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FINDING THE TRACES OF A GIVEN PLANE: ANALYTICALLY AND THROUGH GRAPHICAL CONSTRUCTIONS

BY

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Abstract. This paper determines explicitely the traces of a plane [P] given by three of its points, using two different approaches: by means of analytic geometry and descriptive geometry methods, respectively.

Key words: traces, planes, analytic geometry, descriptive geometry.

1. Introduction

The problem which is of concern in this paper is stated as follows:

Find the traces of the plane [P] determined by the point A(70, 50, 40) and the straight line(BC), where B(100, 20, 50) and C(60, 80, 10).

In what follows, this problem will be solved first by using analytic geometry and then by methods of descriptive geometry, a comparison between these approaches being drawn as a conclusion.

2. Solving Methods

2.1. *Analytical Geometry Method*

Having in view that a line is completely determined by two of its points, we solve the general problem of determining the traces of a plane given by three points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$. Then, we particularize the results for the concrete problem stated above.

We start by determining the trace (P_h) of the plane [P] on the plane (xOy). First, let us denote:

(1) $$\Delta_x = \begin{vmatrix} y_A & z_A & 1 \\ y_B & z_B & 1 \\ y_C & z_C & 1 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} z_A & x_A & 1 \\ z_B & x_B & 1 \\ z_C & x_C & 1 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix},$$

(2) $$\Delta = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}.$$

Since the equation of [P] is:

(3) $$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

and all points in the plane (xOy) have the z-coordinate equal to 0, one obtains by expanding the determinant along the first row that:

(4) $$(P_h): x\begin{vmatrix} y_A & z_A & 1 \\ y_B & z_B & 1 \\ y_C & z_C & 1 \end{vmatrix} - y\begin{vmatrix} x_A & z_A & 1 \\ x_B & z_B & 1 \\ x_C & z_C & 1 \end{vmatrix} - \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} = 0; \quad z = 0.$$

that is,

(5) $$(P_h): x\begin{vmatrix} y_A & z_A & 1 \\ y_B & z_B & 1 \\ y_C & z_C & 1 \end{vmatrix} + y\begin{vmatrix} z_A & x_A & 1 \\ z_B & x_B & 1 \\ z_C & x_C & 1 \end{vmatrix} = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}; \quad z = 0.$$

Consequently, the trace (P_h) of [P] on the plane (xOy) is characterized by:

(6) $$(P_h): x\Delta_x + y\Delta_y = \Delta; \quad z = 0.$$

Through the circular permutation $x \to y \to z \to x \to y$, one obtains that the trace (P_l) of [P] on the plane (yOz) is given by:

$$(P_l): y\Delta_y + z\Delta_z = \Delta; \quad x = 0 \tag{7}$$

and the trace (P_v) of [P] on the plane (xOz) is given by:

$$(P_v): z\Delta_z + x\Delta_x = \Delta; \quad y = 0. \tag{8}$$

Note that the traces (P_h), (P_l) and (P_v) can also be given in the following parametric forms:

$$(P_h): x = \frac{1}{2}\frac{\Delta}{\Delta_x} - \frac{\Delta}{\Delta_x}t; \quad y = \frac{1}{2}\frac{\Delta}{\Delta_y} + \frac{\Delta}{\Delta_y}t; \quad z = 0, \quad t \in \mathbf{R}, \tag{9}$$

$$(P_v): y = \frac{1}{2}\frac{\Delta}{\Delta_y} - \frac{\Delta}{\Delta_y}t; \quad z = \frac{1}{2}\frac{\Delta}{\Delta_z} + \frac{\Delta}{\Delta_z}t; \quad x = 0, \quad t \in \mathbf{R}, \tag{10}$$

$$(P_l): z = \frac{1}{2}\frac{\Delta}{\Delta_z} - \frac{\Delta}{\Delta_z}t; \quad x = \frac{1}{2}\frac{\Delta}{\Delta_x} + \frac{\Delta}{\Delta_x}t; \quad y = 0, \quad t \in \mathbf{R}\ . \tag{11}$$

Then, we determine the intersections between [P] and the coordinate axes. Since the intersection between [P] and (Ox) is actually the intersection between its trace (P_h) and (Ox) and this point has the y-coordinate equal to *0*, one obtains from Eq (6) that the intersection between [P] and (Ox) is the point P_x given by:

$$P_x = P_x\left(\frac{\Delta}{\Delta_x}, 0, 0\right). \tag{12}$$

Through circular permutations, one obtains that the intersection between [P] and (Oy) is the point P_y given by:

$$P_y = P_y\left(0, \frac{\Delta}{\Delta_y}, 0\right) \tag{13}$$

and the intersection between [P] and (Oz) is the point P_z given by:

(14) $$P_z = P_z\left(0, 0, \frac{\Delta}{\Delta_z}\right).$$

We are now ready to solve the problem stated in the Introduction.

S o l u t i o n

One sees that:

(15) $$\Delta = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} = \begin{vmatrix} 70 & 50 & 40 \\ 100 & 20 & 50 \\ 60 & 80 & 10 \end{vmatrix} = 106000,$$

(16) $$\Delta_z = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = \begin{vmatrix} 70 & 50 & 1 \\ 100 & 20 & 1 \\ 60 & 80 & 1 \end{vmatrix} = 600,$$

(17) $$\Delta_x = \begin{vmatrix} y_A & z_A & 1 \\ y_B & z_B & 1 \\ y_C & z_C & 1 \end{vmatrix} = \begin{vmatrix} 50 & 40 & 1 \\ 20 & 50 & 1 \\ 80 & 10 & 1 \end{vmatrix} = 600,$$

(18) $$\Delta_y = \begin{vmatrix} z_A & x_A & 1 \\ z_B & x_B & 1 \\ z_C & x_C & 1 \end{vmatrix} = \begin{vmatrix} 40 & 70 & 1 \\ 50 & 100 & 1 \\ 10 & 60 & 1 \end{vmatrix} = 800.$$

From Eqs (12)-(14), it follows that the intersections of [P] with the coordinate axes are: $P_x\left(\frac{530}{3}, 0, 0\right)$, $P_y\left(0, \frac{265}{2}, 0\right)$ and $P_z\left(0, 0, \frac{530}{3}\right)$. From Eqs (5)-(7), it also follows that the trace (P_h) is $3x + 4y = 530; z = 0$, the trace (P_v) is $3x + 3z = 530; y = 0$ and the trace (P_l) is $4y + 3z = 530; x = 0$. Alternatively, form Eqs (9)-11, the parametric equations of the traces are:

(18) $$(P_h): x = \frac{530}{6}(1 - 2t); \quad y = \frac{265}{4}(1 + 2t); \quad z = 0, \quad t \in \mathbf{R},$$

(19) $$(P_v): y = \frac{265}{4}(1-2t); \quad z = \frac{530}{6}(1+2t); \quad x = 0, \quad t \in \mathbf{R},$$

(20) $$(P_l): z = \frac{530}{6}(1-2t); \quad x = \frac{530}{6}(1+2t); \quad y = 0, \quad t \in \mathbf{R}.$$

2.2. *Descriptive Geometry Method*

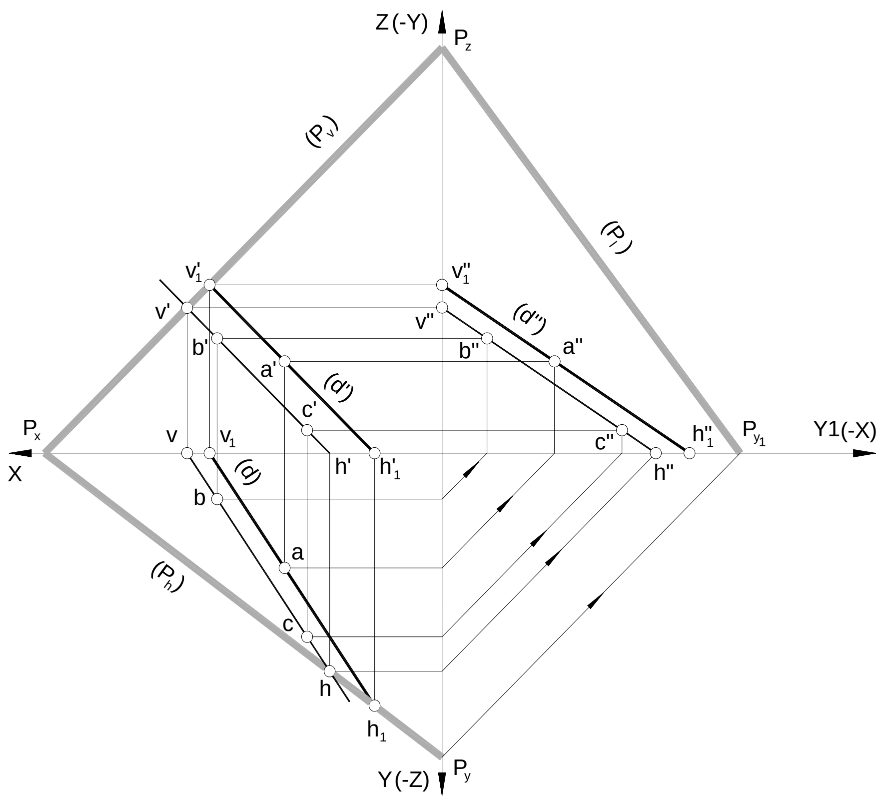

Fig. 1 – Traces of [P] plane

We determine the traces h and v' of the straight line (BC) and then we construct through the point A a line *(D_1)* which is parallel to the given one (Fig. 1). To determine the traces of the plane, we need only one of the traces of the line *(D_1)* (for example, the horizontal trace h_1). First of all, we represent the horizontal trace (P_h) of [P] and then the vertical one, (P_v). We note that $P_x=P_z=176.(6)$ and $P_y=132.50$, which are the values obtained by using the analytical method.

3. Conclusions

As we can see, Descriptive Geometry offers a quick and elegant solution to the problem, while the analytical approach, although more computational, offers general formulas which can be applied to a large class of related problems.

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R E F E R E N C E S

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DETERMINAREA URMELOR UNUI PLAN FOLOSIND METODELE GEOMETRIEI ANALITICE ŞI ALE GEOMETRIEI DESCRIPTIVE

(Rezumat)

Lucrarea îşi propune să prezinte modalităţi diferite de determinare a urmelor unui plan dat prin trei puncte necoliniare, abordând metodele geometriei analitice şi aplicând principiile geometriei descriptive. Deşi ambele metode ajung la acelaşi rezultat privind determinarea urmelor planului [P], geometria descriptivă este mai rapidă şi mai intuitivă decât geometria analitică, care necesită calcule laborioase, dar are o aplicabilitate extinsă. Sperăm că demersul nostru va fi util, interesant şi edificator pentru studenţii facultăţilor tehnice, dar şi celor interesaţi de cele două discipline de studiu.