, framework to investigate the behaviour of a single species model which is subject to adverse outside influence or of self-inhibition is represented by the following autonomous ordinary differential equation

$$x'(t) = x(t)f(x(t)) - h(x(t)) \tag{1}$$

In the above, $x(t)$ denotes the density of species x at time t, $f = f(x)$ represents its per capita growth rate in the absence of external factors and $h = h(x)$ represents the cumulative effects of the outside influence, often materialized in the form of predation from another species, stocking, harvesting or self-inhibition.

Concrete choices for f and h are numerous and well motivated by the particulars of the species to be considered. Perhaps the most popular choice of f is given by $f(x) = a - bx$, $a, b > 0$, giving rise to a logistic growth model. As far as h is assumed to model the effects of predation (and in another similar circumstances, such as modelling the effects of intraspecies competition), it can also be expressed as $h(x) = x\varphi(x)$, where φ is the per capita death rate of species x due to predation.

A particular case of (1), namely the equation

$$x'(t) = x(t)[a - bx(t)] - \frac{\alpha x^2(t)}{\beta + x^2(t)} \tag{2}$$

has been proposed by Ludwig *et al.* (1978) and Murray (2002) to model the outbreak of a spruce budworm population. In (2), a is the linear birth rate of the budworm and $\frac{a}{b}$ is the carrying capacity of the environment, related to the foliage available on trees. The term $\frac{\alpha x^2(t)}{\beta + x^2(t)}$ represents the decrease in the budworm population due to predation, mainly by birds.

Another particular case of (1), namely the equation

$$x'(t) = x(t)[a - bx(t)] - \frac{cx(t)}{d + x(t)} \tag{3}$$

has been used in Tan *et al.* (2012) to derive their model of a single species dynamics in a periodically varying environment, the coefficients *a*, *b*, *c* and *d* being assumed to be continuous and periodic functions rather than constants.

Most populations experience environmental or biological fluctuations, some with unpredictable, stochastic variation, but some changing in a regular, diurnal, seasonal or annual manner due to climatic factors such as temperature, sunlight or humidity. See Cushing, 1986 for further details. See also Nisbet and Gurney, 1976 for a discussion on the ecological and evolutionary consequences of environmental periodicity, the effects of periodic variation in the values of intrinsic growth rate and of the carrying capacity of a delayed logistic model upon system in a limit cycle or at a stable equilibrium being considered.

Impulsive dynamical systems, characterized by the coexistence of continuous and discrete dynamics, are natural choices for the mathematical modelling of systems involving abrupt changes of state or sudden perturbations from external factors. In recent years, impulsive dynamical systems have found their applications in population dynamics (hunting or harvesting predator-prey models, Zhang *et al.*, 2008), agriculture (integrated pest management, Zhang *et al.*, 2007), medicine (vaccination strategies, Stone *et al.*, 2000, immunotherapy, Bunimovich-Mendrazitsky *et al.*, 2008), communication security (signal encryption, Khadra *et al.*, 2003), mechanics (impact mechanical systems, Galyaev *et al.*, 2006), to mention only a few fields.

Following the above considerations, we shall now discuss the dynamics of the model

$$\begin{aligned} & x'(t) = x(t)[a(t) - b(t)x(t)] - x(t)\varphi(t, x(t)), \qquad t > 0, t \neq \tau_k, k \in \mathbf{N}^* \\ & x(\tau_k +) = (1+\lambda_k)x(\tau_k) \end{aligned} \tag{4}$$

the following assumptions being deemed to hold.

(H1) $0 < \tau_1 < \tau_2 < \ldots$ are the time instances at which the impulsive perturbations occur and $\lim_{k\to\infty} \tau_k = +\infty$.

(H2) $(\lambda_k)_{k\in\mathbf{N}^*}$ is a sequence of real numbers such that $\lambda_k > -1$ for $k \in \mathbf{N}^*$.

(H3) There exist $q \in \mathbf{N}^*$ and $T > 0$ such that $\lambda_{k+q} = \lambda_k$ and $\tau_{k+q} = \tau_k + T$ for all $k \in \mathbf{N}^*$

The functional coefficients $a:[0,\infty) \to \mathbf{R}$ and $b:[0,\infty) \to (0,\infty)$ are continuous and T -periodic, while $\varphi:[0,\infty)\times[0,\infty) \to [0,\infty)$ is continuous, T -periodic with respect to the first variable and has a continuous and bounded partial derivative with respect to the second variable. In this regard, let us denote

$$D = \sup_{t\in[0,T], x\in\mathbf{R}} \left| \frac{\partial\varphi}{\partial x}(t,x) \right| .$$

Assume also that

$$\varphi(t,x) \le \varphi(t,0), \quad \text{for all } t \in [0,T] \text{ and } x \ge 0,$$

that is, the per capita death rate of the species x is higher at low densities.

Impulsively perturbed models which are related to (4) have been investigated in Liu *et al.*, 2010, in which a version of the eq. (2) is analyzed, a, b, α, β being replaced by continuous and periodic functions, the influence of proportional and of constant impulsive perturbations upon the permanence of the system being considered, in Tan *et al.* (2012), which studies the existence and global stability of periodic solutions for a particular case of (4) in which $\varphi(t,x(t)) = \dfrac{c(t)}{d(t)+x(t)}$ and in Liu *et al.* (2010), in which the influence of hereditary effects and of constant impulsive perturbations upon the permanence of a logistic model with delay-dependent predation is investigated. A related impulsively perturbed model with stage structure describing a strategy for controlling the apple snail in paddy fields has been analyzed in Zhang *et al.* (2007). See also Luca (2001, 2008).

2. The Existence of Positive Periodic Solutions

First of all, we shall reduce the initial system (4), whose solutions are subject to impulsive perturbations, to another equation whose solutions are not in themselves subject to impulsive perturbations, but whose coefficients are discontinuous.

Let us consider the reduced equation

$$z'(t) = z(t)[a(t) - b(t)\lambda(t)z(t)] - z(t)\varphi(t,\lambda(t)z(t)).$$

where

$$\lambda(t) = \prod_{0<\tau_k<t} (1+\lambda_k).$$

The following Lemmas, proved in Tan *et al.* (2012), provide the link between the solutions of the initial system (4) and those of the reduced eq. (5). At this point, it should be noted that, unlike those of the initial system (4), the solutions of the reduced eq. (5) are continuous functions, fact which simplifies several quantitative estimations provided that they are stated in terms of solutions of (5) rather than in terms of solutions of (4).

Lemma 1. (Tan *et al.*, 2012). *$\lambda(t)z(t)$ is a solution of the initial system (4) if and only if $z(t)$ is a solution of the reduced eq. (5).*

In this regard, since one needs to find a T-periodic solution for (4), it is natural to apply Brouwer's fixed point theorem to a T-mapping which associates to a given initial data the value of the associated solution of (4) at time T . Since, by Lemma 1, the solutions $x(t)$ of (4) are of type $\lambda(t)z(t)$, where $z(t)$ is a solution of (5), one starts by establishing quantitative properties of the said products $\lambda(t)z(t)$.

Lemma 2. (Tan *et al.*, 2012). *Let* $z(t)$ *be a solution of the reduced eq. (5). Then*

$$z_1(t) = \frac{\lambda(t+T)z(t+T)}{\lambda(t)}, \qquad z_2(t) = \frac{\lambda(t-T)z(t-T)}{\lambda(t)},$$

are also solutions of the reduced eq. (5).

Let us denote

$$E = \lambda(T)\exp(\int_0^T a(t)dt), \quad \Lambda_U = \max_{t\in[0,T]} \lambda(t), \quad \Lambda_L = \min_{t\in[0,T]} \lambda(t).$$

Lemma 3. *If*

$$E > \exp(\int_0^T \varphi(t,0)dt), \tag{6}$$

then there exists $z_1^0 > 0$ *such that*

$$\lambda(T)z_1(T) \geq z_1^0,$$

where z_1 *is the solution of (5) with initial data* $z_1(0) = z_1^0$.

Proof. Let $\varepsilon_1 > 0$ such that

$$\lambda(T)\exp\left(\int_0^T (a(t) - \varphi(t,0))dt - \varepsilon_1\right) > 1.$$

Let us also fix

$$\xi_1 \in \left(0, \frac{\varepsilon_1}{\Lambda_U(\int_0^T b(t)dt + DT)}\right), \qquad z_1^0 \in \left(0, \xi_1 \exp\left(-\max_{t\in[0,T]}\int_0^t a(s)ds\right)\right).$$

Let z_1 be the solution of (5) with initial data $z_1(0) = z_1^0$. Then, since $z_1'(t) \leq a(t)z(t)$, it follows that

$$z_1(t) \leq z_1^0 \exp\left(\int_0^t a(s)ds\right) < \xi_1, \quad \text{for } t \in [0,T].$$

Also,

$$z_1'(t) = z_1(t)[a(t) - b(t)\lambda(t)z_1(t) - \varphi(t, \lambda(t)z_1(t))],$$

which implies that

$$\begin{aligned} z_1(T) &= z_1^0 \exp\left(\int_0^T a(t) - b(t)\lambda(t)z_1(t) - \varphi(t,\lambda(t)z_1(t))dt\right) \\ &= z_1^0 \exp\left(\int_0^T \left(a(t) - \varphi(t,0)\right)dt\right) \exp\left(-\int_0^T b(t)\lambda(t)z_1(t)dt\right) \\ &\cdot \exp\left(-\int_0^T \left(\varphi(t,\lambda(t)z_1(t)) - \varphi(t,0)\right)dt\right). \end{aligned}$$

We note that

$$\int_0^T b(t)\lambda(t)z_1(t)dt \le \Lambda_U \xi_1 \int_0^T b(t)dt$$

$$\left|\int_0^T \left(\varphi(t,\lambda(t)z_1(t)) - \varphi(t,0)\right)dt\right| \le D\Lambda_U \xi_1 T$$

and consequently

$$\int_0^T b(t)\lambda(t)z_1(t)dt + \int_0^T \left(\varphi(t,\lambda(t)z_1(t)) - \varphi(t,0)\right)dt \le \Lambda_U \xi_1 \left(\int_0^T b(t)dt + DT\right) < \varepsilon_1.$$

This implies that

$$z_1(T) \ge z_1^0 \exp\left(\int_0^T \left(a(t) - \varphi(t,0)\right)dt - \varepsilon_1\right)$$

and consequently

$$\lambda(T)z_1(T) \ge \lambda(T)z_1^0 \exp\left(\int_0^T \left(a(t) - \varphi(t,0)\right)dt - \varepsilon_1\right) \ge z_1^0.$$

Lemma 4. *If*

$$E < \exp\left(\frac{\Lambda_L}{\Lambda_U}\frac{b_L}{b_U}\right) \quad \text{and} \quad a(t) \ge \varphi(t,0), \quad \text{for } t \in [0,T], \tag{7}$$

then there exists $z_2^0 > 0$ *such that*

$$\lambda(T)z_2(T) \le z_2^0,$$

where z_2 *is the solution of (5) with initial data* $z_2(0) = z_2^0$.

Proof. Let us fix $\delta_2 \in \left(\frac{\ln E}{T}, \frac{1}{T}\frac{\Lambda_L}{\Lambda_U}\frac{b_L}{b_U}\right)$ and denote $l_2 = \frac{\delta_2}{b_L \Lambda_L}$. Note that, by this choice of δ_2, one has

$$\lambda(T)\exp\left(\int_0^T \left(a(t) - \delta_2\right)dt\right) = E\exp\left(-\delta_2 T\right) < 1.$$

Let us now find z_2^0 such that $\frac{1}{z_2^0} + b_U \Lambda_U T = \frac{1}{l_2}$. Since

$$\begin{aligned} z_2'(t) &= z_2(t)[a(t) - b(t)\lambda(t)z_2(t)] - z_2(t)\varphi(t,\lambda(t)z_2(t)) \\ &\ge z_2(t)[a(t) - \varphi(t,0)] - b(t)\lambda(t)z_2^2(t) \\ &\ge -b(t)\lambda(t)z_2^2(t), \end{aligned}$$

it follows that

$$z_2'(t) \geq -b_U \Lambda_U z_2^2(t), \quad t \in [0,T]$$

which implies that

$$\frac{1}{z_2(t)} \leq \frac{1}{z_2^0} + b_U \Lambda_U T = \frac{1}{l_2}, \quad t \in [0,T],$$

and consequently $z_2(t) \geq l_2$ for all $t \in [0,T]$. It now follows that

$$z_2'(t) \leq z_2(t)[a(t) - b(t)\lambda(t)z_2(t)] \leq z_2(t)[a(t) - b_L \lambda_L z_2(t)]$$
$$\leq z_2(t)[a(t) - \delta_2], \quad t \in [0,T].$$

This implies that

$$z_2(T) \leq z_2^0 \exp\left(\int_0^T \left(a(t) - \delta_2\right) dt\right)$$

and consequently

$$\lambda(T) z_2(T) \leq \lambda(T) z_2^0 \exp\left(\int_0^T \left(a(t) - \delta_2\right) dt\right) \leq z_2^0.$$

We are now ready to establish that conditions (6) and (7) employed in Lemmas 3 and 4 are actually sufficient conditions for the existence of periodic solutions of (4).

Theorem 5. *If conditions (6) and (7) are satisfied, then the system (4) has a positive T-periodic solution.*

Proof. Let z_1^0 and z_2^0 be as specified in Lemmas 3 and 4 and let z_1, z_2 be the solutions of (5) with initial data z_1^0 and z_2^0, respectively. Let also $z_0 \in [z_1^0, z_2^0]$ and let z be the solution of (5) with initial data z_0. Then, by a comparison argument, it follows that

$$z_1(t) \leq z(t) \leq z_2(t), \quad \text{for } t \in [0,T].$$

Using Lemmas 3 and 4, it follows that

$$z_1^0 \leq \lambda(T) z_1(T) \leq \lambda(T) z(T) \leq \lambda(T) z_2(T) \leq z_2^0.$$

Let us define

$$L : [z_1^0, z_2^0] \to [z_1^0, z_2^0], \quad L z_0 = \lambda(T) z(T).$$

By the above argument, L is well-defined, and by the continuous dependence of the solutions of (5) on the initial data, L is continuous. Applying the scalar form of Brouwer's fixed point theorem, it follows that there is

$z_*^0 \in [z_1^0, z_2^0]$ such that $Lz_*^0 = z_*^0$, that is, $\lambda(T)z_*(T) = z_*^0$, where z_* is the solution of (5) with initial data z_*^0.

Up to now, we have employed the various properties of the solutions of the reduced eq. (5), although we actually have to find a periodic solution of the initial system (4). We return to investigating the solutions of the initial system (4) via the linking property indicated in Lemma 1, namely by defining the desired periodic solution of the initial system (4) through the formula

$$x_*(t) = \lambda(t)z_*(t).$$

As previously mentioned, by Lemma 1, x_* is a solution of the initial system (4). It remains to show that x_* is T-periodic.

By Lemma 2, $\widetilde{z}(t) = \dfrac{\lambda(t+T)z_*(t+T)}{\lambda(t)}$ is also a solution of the reduced eq. (5). Let us also note that

$$\widetilde{z}(0) = \frac{\lambda(T)z_*(T)}{\lambda(0)} = z_*^0 = z_*(0),$$

and by the uniqueness theorem, $\widetilde{z}(t) = z_*(t), \quad t \in [0,T]$. It now follows that

$$\begin{aligned} x_*(t+T) &= \lambda(t+T)z_*(t+T) = \lambda(t)\frac{\lambda(t+T)z_*(t+T)}{\lambda(t)} \\ &= \lambda(t)\widetilde{z}(t) = \lambda(t)z_*(t) = x_*(t), \quad t \in [0,T], \end{aligned}$$

that is, x_* is T-periodic. This ends the proof of Theorem 5.

3. The Permanence of the System

In this section, we shall study the permanence of (4). To this purpose, we introduce the following definition.

Definition 1. *The system (4) is said to be permanent (uniformly persistent) if there are* $m, M > 0$ *such that for each solution* $x(t)$ *of (4) with positive initial data* $x(0)$*, one has* $m \le x(t) \le M$*, for enough large* t*.*

From a biological viewpoint, if (4) is permanent, then the species x will, in the long term, neither face extinction nor extreme proliferation, its population size varying between bounds not depending on the initial conditions. Further information relating to the mathematical theory of persistence can be found in the comprehensive monograph of Smith and Thieme (2011).

We shall now establish the permanence of (4). In this regard, the following Lemma, proved in Liu *et al.* (2009), establishes the existence and global asymptotic stability of the periodic solution for the logistic equation with T -periodic coefficients.

Lemma 6. (Liu *et al.*, 2009). *Consider the system*

$$\begin{aligned} y'(t) &= y(t)[a(t)-b(t)y(t)], \qquad t>0, t\neq \tau_k, k\in \mathbf{N}^* \\ y(\tau_k+) &= (1+\lambda_k)y(\tau_k) \end{aligned} \tag{8}$$

where the functional coefficients a *and* b *are continuous* T *-periodic functions,* $T>0$ *, and* $b(t)>0$ *for* $t\geq 0$ *. Assume also that* $(\tau_k)_{k\in\mathbf{N}^*}$ *is a sequence of strictly positive numbers and* $(\lambda_k)_{k\in\mathbf{N}^*}$ *is a sequence of real numbers such that* $\lambda_k>-1$ *for* $k\in\mathbf{N}^*$ *for which there exist* $q\in\mathbf{N}^*$ *and* $T>0$ *such that* $\lambda_{k+q}=\lambda_k$ *and* $\tau_{k+q}=\tau_k+T$ *for all* $k\in\mathbf{N}^*$ *. If*

$$\prod_{0<\tau_k<T}(1+\lambda_k)\exp\left(\int_0^T a(t)dt\right)>1\,.$$

then the system (8) has a unique positive T *-periodic solution which is globally asymptotically stable.*

We are now ready to state and prove a sufficient condition for the permanence of (4).

Theorem 7. The system (4) is permanent provided that condition (6) holds.

Proof. Let x be a solution of (4) which starts with strictly positive initial data $x(0)$. It is easy to see that $x(t)>0$ for all $t\geq 0$. We consider the following comparison systems

$$\begin{aligned} u_1'(t) &= u_1(t)[a(t)-b(t)u_1(t)]-u_1(t)\varphi(t,0), \quad t\neq\tau_k, k\in\mathbf{N}^* \\ u_1(\tau_k+) &= (1+\lambda_k)u_1(\tau_k) \end{aligned} \tag{9}$$

and

$$\begin{aligned} u_2'(t) &= u_1(t)[a(t)-b(t)u_2(t)], \quad t\neq\tau_k, k\in\mathbf{N}^* \\ u_2(\tau_k+) &= (1+\lambda_k)u_2(\tau_k) \end{aligned} \tag{10}$$

By Lemma 6, systems (9) and (10) have positive and globally asymptotically stable T -periodic solutions u_{1*} and u_{2*}, respectively. Let now $\varepsilon<\min_{t\in[0,T]}u_{1*}(t)$ and let $\widetilde{u}_1$, $\widetilde{u}_2$ be initial solutions of (9) and (10), respectively, with initial data $x(0)$. By a comparison argument, one obtains using also the global asymptotic stability of u_{1*} and u_{2*} that there is t_1 large enough such that, for all $t\geq t_1$,

$$u_{1*}(t)-\varepsilon<\widetilde{u}_1(t)\le x(t)\le \widetilde{u}_2(t)\le u_{2*}(t)+\varepsilon,$$

and consequently, after denoting

$$m=\min_{t\in[0,T]} u_{1*}(t)-\varepsilon,\ M=\min_{t\in[0,T]} u_{2*}(t)+\varepsilon,$$

it follows that

$$m\le x(t)\le M,\quad \text{for } t\ge t_1,$$

which ends the proof of the permanence result.

Note that the permanence condition (6) can be expressed in the form

$$\ln\lambda(T)+\int_0^T \big(a(t)-\varphi(t,0)\big)dt>0 \tag{11}$$

Let us briefly discuss the biological meaning of (11). Suppose that the species x approaches extinction (that is, $x(t)$ approaches 0). Then, over a period, $\int_0^T a(t)dt$ approximates the total per capita growth of the species x, while $\int_0^T \varphi(t,0)dt$ approximates the total per capita population loss of this species and $\ln\lambda(T)$ is a correction term which accounts for the total effects of the impulsive controls. Consequently, the permanence condition (11) represents the fact that total normalized growth of species x over a period exceeds the total normalized loss of this species in the same amount of time, the species x being then able to avoid extinction.

4. The Global Stability of the Positive Periodic Solutions

Having established sufficient conditions for the existence of the positive periodic solutions, we are now ready to discuss its global asymptotic stability.

Theorem 8. *Apart from the existence conditions (6) and (7), assume that there exists* $\mu>0$ *such that*

$$b(t)+\frac{\partial\varphi}{\partial x}(t,x)\ge\mu\quad \text{for all } t\in[0,T] \text{ and } x\ge m, \tag{12}$$

where m *is a lower permanence constant for (4). Then (4) has a positive* T*-periodic solution which is globally asymptotically stable.*

Proof. Let x_* be the positive T-periodic solution of (4), which exists by virtue of Theorem 5 and let x be any other positive solution of (4). By Theorem 7, there are $m,M>0$ and $t_1>0$ such that

$$m\le x(t)\le M,\quad m\le x_*(t)\le M,\quad \text{for } t\ge t_1, \tag{13}$$

Let us consider the functional $V:[0,\infty)\to[0,\infty)$ defined by $V(t)=\left|\ln x(t)-\ln x_*(t)\right|$. Since, for $k\in\mathbf{N}^*$,

$$V(\tau_k+) = \left|\ln\frac{x(\tau_k+)}{x_*(\tau_k+)}\right| = \left|\ln\frac{(1+\lambda_k)x(\tau_k+)}{(1+\lambda_k)x_*(\tau_k+)}\right| = \left|\ln\frac{x(\tau_k)}{x_*(\tau_k)}\right| = V(\tau_k),$$

it follows that V is continuous. One then has, for all $t \geq t_1$, $t \neq \tau_k$, that

$$D^+V(t) = \operatorname{sgn}(\ln x(t) - \ln x_*(t))\left(\frac{x'(t)}{x(t)} - \frac{x'_*(t)}{x_*(t)}\right),$$

$$= \operatorname{sgn}(x(t) - x_*(t))[-b(t)(x(t) - x_*(t)) + \varphi(t, x_*(t)) - \varphi(t, x(t))]$$

$$= -|x(t) - x_*(t)|\left(b(t) + \frac{\partial\varphi}{\partial x}(t, \xi_t)\right),$$

for some ξ_t between $x(t)$ and $x_*(t)$, by the Mean Value Theorem.

By (12), there is $\mu > 0$ such that, for all $t \geq t_1$, $t \neq \tau_k$,

$$D^+V(t) \leq -\mu|x(t) - x_*(t)|.$$

By the Mean Value Theorem and (13), it follows that, for all $t \geq t_1$, $t \neq \tau_k$,

$$\frac{1}{M}|x(t) - x_*(t)| \leq |\ln x(t) - \ln x_*(t)| \leq \frac{1}{m}|x(t) - x_*(t)|, \tag{14}$$

which implies that, for all $t \geq t_1$, $t \neq \tau_k$,

$$D^+V(t) \leq -\mu m V(t). \tag{15}$$

Fix now $k \in \mathbf{N}^*$ such that $\tau_k > t_1$. By (15), it follows that

$$V(t) \leq V(t_k)\exp(-\mu m(t - \tau_k)), \quad \text{for } t \geq \tau_k,$$

which implies that $\lim_{t\to\infty} V(t) = 0$. Using (14), this yields that

$$\lim_{t\to\infty}|x(t) - x_*(t)| = 0,$$

which ends the proof of Theorem 8.

Note that the existence theorem, Theorem 2.1, the permanence result, Lemma 3.2 and the global stability result, Theorem 3.1 of Tan *et al.*, 2012 can be obtained from our Theorems 5, 7 and 8, respectively by particularizing $\varphi(t, x(t)) = \dfrac{c(t)}{d(t) + x(t)}$. Our results can also be applied for other specific forms forms of φ such as $\varphi(t, x(t)) = \dfrac{c(t)}{1 + d(t)x(t) + \omega(t)x^2(t)}$, introduced by Andrews, 1968 to describe inhibition phenomena which occur in the decomposition of wastewater. Further possible developments include incorporating the effects of the delay which is necessary to reach maturity into the logistic growth term which appears in (4).

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PERMANENȚĂ, PERIODICITATE ŞI STABILITATE PENTRU UN MODEL PRIVIND DINAMICA UNEI SINGURE SPECII SUPUSE LA PERTURBAȚII DE TIP IMPULSIV

(Rezumat)

Un model care descrie dinamica unei specii supuse la perturbații de tip impulsiv este investigat cu scopul de a determina condiții suficiente pentru permanență, pentru existența soluțiilor periodice şi pentru stabilitatea globală a acestora. Mai întâi, este introdusă o ecuație auxiliară, ale cărei soluții sunt funcții continue, dar care încorporează efectele perturbațiilor impulsive, fiind investigată relația dintre soluțiile acesteia şi soluțiile ecuației inițiale. Permanența sistemului este stabilită ulterior cu ajutorul unui argument de comparație, în timp ce existența şi stabilitatea globală a soluțiilor periodice utilizează, pe lângă estimări de comparație, şi teorema de punct fix a lui Brouwer.