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The stochastic evolution of a rumor spreading model with two distinct spread inhibiting and attitude adjusting mechanisms in a homogeneous social network

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A B S T R A C T

In this paper, we propose and analyze from a stability viewpoint a deterministic, ODE-based class of rumor spreading models with two distinct inhibiting and adjusting mechanisms, together with its corresponding stochastic counterpart. For the deterministic model, a threshold parameter R_0 defined *ad hoc*, called the basic influence number, is used to ascertain whether the rumors are prevailing or not. If $R_0 < 1$, the rumor-free equilibrium is found to be locally asymptotically stable, while if $R_0 > 1$ it is shown that there is at least one additional rumor-prevailing equilibrium, which is necessarily locally asymptotically stable. For the stochastic model, we first show that there exists a unique global solution. Subsequently, we investigate the asymptotic behavior of the stochastic system around the equilibria of the deterministic system by constructing suitable Lyapunov functionals. Furthermore, numerical simulations are given to illustrate, support and enhance our theoretical analysis.

1. Introduction

Rumors, the oldest form of mass media, are defined in [1] as being improvised news resulting from a process of collective discussion. Typically, rumors spread at first between close friends or by means of other similar informal encounters, before making their way, often reshaped or distorted, into public discourse.

Certain rumors spread faster since they are targeting deep insecurities, making people uncertain of outcomes, afraid of consequences and distrustful of real, truthful information. For instance, there are rumors in circulation which allege that high-speed 5G cellular technology is to blame for the rapid spread of COVID-19, that the spread of the Zika virus is caused by genetically modified mosquitoes and that microcephaly is caused by vaccines, which, via ever-shifting alleged mechanisms, also cause autism. Other rumors, especially popular in West Africa, allege that Ebola can be contracted from a motorcycle helmet. The "Obama is a Muslim" rumor, built upon questions caused in part by his middle name, was a recurrent theme of the 2008 US presidential elections and reverberated for a long time, as a 2014 poll showed that 54% of the (Republican) subjects believed that "deep down" Obama was a Muslim.

To understand the impact of rumor spreading, it is useful to investigate the dynamical behavior of appropriate mathematical models which are often not unlike disease propagation models. The research on rumor spreading models

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started in the 1960s. In their seminal papers [2,3], Daley and Kendall compared the difference between the basic tenets of a model of virus transmission and those of a model of rumor spreading, proposing their classical compartmental model of rumor spreading. This model keeps track of three compartments: ignorants, spreaders and stiflers and assumes that an ignorant should absolutely become a spreader upon hearing a rumor, which is not always factual.

Since then, the DK model has been actively modified and expanded, leading to a more accurate account of rumor spreading. Maki and Thompson [4] developed a model which allowed for a more nuanced interaction between ignorants, spreaders and stiflers. Zanette [5,6] proposed a rumor propagation model on a small-world network and performed simulations for the MT model. In 2004, Moreno et al. [7] studied the stochastic properties of the MT model on scale-free networks by means of Monte Carlo simulations. In 2008, Kawachi [8] established global behavior patterns for rumor propagation models that also extended the classical DK model. In 2014, Afassinou [9] considered the influence of education rate of the population, which led to the proposal of a susceptible, educated, infected, and recovered (*SEIR*) rumor spread model.

Recently, other mechanisms aiming at a fine-grained description of rumor spreading, such as forgetting mechanisms (Zhao et al. [10]), incubation mechanisms (Al-Tuwairqi et al. [11]), hesitation mechanisms (Xia et al. [12]), punishment and control by the government (Li and Ma [13], Zhao and Wang [14]), memory effects (Zhang and Xu [15]), different probabilities for spreaders to become stiflers (Zhao et al. [16]), have attracted the attention of various authors. In addition, a rumor spreading model accounting for two distinct rumors spreading simultaneously in a target population was proposed and investigated by Wang et al. [17]. In 2017, Zhu and Wang [18] used a modified susceptible–infected–removed (*SIR*) model to characterize the dynamics of rumor diffusion both on homogeneous networks and heterogeneous networks, taking also variation of connectivity into consideration. In 2018, Hu et al. [19] established a Susceptible–Hesitating–Affected–Resistant (*SHAR*) model incorporating different attitudes towards rumor spreading. Zhu and Wang [20] investigated a rumor diffusion model accounting for the uncertainty of human behavior, in a spatio-temporal diffusion framework. Zan [21] studied two types of double-rumors spreading models: the *DSIR* ("double" *SIR*) model and the *C-DSIR* (comprehensive *DSIR*) model. Recently, Horst [22] pointed out that rumor spreading agents change their activities at random points in time, at a rate that depends on the current state of a designated neighborhood and on the average choice throughout the entire population, making a case for the use of stochastic models to describe rumor spreading.

Rumors can destroy confidence, affect self-esteem and even ruin reputations. They may also lead to anxiety, depression, suicidal thoughts and to a host of other issues. Rumors can alienate friends and lead to ostracization and other forms of relational aggression. On a societal level, the rise and ubiquity of social media led to an increase in the exposure of users to unsubstantiated rumors, fake news, defined as fabricated information that mimics news media content in form but not in organizational process or intent (Lazer et al. [23]), extremely biased news, conspiracy theories and to other forms of misinformation, which became more and more pervasive. In online social media, social-cognitive biases such as selective exposure and confirmation bias helped the emergence of echo chamber-like communities consisting of opinionated, like-minded (or perhaps hive-minded) users sharing similar beliefs about polarizing topics and avoiding communication with those who have opposite views. These communities are prone to spreading information aligning with these beliefs in sharing cascades, without a substantive veracity check (Wang et al. [24], Bovet and Makse [25], Liu et al. [26]). By analyzing a dataset of tweets collected during the 5 months preceding the US 2016 presidential elections, Bovet and Makse [25] found that 25% of these tweets spread either fake or extremely biased news. Also, it has been determined in Del Vicario et al. [27] that, for two distinct types of online communities, homogeneity and polarization are the main determinants for predicting the sizes of sharing cascades.

One of important responsibilities of any governmental body is then to put in place effective measures to control the spread of rumors, thereby containing and minimizing any harm they may cause. However, inhibiting and clarifying all sorts of ubiquitous rumors on social networks is often a long-term process, requiring a stable, sustained budget input. In our model of rumor spreading, we attempt to introduce two spread inhibiting and attitude adjusting mechanisms, which depend only upon governmental input, acting towards reducing rumor spreading from spreaders to newcomers and changing the attitude of spreaders, respectively. We then discuss the effects of the qualitative properties of the inhibiting and adjusting functions, respectively, upon rumor spreading, with a view towards contributing to a better understanding of the mechanisms of rumor spreading and providing the governmental bodies and the public with significant insights for rumor control.

This paper is organized as follows. In Section 2, we introduce a deterministic model for rumor spreading which accounts for two inhibiting and adjusting mechanisms. In Section 3, the stability properties of the deterministic model are characterized in terms of a basic influence number, viewed as a threshold parameter. In Section 4, we propose the corresponding stochastic model and discuss its asymptotic behavior around the equilibria of the deterministic system. In Section 5, we illustrate, support and enhance our results via numerical simulations. Several concluding remarks are given in Section 6.

2. The deterministic model

It is well known that an undirected graph $G = (V, E)$, where V represents a set of nodes and E represents a set of sides, can be obtained from a social network consisting of N individuals if the individuals and the contacts between them are considered as nodes and sides, respectively. The size of the total population which is subject to mechanisms for rumor

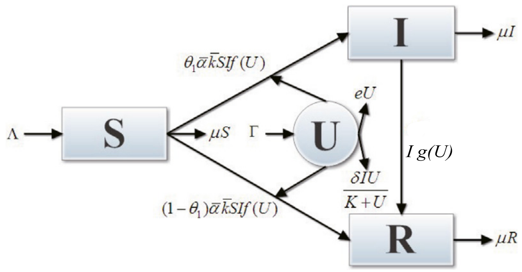

Fig. 1. Structure of the model with inhibiting mechanisms for the rumor spreading process.

control is understood to be variable and is denoted at time t by $N(t)$. This population is subdivided into three categories: newcomer (S), spreader (I) and stifler (R). Upon hearing a rumor, the newcomer, who has no previous information about that rumor, may subsequently display two different attitudes. Specifically, certain individuals will choose not to spread the rumor or not to believe it (stiflers), while other individuals will actively spread it (spreaders). After a newcomer hears the rumor through the contact with a spreader at a rate $\bar{\alpha}$, it then has two possible choices: to become a spreader with probability θ_1, or to become a stifler with probability $1-\theta_1$. Once a rumor spreads in a social network, the government then has the authority to use inhibiting and adjusting mechanisms, whose strength and resources are quantified through the use of the inhibitor variable U, to contain the various damages caused by the rumors.

The movement of individuals from one class to another, given in Fig. 1, is supposed to be unidirectional, which means the flowchart is irreversible. We assume that the flow into the newcomer class is constant and denoted by Λ and denote the constant leaving rate of each non-inhibitor compartment by μ. The constant Γ represents the allotted budgeting rate by the government for adjusting and inhibiting mechanisms and e is the decay rate of those mechanisms. At any time, certain spreaders adjust their attitude towards spreading rumors as a result of the constant governmental work to improve the mechanisms for clarifying and inhibiting rumors, thereby becoming stiflers at a rate $g(U)$.

For the sake of simplicity, we consider only rumors spreading through human contacts, rather than rumors spreading through the media, which motivate our use of an augmented *SIR*-like model. We now introduce our mathematical model, which involves different attitudes towards rumors and two distinct mechanisms for rumor control, in the following form

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \bar{\alpha}\bar{k}SIf(U) - \mu S,\\
\frac{dI}{dt} &= \theta_1\bar{\alpha}\bar{k}SIf(U) - g(U)I - \mu I,\\
\frac{dU}{dt} &= \Gamma - eU - \frac{\delta IU}{K+U},\\
\frac{dR}{dt} &= (1-\theta_1)\bar{\alpha}\bar{k}SIf(U) + g(U)I - \mu R.
\end{aligned} \tag{1}$$

Here, the functions f and g stand for the effects of rumor control mechanisms upon attenuating rumor spreading to newcomers and changing the attitudes of spreaders, respectively. Also, $\bar{k}(\geq 2)$ denotes the average degree of the network. The constants δ and K quantify the usage of the inhibitor, δ being the maximal uptake rate of I and K being a half-saturation parameter. The resulting uptake rate of the inhibitor $\frac{\delta U}{K+U}$ is increasing (the more the inhibitor is available, the higher the uptake rate is) and saturates for large U. Assume that the C^1 function $f : [0,\infty) \to \mathbf{R}$ satisfies

(f.i) $f(U) \geq 0, f(0) = 1$;

(f.ii) f is non-increasing on $[0,\infty)$.

Assumption **(f.i)** represents the fact that the interaction between newcomers and spreaders leads to a decrease in the number of newcomers, some of them becoming either spreaders of stiflers, and that the function f is normalized ($f(0) = 1$ means that the mechanisms for the inhibition of rumor spreading are not active), the strength of the interaction between newcomers and spreaders being also quantified through the use of the parameter $\bar{\alpha}$. In fact, $f(0) = 1$ is a modeling assumption (a description of the control mechanism), which is not actually needed for any of our proofs. Assumption **(f.ii)** describes that the attempts to control rumor spreading do not backfire (more of the inhibitor U leads to less rumor spreading). Also, the C^1 function $g : [0,\infty) \to \mathbf{R}$ satisfies

(g.i) $g(U) \geq 0$;

(g.ii) g is non-decreasing on $[0,\infty)$;

(g.iii) $\overline{\lim}_{U\to\infty} g(U) \le \bar{g}$.

Assumption **(g.i)** represent the fact that the use of rumor control mechanisms does indeed change the attitude of spreaders, while assumption **(g.ii)** signifies that the attempts to adjust attitudes do not backfire either (more of the inhibitor U leads to more spreaders changing their attitudes). Assumption **(g.iii)** describes the fact that the attitude adjustment mechanisms eventually saturate. Let us also assume that

(fg.i) $fg + \mu f$ is non-decreasing on $[0, \infty)$.

While assumptions **(f.i-ii)** and **(g.i-iii)** stem from social considerations, the motivation for assumption **(fg.i)** is purely mathematical, as it is employed only to ensure the local stability of the rumor-prevailing equilibrium. If $f \equiv 1$, then **(fg.i)** is trivially satisfied, since it reduces just to g being non-decreasing on $[0, \infty)$ (that is, to **(g.ii)**).

Remark 2.1. An example of a function f modeling the inhibiting mechanisms, which satisfies assumptions **(f.i)** and **(f.ii)** is $f(U) = \frac{A+Be^{-U}}{A+B}$, $A \ge 0, B > 0$, while an example of a function g modeling the adjusting mechanisms which satisfy assumptions **(g.i-iii)** is $g(U) = \frac{AU}{B+U}$, $A, B > 0$. However, there are many other possible examples, as the assumptions have been kept to a minimum. Further efforts should go into determining more specific forms or sets of assumptions for each function, which will allow us to further refine and analyze our stability results.

Remark 2.2. We view U as a variable "proxying" (sort of) the rumor control. That is, U can model the resources of an organization mandated by the government to deal with rumor control (but we do not want to limit ourselves just to this interpretation). There is a (constant) growth of resources due to government budgeting, an uptake of resource in which each subject (spreader) is allotted (in average) an amount of resources which increases as more resources become available and a decay of resource (interpretable, for instance, as administrative costs).

3. A qualitative analysis of the deterministic model

3.1. The positive invariance

Having in view the social significance of the variables, we are interested only in solutions that are nonnegative and bounded. It can be easily proved that the solutions of the system (1) which start with nonnegative initial data, that is, with

$$S(0) \ge 0,\ I(0) \ge 0,\ U(0) \ge 0,\ R(0) \ge 0$$

stay nonnegative for all $t \ge 0$. From (1), we see that

$$\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} = \Lambda - \mu N(t),$$

which implies that

$$N(t) = \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t} + \frac{\Lambda}{\mu}$$

for all $t \ge 0$, which ensures the boundedness of S, I and R. Also,

$$\left.\frac{dU(t)}{dt}\right|_{U(t)=\frac{\Gamma}{e}} \le 0.$$

Hence, U is bounded as well and a positively invariant set of (1) is

$$\Omega = \left\{(S,\ I,\ U,\ R) \in \mathbf{R}^4 : 0 \le S + I + R \le \frac{\Lambda}{\mu} \text{ and } 0 \le U \le \frac{\Gamma}{e}\right\}.$$

3.2. The basic influence number

It is easy to see that the system (1) has a rumor-free equilibrium E_0, given by

$$E_0 = \left(\frac{\Lambda}{\mu},\ 0,\ \frac{\Gamma}{e},\ 0\right).$$

To discuss the dynamics of the model via an approach which has been proven highly successful in Mathematical Epidemiology and to rearrange the system (1) to fit the framework of the next generation method laid out in [28], let us first notice that I and R are "infected-like" compartments, while S and U are "noninfected-like" compartments.

Let us denote $X = (I,\ R)^T$, $Y = (S,\ U)^T$ and $\alpha = \bar{\alpha}\bar{k}$. Then the system (1) can be rearranged as

$$\frac{dX}{dt} = \mathrm{F}(X, Y) - \mathrm{V}(X, Y),$$
$$\frac{dY}{dt} = h(X, Y)$$

in which

$$\mathrm{F}(X, Y) = \begin{pmatrix} \theta_1 \alpha S I f(U) \\ 0 \end{pmatrix}, \quad \mathrm{V}(X, Y) = \begin{pmatrix} (g(U) + \mu)I \\ -(1-\theta_1)\alpha S I f(U) - g(U)I + \mu R \end{pmatrix}.$$

We thereby obtain that

$$F = D\mathrm{F}|_{E_0} = \begin{pmatrix} \theta_1 \alpha \frac{\Lambda}{\mu} f(\frac{\Gamma}{e}) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$V = D\mathrm{V}|_{E_0} = \begin{pmatrix} g(\frac{\Gamma}{e}) + \mu & 0 \\ -(1-\theta_1)\alpha \frac{\Lambda}{\mu} f(\frac{\Gamma}{e}) - g(\frac{\Gamma}{e}) & \mu \end{pmatrix}.$$

Hence, a threshold parameter for the stability of the system (1), called *ad hoc* the basic influence number of the system (1), is the spectral radius of matrix FV^{-1}, given by

$$\mathrm{R}_0 = \frac{\theta_1 \alpha \Lambda f(\frac{\Gamma}{e})}{\mu(g(\frac{\Gamma}{e}) + \mu)}. \tag{2}$$

Remark 3.1. Note that, in the above fraction, the numerator is an non-increasing function of Γ due to **(f.ii)**, while the denominator is an increasing function of Γ, due to **(g.ii)**. As a result, R_0 is a non-increasing function of Γ, which leads to the conclusion that the higher the allotted budgeting rate for the rumor control mechanisms Γ is, the lower the average influence of a spreader in a totally susceptible population R_0 becomes.

3.3. The existence of a positive equilibrium

To investigate the possible existence of a positive (or rumor-prevailing) equilibrium E^*,

$$E^* = (S^*,\ I^*,\ U^*,\ R^*),$$

let us first observe that the equilibrium relations take the form

$$\begin{aligned} \Lambda - \mu S^* - \alpha S^* I^* f(U^*) &= 0, \\ \theta_1 \alpha S^* I^* f(U^*) - g(U^*)I^* - \mu I^* &= 0, \\ \Gamma - eU^* - \frac{\delta I^* U^*}{K + U^*} &= 0, \\ (1-\theta_1)\alpha S^* I^* f(U^*) + g(U^*)I^* - \mu R^* &= 0. \end{aligned} \tag{3}$$

By straightforward algebraic manipulations, one sees that

$$\begin{aligned} S^* &= \frac{\Lambda}{\mu} - \frac{(g(U^*) + \mu)}{\theta_1 \mu} I^*, \\ I^* &= \frac{(\Gamma - eU^*)(K + U^*)}{\delta U^*}, \\ R^* &= \frac{(1-\theta_1)\alpha S^* I^* f(U^*) + g(U^*)I^*}{\mu} \end{aligned}. \tag{4}$$

Substituting the explicit formulas in terms of U^* given by (4) into the second equation of (3), one sees that

$$\left(\frac{\mathrm{R}_0}{f(\frac{\Gamma}{e})} \frac{g(\frac{\Gamma}{e}) + \mu}{g(U^*) + \mu} - \frac{\alpha(\Gamma - eU^*)(K + U^*)}{\mu \delta U^*} \right) f(U^*) = 1. \tag{5}$$

Let us define a continuous function τ by

$$\tau : (0, \infty) \to \mathrm{R}, \quad \tau(U) = \left(\frac{\mathrm{R}_0}{f(\frac{\Gamma}{e})} \frac{g(\frac{\Gamma}{e}) + \mu}{g(U) + \mu} - \frac{\alpha(\Gamma - eU)(K + U)}{\mu \delta U} \right) f(U).$$

Since $\lim_{U\to 0^+}\tau(U) = -\infty$ and $\tau(\frac{\Gamma}{e}) = \mathsf{R}_0$, there is a solution U^* of (5), not necessarily unique, provided that $\mathsf{R}_0 > 1$. This, in turn, leads to the existence of (at least) one rumor-prevailing equilibrium E^*.

The uniqueness of U^* (or, equivalently, of the positive equilibrium E^*) depends on the concrete forms of f and g and may or may not hold for arbitrary functions. However, it is easy to note that $\tau'(U) > 0$ for $U \in (0, \frac{\Gamma}{e})$ ensures both the uniqueness of E^* when $\mathsf{R}_0 > 1$ and its nonexistence when $\mathsf{R}_0 \le 1$.

3.4. The stability of the equilibria

Theorem 3.2. *The rumor-free equilibrium* $E_0 = \left(\frac{\Lambda}{\mu},\ 0,\ \frac{\Gamma}{e},\ 0\right)$ *is locally asymptotically stable provided that* $\mathsf{R}_0 < 1$.

Proof. The Jacobian matrix of the system (1) at E_0 is

$$J(E_0) = \begin{bmatrix} -\mu & -\alpha\frac{\Lambda}{\mu}f(\frac{\Gamma}{e}) & 0 & 0 \\ 0 & \theta_1\alpha\frac{\Lambda}{\mu}f(\frac{\Gamma}{e}) - g(\frac{\Gamma}{e}) - \mu & 0 & 0 \\ 0 & -\frac{\delta\Gamma}{\Gamma+eK} & -e & 0 \\ 0 & (1-\theta_1)\alpha\frac{\Lambda}{\mu}f(\frac{\Gamma}{e}) + g(\frac{\Gamma}{e}) & 0 & -\mu \end{bmatrix} \tag{6}$$

We observe that $J(E_0)$ has three negative eigenvalues $\lambda_{1,2} = -\mu$, $\lambda_3 = -e$, the remaining eigenvalue λ_4 being given by

$$\lambda_4 = \theta_1\alpha\frac{\Lambda}{\mu}f(\frac{\Gamma}{e}) - g(\frac{\Gamma}{e}) - \mu = \left(g(\frac{\Gamma}{e}) + \mu\right)(\mathsf{R}_0 - 1).$$

Since $\mathsf{R}_0 < 1$, one sees that $\lambda_4 < 0$ and consequently E_0 is locally asymptotically stable. Note also that E_0 is unstable if $\mathsf{R}_0 > 1$. This completes the proof. $\square$

Via similar computations, it is seen that the Jacobian matrix of the system (1) at E^* has a negative eigenvalue $\lambda_1 = -\mu$, the other three eigenvalues being the roots of the equation

$$\begin{vmatrix} \lambda + a_{11} & a_{12} & a_{13} \\ a_{21} & \lambda & a_{23} \\ 0 & a_{32} & \lambda + a_{33} \end{vmatrix} = 0, \tag{7}$$

where

$$\begin{aligned} a_{11} &= \alpha I^* f(U^*) + \mu,\ a_{12} = \frac{g(U^*)+\mu}{\theta_1},\ a_{13} = \alpha S^* I^* f'(U^*), \\ a_{21} &= -\theta_1\alpha I^* f(U^*),\ a_{23} = -\theta_1\alpha S^* I^* f'(U^*) - I^* g'(U^*), \\ a_{32} &= \frac{\delta U^*}{K+U^*},\ a_{33} = e + \frac{\delta K I^*}{(K+U^*)^2}. \end{aligned}$$

Note that

$$\begin{aligned} a_{23} &= -\theta_1\alpha\frac{g(U^*)+\mu}{\theta_1\alpha f(U^*)} I^* f'(U^*) - I^* g'(U^*) \\ &= -\frac{I^*}{f(U^*)}\left[g(U^*)f'(U^*) + g'(U^*)f(U^*) + \mu f'(U^*)\right] \\ &= -\frac{I^*}{f(U^*)}(fg + \mu f)'\big|_{U(t)=U^*} \end{aligned}$$

and consequently $a_{23} \le 0$, by assumption (**fg.i**). This implies that

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where

$$\begin{aligned} c_1 &= a_{11} + a_{33} > 0, \\ c_2 &= a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{32} > 0, \\ c_3 &= a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} > 0. \end{aligned}$$

Applying the Routh–Hurwitz criterion, the remaining eigenvalues are negative or have negative real part if and only if

$$c_1c_2 > c_3.$$

However, it is obvious that

$$c_1c_2 - c_3 = a_{11}^2a_{33} + a_{11}a_{33}^2 - a_{11}a_{12}a_{21} - a_{23}a_{32}a_{33} - a_{13}a_{21}a_{32} > 0.$$

Hence, one obtains the following result.

Theorem 3.3. *If* $\mathsf{R}_0 > 1$, *then the rumor-free equilibrium* E_0 *is unstable and there is at least one rumor-prevailing equilibrium* $E^* = (S^*, I^*, U^*, R^*)$, *which is necessarily locally asymptotically stable.*

Remark 3.4. Note that, in some sense, assumption (**fg.i**) is not necessarily optimal, since it ensures that each of the three terms (the sign in front of them being included) involved in the expressions of c_2 and c_3 are positive, condition which is strictly stronger than their sum being positive. Consequently, E^* may be locally asymptotically stable even if (**fg.i**) is not satisfied.

Remark 3.5. From Theorems 3.2 and 3.3, it is seen that R_0 is indeed a threshold parameter as far as the stability of the model (1) is concerned, separating the extinction of the rumor from its prevailment. Let us denote $\underline{\lim}_{x\to\infty} f(x) = \underline{f}$. If

$$\theta_1 \alpha \Lambda \underline{f} > \mu(\bar{g} + \mu),$$

(which happens, for instance, if the newcomers recruitment rate Λ or the percentage of newcomers who become spreaders θ_1 are too high, or if the maximal attitude adjustment rate $\bar{g}$ is too low, or the maximal transmission attenuation rate $\underline{f}$ is still too high), then

$$\mathsf{R}_0 \geq \frac{\theta_1 \alpha \Lambda \underline{f}}{\mu(\bar{g} + \mu)} > 1, \tag{8}$$

which leads to the (perhaps unpleasant) conclusion that the rumor will persist regardless of the budget devoted to rumor control. However, it can also be seen that, in such a situation, efforts to improve any of the mechanisms (lowering $\underline{f}$ or increasing $\bar{g}$) can bring back R_0 below 1 and ensure the success of rumor control.

Also, at the opposite end of the spectrum, if

$$\theta_1 \alpha \Lambda < \mu^2$$

(which happens, for instance, when the newcomers are generally not inclined to become spreaders or the recruitment rate of newcomers is low), then

$$\mathsf{R}_0 \leq \frac{\theta_1 \alpha \Lambda}{\mu^2} < 1, \tag{9}$$

and the rumor will disappear on its own, no budget for rumor control measures being necessary.

Remark 3.6. Let us think for the moment of R_0 as a function of Γ, $\mathsf{R}_0 = \mathsf{R}_0(\Gamma)$, and suppose that

$$\frac{\theta_1 \alpha \Lambda}{\mu(g(0) + \mu)} > 1, \quad \frac{\theta_1 \alpha \Lambda \underline{f}}{\mu(\bar{g} + \mu)} < 1,$$

that is,

$$\lim_{\Gamma \downarrow 0} \mathsf{R}_0(\Gamma) > 1, \quad \lim_{\Gamma \to \infty} \mathsf{R}_0(\Gamma) < 1.$$

Let us also suppose that at least one of the functions f, g is strictly monotonic, which leads to $\mathsf{R}_0(\Gamma)$ being strictly decreasing. In this situation, there is a unique Γ_c which solves the equation $\mathsf{R}_0(\Gamma) = 1$. This Γ_c can be thought as a sharp lower bound for the least amount of funding necessary to extinguish the rumor. However, the equation $\mathsf{R}_0(\Gamma) = 1$ might be transcendental (and it is, in our examples, where f contains an exponential and g is a rational function), and consequently Γ_c can sometimes be only approximated, not determined explicitly. Also, there is the caveat that $\Gamma > \Gamma_c$ guarantees the elimination of the rumor only **in the long term**, which may or may not be acceptable in a social setting, for which time is also a concern.

4. The stochastic model

Since the dynamics of R does not affect the spread of rumors directly, we can discard the growth equation of stiflers and then examine the effect of external perturbations occurring in the growth equations of newcomers, spreaders and of the government control mechanisms. We then employ the following stochastic model, derived from its deterministic counterpart (1), to incorporate the effect of randomly fluctuating environments

$$\begin{aligned} dS &= (\Lambda - \alpha SIf(U) - \mu S)\,dt + \sigma_1 S dW_1(t), \\ dI &= (\theta_1 \alpha SIf(U) - g(U)I - \mu I)\,dt + \sigma_2 I dW_2(t), \\ dU &= \left(\Gamma - eU - \frac{\delta IU}{K+U}\right) dt + \sigma_3 U dW_3(t). \end{aligned} \tag{10}$$

Here, W_i $(i = 1, 2, 3)$ are mutually independent standard Brownian motions defined over a complete probability space $(\Omega, \mathsf{F}, \mathsf{F}_t, P)$ with a filtration $\{\mathsf{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing, while

$\{\mathsf{F}_0\}$ contains all P-null sets). In the above, $\sigma_i^2 \geq 0$ represent the intensities of W_i, $i = 1,\ 2,\ 3$. If $a, b \in \mathbf{R}$, we shall denote $\max(a, b)$ by $a \vee b$ and $\min(a, b)$ by $a \wedge b$, while a.s. will be used as a shorthand for almost surely. We shall denote the indicator function of a set S by I_S. Also, for easier reference, we shall call "(1)′" the model which is obtained from the initial deterministic model (1) by discarding the growth equation for stiflers.

Definition 4.1. Consider the n-dimensional stochastic differential equations

$$du(t) = A(t,\ u)dt + B(t,\ u)dW(t), \quad \text{for } t \geq t_0. \tag{11}$$

Let $V(t,\ u) \in C^{1,2}$ be a nonnegative continuously differentiable function, once with respect to t and twice with respect to u. Then the differential operator $\mathbf{L}$ for the function $V(t, u)$ corresponding to the stochastic differential Eqs. (11) with drift and diffusion coefficients $A(t,\ u)$ and $B(t,\ u)$, respectively, is given by

$$\mathbf{L}V(t,\ u) = \frac{\partial V(t,\ u)}{\partial t} + A^T \frac{\partial V(t,\ u)}{\partial u} + \frac{1}{2}\text{trace}\left[B^T \frac{\partial^2 V(t,\ u)}{\partial^2 u} B\right].$$

4.1. Existence of a unique global solution

Denote

$$\mathbf{R}_+^3 = \left\{(x_1, x_2, x_3) | x_i > 0,\ i = 1, 2, 3\right\}.$$

We observe that the stochastically perturbed system (10) is mathematically (and socially) well-posed, as it has a unique global solution which is positivity-preserving.

Theorem 4.2. *For any initial value $\big(S(0), I(0), U(0)\big) \in \mathbf{R}_+^3$, there exists a unique global solution $\big(S(t), I(t), U(t)\big)$ of the system (10) for $t \geq 0$ that will remain in $\mathbf{R}_+^3$ with probability 1.*

The proof of this Theorem can be found in Appendix. We can now further discuss the behavior of these solutions by investigating the dynamics of the stochastic model (10) around the equilibria of the corresponding deterministic, reduced model (1)′.

4.2. Asymptotic behavior around the rumor-free equilibrium of the deterministic model

As mentioned in the previous section, the deterministic system (1) has a rumor-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, \frac{\Gamma}{e}, 0)$, which is locally asymptotically stable provided that $\mathsf{R}_0 < 1$. However, for the stochastic system (10), E_0 is no longer the rumor-free equilibrium, due to the stochastic perturbations, which implies that the solutions cannot converge to E_0. In this section, we shall investigate the asymptotic behavior of the stochastic system (10) around $E_0' = (\frac{\Lambda}{\mu}, 0, \frac{\Gamma}{e})$, the corresponding equilibrium of (1)′.

Theorem 4.3. *If*

$$\mathsf{R}_0 < \frac{2\mu^2}{(\bar{g} + 2\mu)(g(\frac{\Gamma}{e}) + \mu)} f(\frac{\Gamma}{e})$$

and the following conditions hold

$$\sigma_1^2 < \mu,\ \sigma_2^2 < 2\mu,\ \sigma_3^2 < e,$$

then for any given initial value $\big(S(0), I(0), U(0)\big) \in \mathbf{R}_+^3$, the unique global solution $\big(S(t), I(t), U(t)\big)$ of the system (10) satisfies

$$\limsup_{t\to+\infty} \frac{1}{t} E \int_0^t \left[\big(S(s) - \frac{\Lambda}{\mu}\big)^2 + I^2(s) + \big(U(s) - \frac{\Gamma}{e}\big)^2\right] ds \leq \frac{\theta_1^2 \sigma_1^2 (\frac{\Lambda}{\mu})^2 + c_2 \sigma_3^2 (\frac{\Gamma}{e})^2}{\kappa},$$

in which

$$\kappa = \min\left\{\theta_1^2(\mu - \sigma_1^2),\ \mu - \frac{1}{2}\sigma_2^2,\ c_2(e - \sigma_3^2)\right\}$$

and

$$c_2 = -\frac{(\bar{g} + 2\mu)(g(\frac{\Gamma}{e}) + \mu)}{\alpha\delta f(\frac{\Gamma}{e})\frac{\Gamma}{e}} \left(\mathsf{R}_0 - \frac{2\mu^2}{(\bar{g} + 2\mu)(g(\frac{\Gamma}{e}) + \mu)} f(\frac{\Gamma}{e})\right).$$

The proof of this Theorem can be found in Appendix. As seen from Theorem 4.3, if the value of the basic reproduction number R_0 is not too high and the intensities of the stochastic perturbations σ_1^2, σ_2^2, σ_3^2 are low, then the solutions of the stochastic model (10) will oscillate around the rumor-free equilibrium of the deterministic model (1)′. That is, if the spreaders are not very convincing and the strength of the stochastic perturbations is limited, then the solutions of the stochastic model (10) will still be close enough to the rumor-free equilibrium, most of the time. Note that, strictly speaking, Theorem 4.3 does not require that $\mathsf{R}_0 < 1$.

4.3. Asymptotic behavior around the rumor-prevailing equilibrium of the deterministic model

In this subsection, we assume that $\mathsf{R}_0 > 1$, which implies that the deterministic system (1) has a (not necessarily unique) rumor-prevailing equilibrium $E^* = (S^*, I^*, U^*, R^*)$, which is locally asymptotically stable. However, E^* is not a rumor-prevailing equilibrium for the stochastic system (10) anymore, due to the stochastic perturbations. We shall then investigate the asymptotic behavior of the system (10) around E^*.

Theorem 4.4. *If $\mathsf{R}_0 > 1$ and the following conditions hold*

(C1) $\sigma_1^2 < \mu - \frac{1}{\theta_1}\left[g(U^*) + c_3\Big(\Lambda + \alpha\big(1 - f(U^*)\big)\Big)\right]$;

(C2) $\sigma_2^2 < \mu - (1+\theta_1)g(U^*) - (\theta_1 S^* + I^*)(\bar{g} - g(U^*)) - c_3 g(U^*) - c_3\theta_1\alpha\big[1 - f(U^*)\big] - \delta U^*$;

(C3) $\sigma_3^2 < e(1+U^*) - \Gamma$,

then for any given initial value $\big(S(0), I(0), U(0)\big) \in \mathbf{R}_+^3$, the unique global solution $\big(S(t), I(t), U(t)\big)$ of the system (10) satisfies

$$\limsup_{t\to+\infty} \frac{1}{t} E\int_0^t \left[\big(S(s) - S^*\big)^2 + \big(I(s) - I^*\big)^2(s) + \big(U(s) - U^*\big)^2\right] ds \le \frac{\Theta}{\rho},$$

in which

$$\begin{aligned}
\Theta \doteq\,& \theta_1 g(U^*)(S^{*^2} + I^{*^2}) + 2g(U^*)I^{*^2} + \left[\big(\theta_1 S^* + I^*\big)\big(\bar{g} - g(U^*)\big)\right](\frac{1}{2} + I^{*^2}) \\
&+ (\theta_1^2\sigma_1^2 S^{*^2} + \sigma_2^2 I^{*^2}) + c_3 g(U^*)\left(\frac{1}{2} + I^{*^2}\right) + c_3\big(\bar{g} - g(U^*)\big)I^* \\
&+ \frac{1}{2}c_3 I^*\sigma_2^2 + c_3\theta_1\alpha\big(1 - f(U^*)\big)\big(S^{*^2} + I^{*^2}\big) + c_3\theta_1\Lambda\left(S^{*^2} + \frac{1}{2S^{*^2}}\right) \\
&+ \sigma_3^2 U^{*^2} + \frac{\delta I^* U^*}{K + U^*}\left(U^{*^2} + \frac{1}{2}\right) + \delta U^*(I^{*^2} + \frac{1}{2}).
\end{aligned}$$

$$\begin{aligned}
\rho = \min\Big\{ & -\theta_1^2\mu + \theta_1 g(U^*) + \theta_1\sigma_1^2 + c_3\theta_1\alpha\big(1 - f(U^*)\big) + c_3\theta_1\Lambda, \\
& -\mu + (1+\theta_1)g(U^*) + (\theta_1 S^* + I^*)(\bar{g} - g(U^*)) + c_3 g(U^*) + \sigma_2^2 \\
& + c_3\theta_1\alpha\big[1 - f(U^*)\big] + \delta U^*,\ \Gamma - e(1+U^*) + \sigma_3^2 \Big\}
\end{aligned}$$

and

$$c_3 = \frac{g(U^*) + 2\mu}{\alpha f(U^*)}.$$

The proof of this Theorem can be found in Appendix. In a similar vein to what was already observed above, Theorem 4.4 establishes that if the deterministic model (1)′ has a rumor-prevailing equilibrium and the intensities of the stochastic perturbations $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are low, then the solutions of the stochastic model (10) will oscillate around the rumor-prevailing equilibrium of the deterministic model (1)′. That is, if the strength of the stochastic perturbations is limited, then the solutions of the stochastic model (10) will still be close enough to the rumor-prevailing equilibrium, most of the time.

Remark 4.5. Conditions **(C1)**–**(C3)** have a theoretical value only, as they cannot be verified *a priori* (i.e., they explicitly depend upon the coordinates of the rumor-prevailing equilibrium, for which explicit expressions are not available). However, they take a somewhat simpler form if $f \equiv 1$ (the mechanisms for the inhibition of rumor spreading are not active).

5. Numerical simulations

In this section, we shall perform several numerical simulations in order to illustrate and enhance our modeling considerations as well as the abstract findings presented in the previous sections.

5.1. The simulation of deterministic rumor spreading process

For our numerical simulations, we employ a number of parameter values taken from the available literature and estimate several others, as shown in Table 1. 780624 780624

We first choose $f(U) = 0.5(1+e^{-U})$, $g(U) = \frac{U}{5(1+U)}$ and $\mu = 0.25$, so that $fg + \mu f$ is an increasing function, as shown in Fig. 2. Fig. 3 illustrates the bilinear dependence of the basic influence number R_0 upon the rate of flow into the newcomer class Λ and upon the average contact rate $\alpha = \bar{k}\bar{\alpha}$, the other parameters being fixed ($\Gamma = 1$ and $e = 0.5$).

Table 1
Parameters and their values.

Parameter	Value	Source
Λ	2–6	[19,29]
α	10^{-5}–0.6	[19,29,30]
μ	0.25	[19]
θ_1	0.2	[19]
$f(U)$	$0.5(1+e^{-U})$	estimated
$g(U)$	$0.2\frac{U}{1+U}$	estimated
Γ	0.5–1.5	estimated
e	10^{-5}–1	estimated
δ	0.5	estimated
K	1	estimated

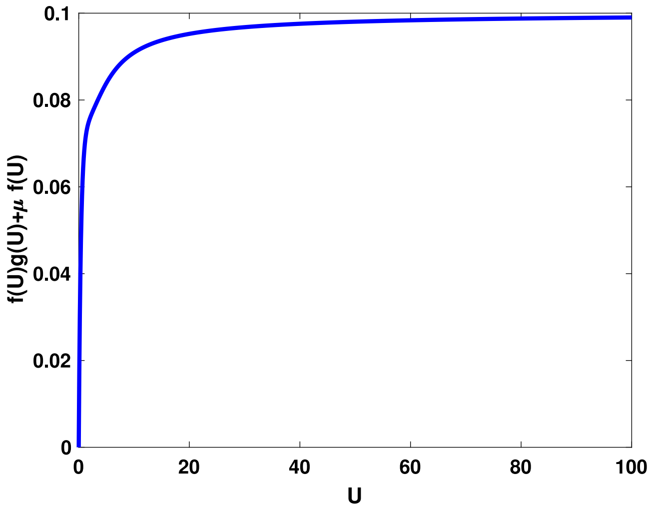

Fig. 2. The graph of $f(U)g(U)+\mu f(U)$.

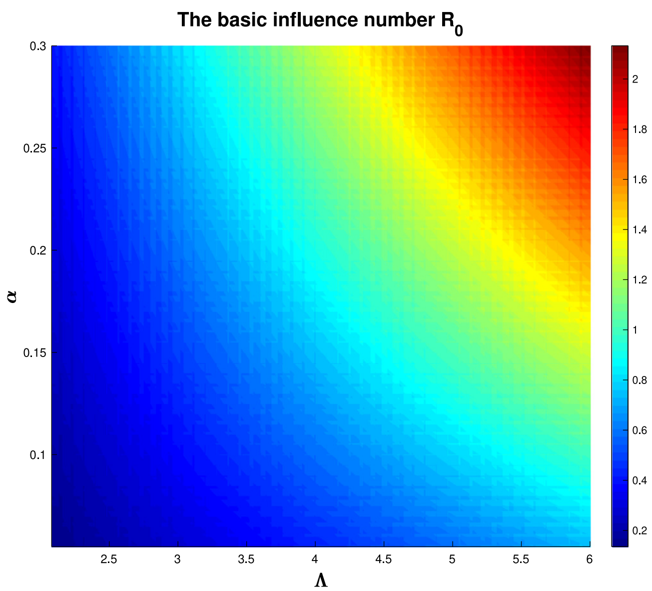

Fig. 3. The basic influence number. $\Lambda \in [2,6]$, $\alpha \in [0.05, 0.3]$, the values of the other parameters being given in Table 1.

As seen from the expression of R_0 given by (2), the parameters Λ, α, Γ and e play a vital role for rumor spreading analysis, and so do the explicit forms of f and g. In what follows, we establish the contributions of the variances of Λ, α, Γ

Fig. 4. First-order and total-effect Sobol sensitivities of the basic influence number R_0 with respect to input parameters (Λ, α, Γ and e). We use 4000 sample points, $\Lambda \in [2, 6]$, $\alpha \in [10^{-5}, 0.6]$, $\Gamma \in [0.5, 1.5]$ and $e \in [10^{-5}, 1]$, the values of the other parameters being given in Table Table 1.

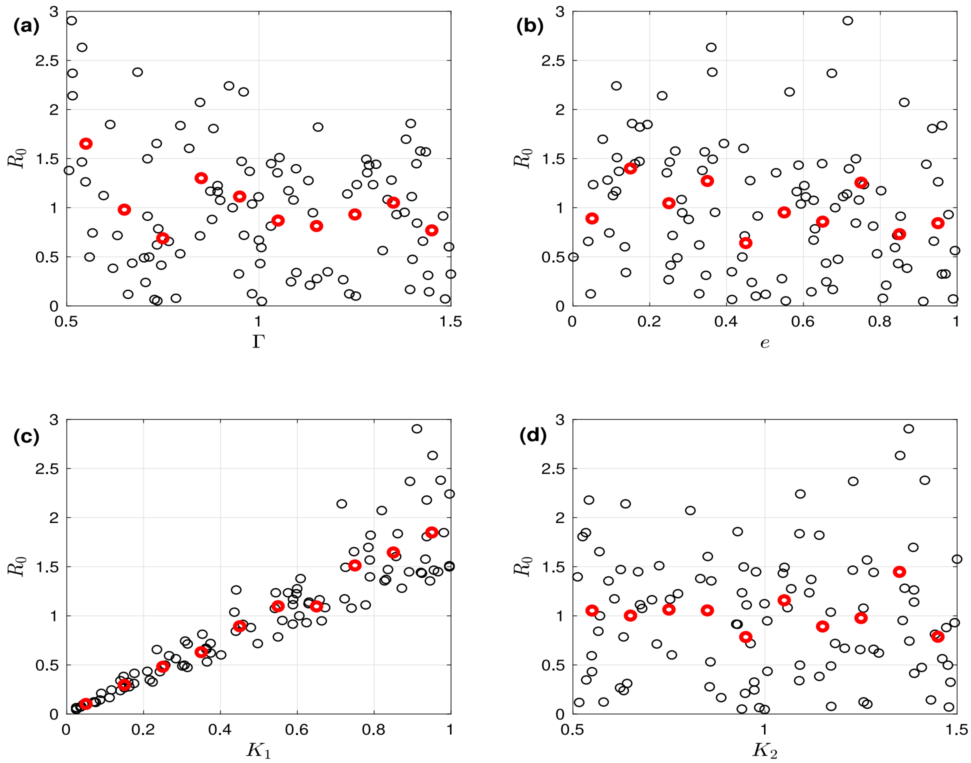

Fig. 5. Mean values of the basic influence number R_0 for random values of (a) $\Gamma \in (0.5, 1.5)$, (b) $e \in (0, 1)$, (c) $K_1 \in (0, 1)$ and (d) $K_2 \in (0.5, 1.5)$. Here, $f(u) = K_1(1+e^{-u})$ and $g(u) = \frac{0.2u}{K_2+u}$. For each input parameter, 100 sample points are randomly picked. Black circles represent random values of R_0, while red circles represent mean values of R_0.

and e to the variance of R_0, as shown in Fig. 4. To this purpose, we use the Sobol method (Sobol [31], Dimitriu et al. [32]), which is a model-independent approach to performing global sensitivity analysis based on variance decomposition. By analyzing the first-order and the total-effect sensitivity indices, it is seen that the variance of α provides the main contribution to the variance of R_0.

Fig. 5 depicts mean values of the basic influence number R_0 for random values of parameters. Here, $f(u) = K_1(1+e^{-u})$ and $g(u) = \frac{0.2u}{K_2+u}$, while $\Gamma \in (0.5, 1.5)$, $e \in (0, 1)$, $K_1 \in (0, 1)$ and $K_2 \in (0.5, 1.5)$, respectively. Obviously, as shown in Fig. 5(c), as $\underline{f}$ (that is, K_1, for our example of f) increases, so does the mean value of R_0. For other randomly selected input

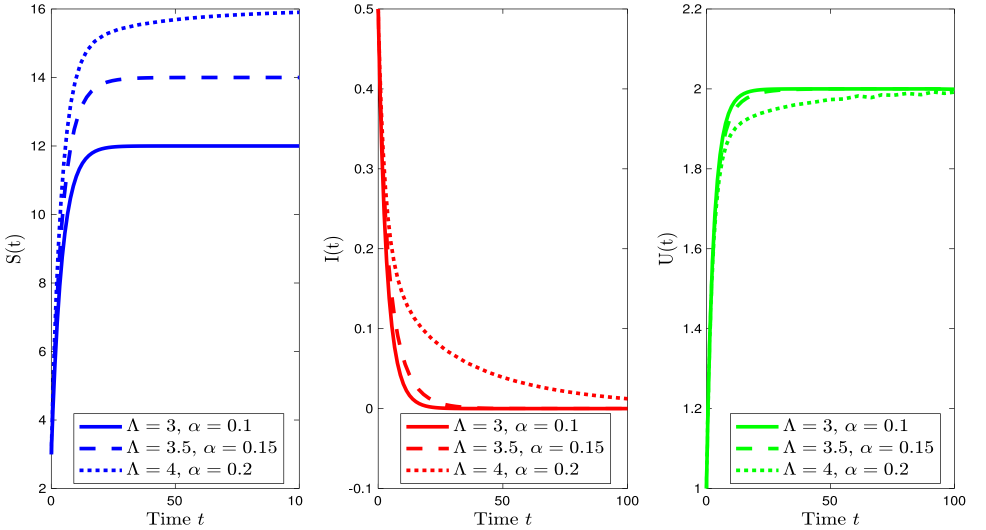

Fig. 6. The evolution of the newcomer population, of the spreader population and of the inhibitor for a low initial density of spreaders. Here, $S(0) = 3$, $I(0) = 0.5$, $U(0) = 1$, $(\Lambda, \alpha) \in \{(3, 0.1), (3.5, 0.15), (4, 0.2)\}$, the values of the other parameters being given in Table 1. The corresponding values of R_0 are 0.37, 0.61 and 0.91, respectively.

parameters, the mean value of R_0 oscillates up and down around 1, as shown in Fig. 5(a), (b) and (d), illustrating the influences of Γ, e and K_2, respectively.

Fig. 6 illustrates the evolution of the total densities of newcomers, spreaders and of the inhibitor for the following parameter values: $(\Lambda, \alpha) = (3, 0.1)$, or $(3.5, 0.15)$, or $(4, 0.2)$, the other parameter values being given in Table 1.

According to Theorem 3.2, a rumor outbreak does not occur when $\mathsf{R}_0 < 1$. As observed from Figs. 6 and 7, the density of spreaders declines to zero, which implies that the rumor dies out. For a low initial density of spreaders, (Fig. 6), the densities of newcomers and of the inhibitor, respectively, steadily increase until they reach their respective equilibrium states. However, for a high initial density of spreaders, (Fig. 7), the densities of newcomers and of the inhibitor, respectively, sharply descend at first and then increase to their respective equilibrium states.

Fig. 8 shows different rumor propagation outcomes for different initial conditions, with $S(0)$ ranging from 2 to 20, $I(0)$ changing from 0.5 to 5, and $I(0)$ varying from 1 to 10. The system no longer has a rumor-prevailing equilibrium and the rumor-free equilibrium is stable, which is in agreement with the results of our analysis.

To illustrate the existence, uniqueness and stability of the rumor-prevailing equilibrium, we choose $\Lambda = 5.8$ and $\alpha = 0.28$, the other parameter values being given in Table 1. We then determine the unique rumor-prevailing equilibrium as having coordinates $(9.123, 2.012, 0.996)$, and being stable, as shown in Fig. 9. This validates our theoretical result given in Theorem 3.3.

In Fig. 10, we use the pairs (Γ, α) employed above, that is, $(\Gamma, \alpha) = (5.8, 0.28)$ and $(\Gamma, \alpha) = (4, 0.2)$, along with $f(U) = 0.3(1 + e^{-U})$, to illustrate that a different value of $f(0)$ not satisfying the normalizing condition $f(0) = 1$ in the assumption (**f.i**) does not influence the local asymptotic stability of E_0 and E^*.

5.2. The simulation of stochastic rumor spreading process

To illustrate the asymptotic behavior of the solutions of the stochastic system around the rumor-free equilibrium, we choose $\Lambda = 2$, $\alpha = 0.1$, the values of the other parameters being given in Table 1. It is then seen that

$$\mathsf{R}_0 = 0.2369 < \frac{2\mu^2}{(\bar{g} + 2\mu)(g(\frac{\Gamma}{e}) + \mu)} f(\frac{\Gamma}{e}) = 0.2644.$$

Also, let $\sigma_1 = 0.2$, $\sigma_2 = 0.5$, $\sigma_3 = 0.6$ and note that $\sigma_1^2 = 0.04 < \mu = 0.25$, $\sigma_2^2 = 0.25 < 2\mu = 0.5$, $\sigma_3^2 = 0.36 < e = 0.5$, the hypotheses of Theorem 4.3 being satisfied.

Figs. 11 and 12 illustrate the time evolution of the total densities of each group (newcomers, spreaders and inhibitors) in the stochastic rumor spreading model for different initial conditions. To illustrate the asymptotic behavior of the solutions of the stochastic system around the rumor-prevailing equilibrium of the deterministic model, we set $f(U) \equiv 1$, $\Lambda = 2$, $\alpha = 10$, $\mu = 0.5$, $e = 1.2$, $\theta_1 = 0.5$, $\delta = 0.5$, $\Gamma = 1$, $K = 1$ and $g(U) = \frac{0.1U}{1+U}$. For these values, it is seen that $\mathsf{R}_0 = 36.67$. We also choose $\sigma_1 = 0.1$, $\sigma_2 = 0.2$ and $\sigma_3 = 0.4$, for which conditions $(\mathbf{C1}) - (\mathbf{C3})$ are satisfied. Fig. 13 illustrates the asymptotic dynamics of the total densities of three groups (newcomer, spreader and inhibitor) in the stochastic rumor spreading model for different initial conditions.

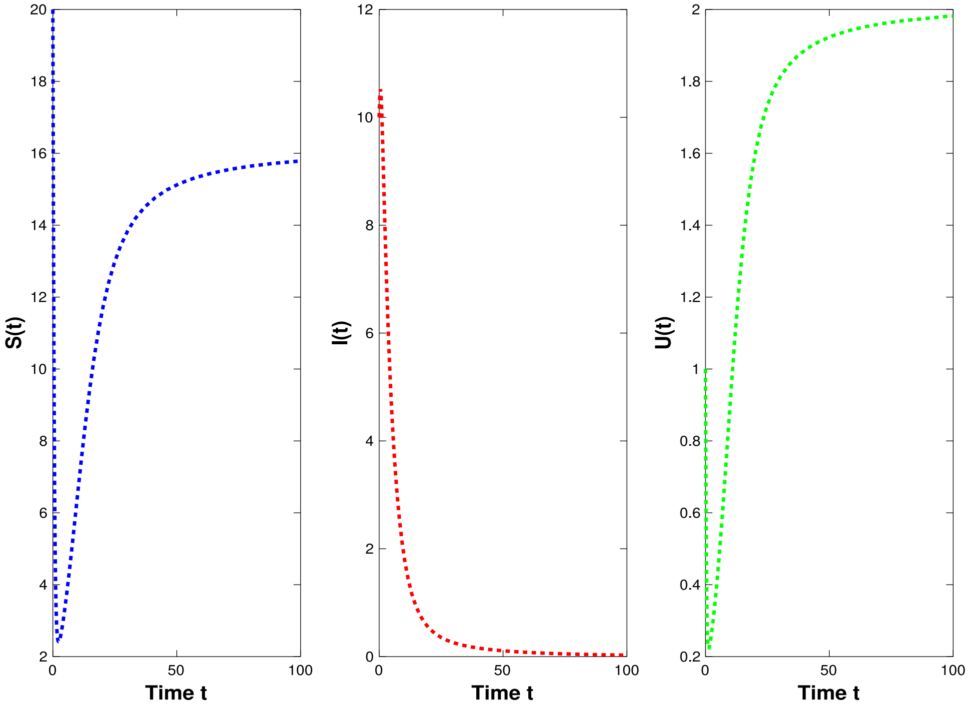

Fig. 7. The evolution of the newcomer population, of the spreader population and of the inhibitor for a high initial density of spreaders. Here, $S(0) = 20$, $I(0) = 10$, $U(0) = 1$, $\Lambda = 4$, $\alpha = 0.2$, the values of the other parameters being given in Table 1, which leads to $R_0 = 0.91$.

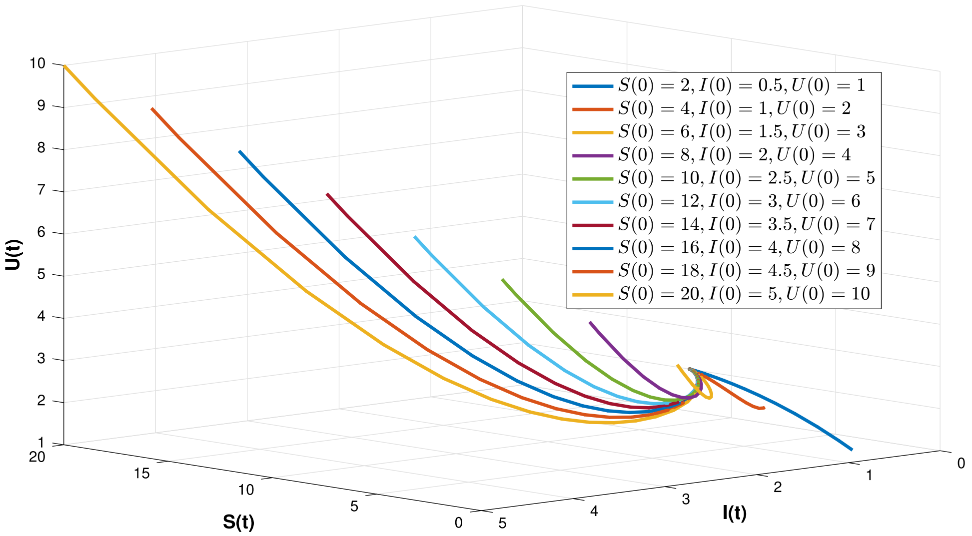

Fig. 8. The evolution of the newcomer population, of the spreader population and of the inhibitor for $\Lambda = 3$, $\alpha = 0.1$, which leads to $R_0 = 0.37$, and for several distinct initial conditions. The solutions converge to the rumor-free equilibrium, with coordinates $(12, 0, 2)$.

6. Conclusions

By extending the traditional DK rumor model to account for the effects of the attitude adjusting and spread inhibiting mechanisms depending upon governmental input, we propose deterministic and stochastic models, respectively, to discuss the control of rumor spreading on social networks. Although our starting deterministic model can be characterized as being an augmented *SIR* model, the use of a fourth variable, the inhibitor, which can be thought as being budget-related rather than having an epidemiological significance, transcends, in some sense, the epidemiological framework. Also, the

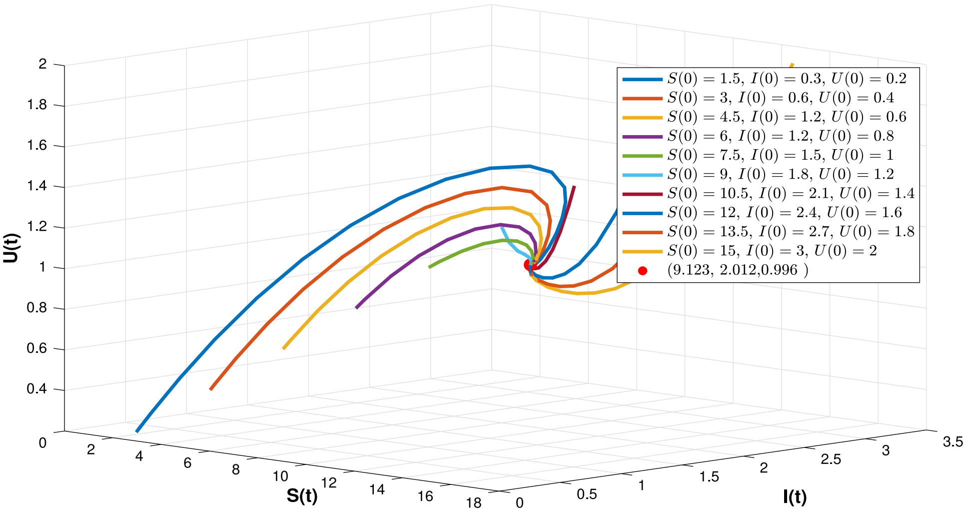

Fig. 9. The evolution of the newcomer population, of the spreader population and of the inhibitor for $\Lambda = 5.8$, $\alpha = 0.28$, which leads to $\mathrm{R}_0 = 1.93$, and for several distinct initial conditions. The solutions converge to the (unique) rumor-prevailing equilibrium, with coordinates (9.123, 2.012, 0.996).

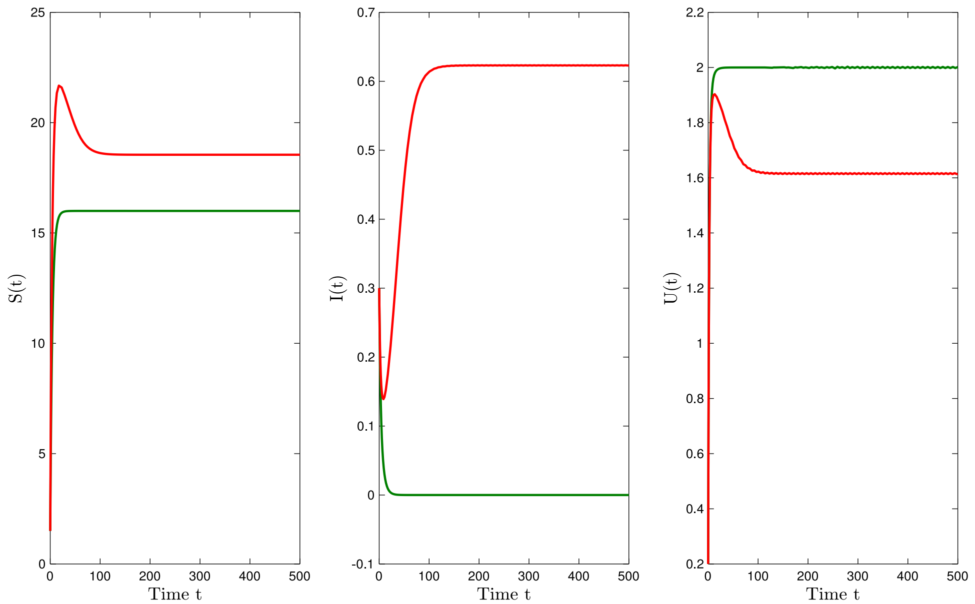

Fig. 10. A numerical example illustrating the fact that a different value of $f(0)$ not satisfying the normalizing condition $f(0) = 1$ in the assumption (**f.i**) does not influence the local asymptotic stability of E_0 and E^*. Here, $f(U) = 0.3(1 + e^{-U})$, $S(0) = 1.5$, $I(0) = 0.3$, $U(0) = 0.2$ and $(\Lambda, \alpha) = (4, 0.2)$ (green, $\mathrm{R}_0 = 0.5687$), $(\Lambda, \alpha) = (5.8, 0.28)$ (red, $\mathrm{R}_0 = 1.1544$), the values of the other parameters being given in Table 1.

use of the possibly nonlinear attitude changing function g makes the coupling between the equations of the model slightly more involved. For those reasons, we have been unable to obtain global stability results.

First of all, we find an explicit expression for our threshold parameter, the basic influence number R_0, and then investigate the stability of rumor-free and rumor-prevailing equilibria, respectively, of the deterministic model, in terms of R_0, being also observed that R_0 is a non-increasing function of the budgeting rate Γ, that is, the higher the budgeting rate is, the lower the average influence of a spreader in a totally susceptible population becomes. Depending upon the interplay of the parameters of the model, it is observed that, in certain situations, the rumor will persist regardless of the budget allotted to rumor control, while in others the rumor will disappear by itself even without exterior intervention, no budget for rumor control measures being necessary. For all the other situations in-between, a sharp lower bound for the amount of funding Γ_c, necessary to make the rumor go away, is determined.

Second, to account for the effect of random external fluctuations, we augment our model by considering stochastic perturbation of white noise type. Here, the stability properties obtained in the deterministic case are replaced by

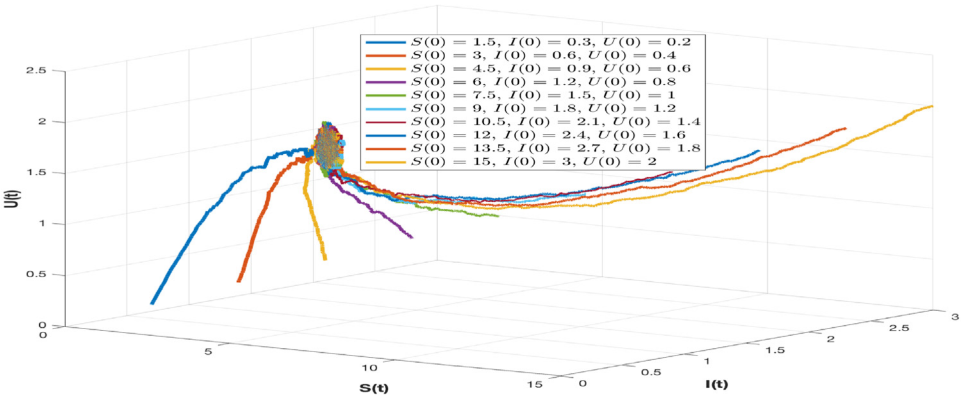

Fig. 11. Sample paths for the stochastic process $(S(t), I(t), U(t))$ of the stochastic model (10) that fluctuate about the steady state E_0, for different initial conditions. The parameter values are as used for Fig. 8, except for noise intensities $\sigma_1 = 0.2$, $\sigma_2 = 0.5$, $\sigma_3 = 0.6$ and the rate of input flow of newcomers $\Gamma = 2$.

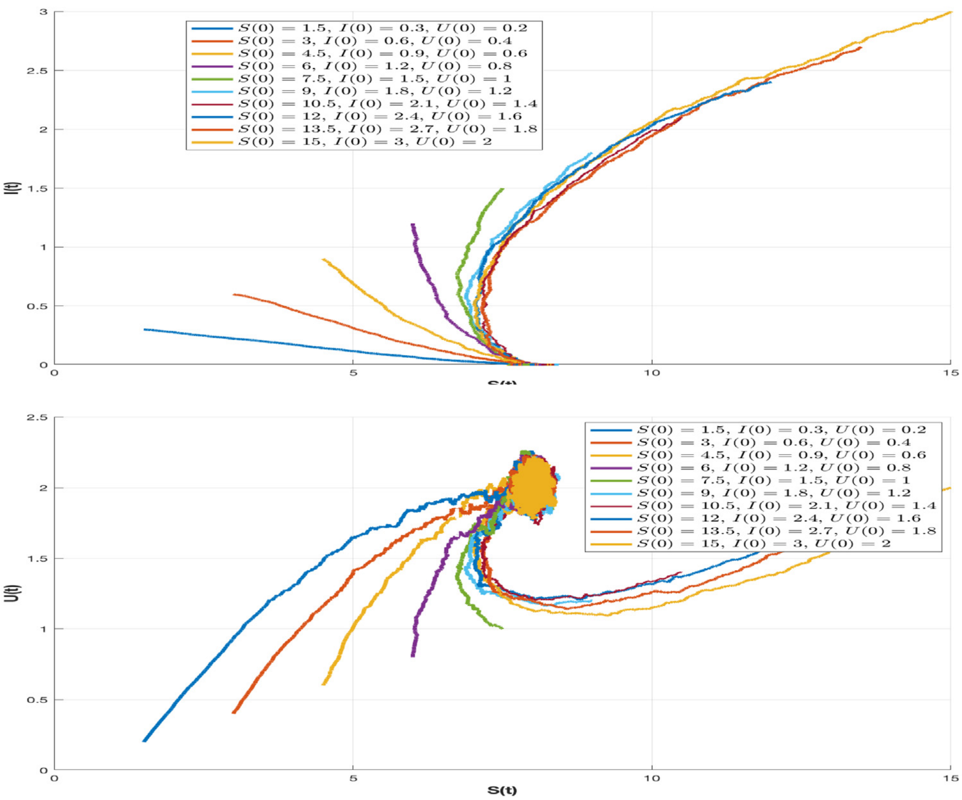

Fig. 12. Plane projections of Fig. 11. The newcomer population S persists, while the spreader population I is driven to extinction.

estimations in terms of expected values. It is proved that if the spreaders are not very convincing (R_0 is low enough) and the strength of the stochastic perturbations is limited, then the solutions of the stochastic model will still be close enough to the rumor-free equilibrium, most of the time.

It is also proved that if the deterministic model has at least one rumor-prevailing equilibrium ($\mathsf{R}_0 > 1$) and, similarly to the above, the strength of the stochastic perturbations is limited, then the solutions of the stochastic model will still be close enough to this rumor-prevailing equilibrium, most of the time. However, a shortcoming of the necessary conditions in this case is that they are expressed in terms of the coordinates of the rumor-prevailing equilibrium, for which explicit expressions are not available (i.e., they cannot be verified *a priori*). Still, this is also not unexpected, since there can be multiple rumor-prevailing equilibria in this case, as condition $\mathsf{R}_0 > 1$ does not ensure uniqueness, and it is then natural for a generic condition to depends somehow on the equilibrium it refers to.

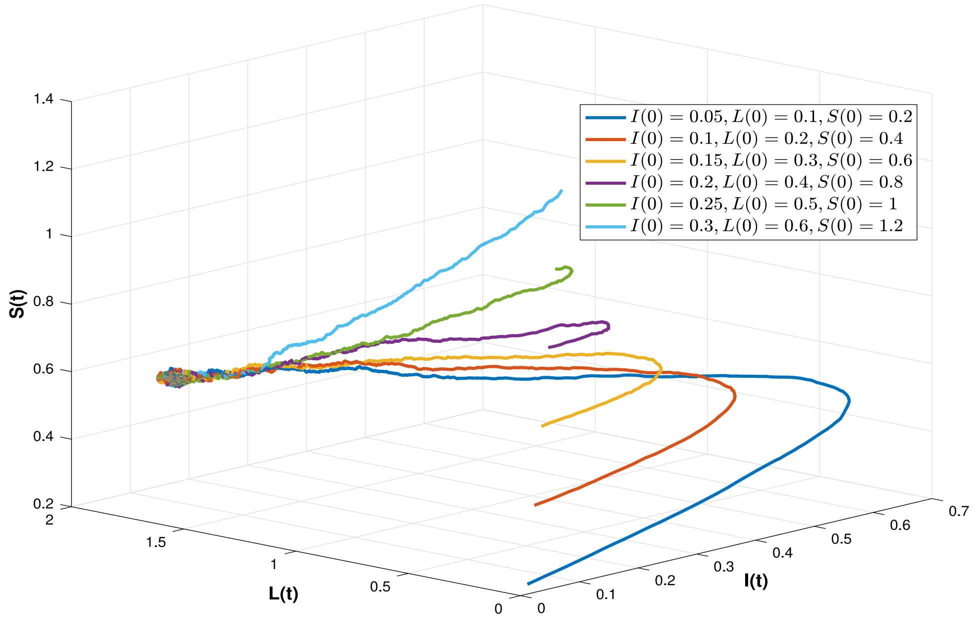

Fig. 13. Increasing α to 10 and slightly decreasing noise intensities $\sigma_i (i = 1, 2, 3)$ while keeping all of other parameter values the same as for Fig. 11 leads to the persistence of both the newcomer population S and spreader population I for several distinct initial conditions.

In order to complement our analysis, numerical simulations are performed to illustrate and enhance our mathematical findings. To establish the contributions of the variances of the parameters to the variance of R_0, a Sobol sensitivity analysis has been performed. By analyzing the first-order and the total-effect sensitivity indices, it is seen that the variance of α, the "normalized" contact rate between newcomers and spreaders, provides the main contribution to the variance of R_0. Further, the evolution of the newcomer population, of the spreader population and of the inhibitor are depicted for both how and high initial densities of spreaders. This study does not consider the influence of the network structure on rumor spreading, which is left as a possible avenue of further research.

CRediT authorship contribution statement

Ming Li: Conceptualization, Writing - original draft, Data curation, Editing. **Hong Zhang:** Methodology, Writing - original draft, Visualization, Editing, Supervision, Funding acquisition. **Paul Georgescu:** Methodology, Writing - original draft, Editing, Supervision, Writing - revision. **Tan Li:** Conceptualization, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Proof of Theorem 4.2. Since all functional coefficients of the model (10) satisfy local Lipschitz conditions, this system has a unique local solution on $[0, \tau_e)$, where τ_e is the explosion time. To show that this unique solution is global, we prove that $\tau_e = +\infty$ a.s. In fact, choose $m_0 >$ large enough such that $S(0)$, $I(0)$, $U(0)$ all belong to $(\frac{1}{m_0}, m_0)$. For each integer $m \geq m_0$, we define the stopping time as follows

$$\tau_m = \inf\Big\{t \in [0, \tau_e) : \ \min(S(t), I(t), U(t)) \leq \frac{1}{m_0} \text{ or } \max(S(t), I(t), U(t)) \geq m_0\Big\}.$$

Clearly, τ_m is increasing as $m \to \infty$. Let $\tau_0 = \lim_{m\to\infty} \tau_m$, where $\tau_0 \leq \tau_e$ a.s. To complete the proof, it suffices to show that $\tau_0 = +\infty$ a.s., which would imply that $\tau_e = +\infty$ a.s.

If this statement is false, then there exists a pair of constants $T > 0$ and $\nu \in (0, 1)$ such that $P\{\tau_0 \leq T\} > \nu$. Consequently, there is a positive constant $m_1 \geq m_0$ such that $P\{\tau_m \leq T\} \geq \nu$ for any integer $m \geq m_1$. Denote $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(x) = x + 1 - \ln x$. Define also $V : \mathbf{R}^3_+ \to \mathbf{R}_+$ by

$$V(S, I, U) = \varphi(S) + \varphi(I) + \varphi(U).$$

In what follows, we shall shorten $V(S(t), I(t), U(t))$ as $V(t)$. Applying Ito's formula, we have

$$\begin{aligned} dV\big(t\big) =&\mathbf{L}Vdt + (S(t) - 1)\sigma_1 dW_1 + (I(t) - 1)\sigma_2 dW_2 \\ &+ (U(t) - 1)\sigma_3 dW_3, \end{aligned}$$

where

$$\begin{aligned} \mathbf{L}V =& \left(1 - \frac{1}{S}\right)(\Lambda - \alpha SIf(U) - \mu S) + \left(1 - \frac{1}{I}\right)(\theta_1 \alpha SIf(U) - g(U)I - \mu I) \\ &+ \left(1 - \frac{1}{U}\right)\left(\Gamma - eU - \frac{\delta IU}{K + U}\right) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \end{aligned}$$

Using **(g.iii)**, this implies that

$$\begin{aligned} \mathbf{L}V \leq& \Lambda + \Gamma + \bar{g} + 2\mu + e + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \left(\alpha + \frac{\delta}{K}\right) I \\ \leq& \Lambda + \Gamma + \bar{g} + 2\mu + e + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &+ 2\left(\alpha + \frac{\delta}{K}\right)(I + 1 - \ln I) \\ \leq& k_1 + k_2 V, \end{aligned}$$

in which $k_1 = \Lambda + \Gamma + \bar{g} + 2\mu + e + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ and $k_2 = 2(\alpha + \frac{\delta}{K})$. Hence,

$$dV \leq (k_1 + k_2 V)dt + (S - 1)\sigma_1 dW_1 + (I - 1)\sigma_2 dW_2 + (U - 1)\sigma_3 dW_3.$$

Integrating this inequality from 0 to $\tau_m \wedge T$ and then taking expectation, we obtain

$$EV(\tau_m \wedge T) \leq V(0) + k_1(\tau_m \wedge T) + k_2 \int_0^{\tau_m \wedge T} EV(t)dt,$$

which implies by Gronwall's inequality that

$$EV(\tau_m \wedge T) \leq \Big(V(0) + k_1(\tau_m \wedge T)\Big)e^{k_2(\tau_m \wedge T)}.$$

Let $\Omega_m = \{\tau_m \leq T\}$ for all $m \geq m_1$, and then $P(\Omega_m) > \nu$. Note that for every $\omega \in \Omega_m$, one of $S(\tau_m, \omega)$, $I(\tau_m, \omega)$ or $U(\tau_m, \omega)$ equals either m or $\frac{1}{m}$. Then

$$V\big(S(\tau_m, \omega), I(\tau_m, \omega), U(\tau_m, \omega)\big) > \min(m + 1 - \ln m, \frac{1}{m} + 1 + \ln m).$$

Finally,

$$\begin{aligned} +\infty > \Big(V(0) + k_1\tau_m\Big)e^{k_2\tau_m} \geq& E\Big(I_{\Omega_m}(\omega)V\Big(S(\tau_m, \omega), I(\tau_m, \omega), U(\tau_m, \omega)\Big)\Big) \\ =& P(\Omega_m)V\big(S(\tau_m, \omega), I(\tau_m, \omega), U(\tau_m, \omega)\big) \\ >& \nu(\min(m + 1 - \ln m, \frac{1}{m} + 1 + \ln m)) \to +\infty \ (m \to \infty), \end{aligned}$$

which leads to a contradiction. Then $\tau_0 = +\infty$ a.s., which implies that $\tau_e = +\infty$ a.s. This completes the proof. □

Proof of Theorem 4.3. Let define a functional V_1 by

$$V_1(S, I, U) = \frac{\big(\theta_1(S - \frac{\Lambda}{\mu}) + I\big)^2}{2} + c_1 I + \frac{c_2(U - \frac{\Gamma}{e})^2}{2},$$

where c_1 and c_2 are positive constants that will be determined later. Applying Ito's formula, we obtain

$$\begin{aligned} dV_1\big(t\big) =& \mathbf{L}V_1dt + \left(\theta_1(S-\frac{\Lambda}{\mu})+I\right)(\theta_1\sigma_1SdW_1+\sigma_2IdW_2)+c_1\sigma_2IdW_2 \\ &+ c_2\sigma_3U\left(U-\frac{\Gamma}{e}\right)dW_3, \end{aligned}$$

in which

$$\begin{aligned} \mathbf{L}V_1 =& \left(\theta_1(S-\frac{\Lambda}{\mu})+I\right)[\theta_1(\Lambda-\alpha SIf(U)-\mu S)+\theta_1\alpha SIf(U)-g(U)I-\mu I] \\ &+\ c_1\left[\theta_1\alpha SIf(U)-g(U)I-\mu I\right]+c_2\left(U-\frac{\Gamma}{e}\right)\left(\Gamma-eU-\frac{\delta IU}{K+U}\right) \\ &+\ \frac{1}{2}\left(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2+c_2\sigma_3^2U^2\right). \end{aligned}$$

Having in view that

$$\frac{1}{2}a^2 \le (a-b)^2+b^2, \quad \forall a,b\in\mathbf{R}, \tag{12}$$

one obtains that

$$\begin{aligned} \mathbf{L}V_1 \le& \left(\theta_1(S-\frac{\Lambda}{\mu})+I\right)\left[\theta_1\mu(\frac{\Lambda}{\mu}-S)-(g(U)+\mu)I\right] \\ &+\ c_1\left[\theta_1\alpha SIf(U)-(g(U)+\mu)I\right]+c_2\left(U-\frac{\Gamma}{e}\right)\left(e(\frac{\Gamma}{e}-U)-\frac{\delta IU}{K+U}\right) \\ &+\ \theta_1^2\sigma_1^2\left((S-\frac{\Lambda}{\mu})^2+(\frac{\Lambda}{\mu})^2\right)+\frac{1}{2}\sigma_2^2I^2+c_2\sigma_3^2\left((U-\frac{\Gamma}{e})^2+(\frac{\Gamma}{e})^2\right). \end{aligned}$$

By rearranging the right-hand side, this leads to

$$\begin{aligned} \mathbf{L}V_1 \le& -\mu\theta_1^2(S-\frac{\Lambda}{\mu})^2-\theta_1(g(U)+\mu)SI+\theta_1(g(U)+\mu)\frac{\Lambda}{\mu}I+\theta_1\Lambda I-\theta_1\mu SI \\ &-\ (g(U)+\mu)I^2+c_1\left[\theta_1\alpha SIf(U)-(g(U)+\mu)I\right]-ec_2(U-\frac{\Gamma}{e})^2 \\ &-\ c_2(U-\frac{\Gamma}{e})\frac{\delta IU}{K+U}+\theta_1^2\sigma_1^2\left((S-\frac{\Lambda}{\mu})^2+(\frac{\Lambda}{\mu})^2\right)+\frac{1}{2}\sigma_2^2I^2 \\ &+\ c_2\sigma_3^2\left((U-\frac{\Gamma}{e})^2+(\frac{\Gamma}{e})^2\right) \\ =& -(\mu\theta_1^2-\theta_1^2\sigma_1^2)(S-\frac{\Lambda}{\mu})^2-(g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2-c_2(e-\sigma_3^2)(U-\frac{\Gamma}{e})^2 \\ &+\ \theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2+c_2\sigma_3^2(\frac{\Gamma}{e})^2+[-\theta_1(g(U)+\mu)-\theta_1\mu+c_1\theta_1\alpha f(U)]SI \\ &+\ \left[\theta_1(g(U)+\mu)\frac{\Lambda}{\mu}+\theta_1\Lambda-c_1(g(U)+\mu)-c_2\frac{\delta U^2}{K+U}+c_2\frac{\Gamma}{e}\frac{\delta U}{K+U}\right]I \end{aligned}$$

Using **(f.i), (f.ii), (g.i)** and **(g.iii)**, one sees that

$$\begin{aligned} \mathbf{L}V_1 \le& -\theta_1^2(\mu-\sigma_1^2)(S-\frac{\Lambda}{\mu})^2-(g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2-c_2(e-\sigma_3^2)(U-\frac{\Gamma}{e})^2 \\ &+\ \theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2+c_2\sigma_3^2(\frac{\Gamma}{e})^2+\big[-\theta_1(g(U)+\mu)-\theta_1\mu+c_1\theta_1\alpha f(0)\big]SI \\ &+\ \left[\theta_1(g(U)+\mu)\frac{\Lambda}{\mu}+\theta_1\Lambda-c_1\mu+c_2\delta(K+\frac{\Gamma}{e})\right]I \\ \le& -\theta_1^2(\mu-\sigma_1^2)(S-\frac{\Lambda}{\mu})^2-(g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2-c_2(e-\sigma_3^2)(U-\frac{\Gamma}{e})^2 \\ &+\ \theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2+c_2\sigma_3^2(\frac{\Gamma}{e})^2+\big[-2\theta_1\mu+c_1\theta_1\alpha\big]SI \\ &+\ \left[\theta_1(\bar{g}+\mu)\frac{\Lambda}{\mu}+\theta_1\Lambda-c_1\mu+c_2\delta(K+\frac{\Gamma}{e})\right]I. \end{aligned}$$

Now, let us choose $c_1 = \frac{2\mu}{\alpha} > 0$, so that

$$-2\theta_1\mu + c_1\theta_1\alpha = 0.$$

We observe that $\mathsf{R}_0 < \frac{2\mu^2}{(\bar{g}+2\mu)(g(\frac{\Gamma}{e})+\mu)} f(\frac{\Gamma}{e})$ implies that

$$\begin{aligned}&\theta_1(\bar{g}+\mu)\frac{\Lambda}{\mu} + \theta_1\Lambda - c_1\mu\\ =&\frac{(\bar{g}+2\mu)(g(\frac{\Gamma}{e})+\mu)}{\alpha f(\frac{\Gamma}{e})}\left(\mathsf{R}_0 - \frac{2\mu^2}{(\bar{g}+2\mu)(g(\frac{\Gamma}{e})+\mu)} f(\frac{\Gamma}{e})\right) < 0.\end{aligned}$$

We then choose c_2 so that

$$\theta_1(\bar{g}+\mu)\frac{\Lambda}{\mu} + \theta_1\Lambda - c_1\mu + c_2\delta\frac{\Gamma}{e} = 0,$$

which leads to

$$c_2 = -\frac{(\bar{g}+2\mu)(g(\frac{\Gamma}{e})+\mu)}{\alpha\delta f(\frac{\Gamma}{e})(K+\frac{\Gamma}{e})}\left(\mathsf{R}_0 - \frac{2\mu^2}{(\bar{g}+2\mu)(g(\frac{\Gamma}{e})+\mu)} f(\frac{\Gamma}{e})\right) > 0.$$

Hence,

$$\begin{aligned}dV_1 \le& -\theta_1^2(\mu-\sigma_1^2)(S-\frac{\Lambda}{\mu})^2 - (g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2 - c_2(e-\sigma_3^2)(U-\frac{\Gamma}{e})^2\\ &+ \theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2 + c_2\sigma_3^2(\frac{\Gamma}{e})^2 + \left(\theta_1(S-\frac{\Lambda}{\mu}) + I\right)(\theta_1\sigma_1 S dW_1 + \sigma_2 I dW_2)\\ &+ c_1\sigma_2 I dW_2 + c_2\sigma_3 U(U-\frac{\Gamma}{e})dW_3.\end{aligned} \tag{13}$$

Integrating both sides of (13) from 0 to t and then taking expectation we obtain

$$\begin{aligned}EV_1(t) \le& EV_1(0) + \left(\theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2 + c_2\sigma_3^2(\frac{\Gamma}{e})^2\right)t\\ &+ E\int_0^t \Big[-\theta_1^2(\mu-\sigma_1^2)\big(S(s)-\frac{\Lambda}{\mu}\big)^2 - (g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2(s)\\ &\qquad -c_2(e-\sigma_3^2)\big(U(s)-\frac{\Gamma}{e}\big)^2\Big]ds.\end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}E\int_0^t &\Big[\theta_1^2(\mu-\sigma_1^2)\big(S(s)-\frac{\Lambda}{\mu}\big)^2 + (g(U)+\mu-\frac{1}{2}\sigma_2^2)I^2(s)\\ &+c_2(e-\sigma_3^2)\big(U(s)-\frac{\Gamma}{e}\big)^2\Big]ds \le \theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2 + c_2\sigma_3^2(\frac{\Gamma}{e})^2.\end{aligned}$$

If

$$\kappa = \min\big\{\theta_1^2(\mu-\sigma_1^2),\ \mu-\frac{1}{2}\sigma_2^2,\ c_2(e-\sigma_3^2)\big\},$$

then

$$\limsup_{t\to+\infty}\frac{1}{t}E\int_0^t\Big[\big(S(s)-\frac{\Lambda}{\mu}\big)^2 + I^2(s) + \big(U(s)-\frac{\Gamma}{e}\big)^2\Big]ds \le \frac{\theta_1^2\sigma_1^2(\frac{\Lambda}{\mu})^2 + c_2\sigma_3^2(\frac{\Gamma}{e})^2}{\kappa}.$$

This completes the proof. □

Proof of Theorem 4.4. Let us define a functional V_2 by

$$V_2(S,I) = \frac{\big(\theta_1(S-S^*)+I-I^*\big)^2}{2}$$

It follows from Ito's formula that

$$dV_2\big(S(t),I(t)\big) = \mathbf{L}V_2 dt + \big(\theta_1(S-S^*)+I-I^*\big)(\theta_1\sigma_1 S dW_1 + \sigma_2 I dW_2), \tag{15}$$

in which

$$\begin{aligned}
\mathbf{L}V_2 =&\big(\theta_1(S-S^*)+I-I^*\big)\big(\theta_1(\Lambda-\alpha SIf(U)-\mu S)+\theta_1\alpha SIf(U)-g(U)I-\mu I\big)\\
&+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2)\\
=&\big(\theta_1(S-S^*)+I-I^*\big)\big(\theta_1(\Lambda-\mu S)-(g(U)+\mu)I\big)\\
&+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2)
\end{aligned}$$

Since, by the first two equilibrium relations in (3),

$$\theta_1\Lambda=\theta_1\mu S^*+g(U^*)I^*+\mu I^*,$$

it is seen by **(g.ii)** and **(g.iii)** that

$$\begin{aligned}
\mathbf{L}V_2 =&\big(\theta_1(S-S^*)+I-I^*\big)\Big(-\theta_1\mu(S-S^*)-(g(U^*)+\mu)(I-I^*)+\big(g(U^*)-g(U)\big)I\Big)\\
&+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2)\\
=&-\theta_1^2\mu(S-S^*)^2-(g(U^*)+\mu)(I-I^*)^2-[\theta_1(g(U^*)+\mu)+\theta_1\mu](S-S^*)(I-I^*)\\
&+\big(\theta_1(S-S^*)+I-I^*\big)\big(g(U^*)-g(U)\big)I+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2).\\
\le&-\theta_1^2\mu(S-S^*)^2-(g(U^*)+\mu)(I-I^*)^2-[\theta_1(g(U^*)+\mu)+\theta_1\mu](S-S^*)(I-I^*)\\
&+\theta_1SIg(U^*)+g(U^*)I^2+\Big[\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big]I+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2).
\end{aligned}$$

Since

$$ab\le(a-a_1)^2+(b-b_1)^2+a_1^2+b_1^2,\qquad \forall a,b,a_1,b_1\in\mathbf{R}, \tag{16}$$

$$a\le\frac{1}{2}+(a-a_1)^2+a_1^2,\qquad \forall a,a_1\in\mathbf{R}, \tag{17}$$

it is seen using also (12) that

$$\begin{aligned}
\mathbf{L}V_2\le&-\theta_1^2\mu(S-S^*)^2-(g(U^*)+\mu)(I-I^*)^2-[\theta_1(g(U^*)+\mu)+\theta_1\mu](S-S^*)(I-I^*)\\
&+\theta_1g(U^*)\Big[(S-S^*)^2+(I-I^*)^2+S^{*^2}+I^{*^2}\Big]+2g(U^*)(I-I^*)^2+2g(U^*)I^{*^2}\\
&+\Big[\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big]\left(\frac{1}{2}+(I-I^*)^2+I^{*^2}\right)+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2).
\end{aligned}$$

By rearranging the right-hand side of the above inequality and using (12), it is seen that

$$\begin{aligned}
\mathbf{L}V_2\le&-\big(\theta_1^2\mu-\theta_1g(U^*)\big)(S-S^*)^2\\
&-\Big(\mu-\theta_1g(U^*)-g(U^*)-\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big)(I-I^*)^2\\
&-[\theta_1(g(U^*)+\mu)+\theta_1\mu](S-S^*)(I-I^*)+\theta_1g(U^*)(S^{*^2}+I^{*^2})+2g(U^*)I^{*^2}\\
&+\Big[\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big]\left(\frac{1}{2}+I^{*^2}\right)+\frac{1}{2}(\theta_1^2\sigma_1^2S^2+\sigma_2^2I^2).\\
\le&-\big(\theta_1^2\mu-\theta_1g(U^*)-\theta_1^2\sigma_1^2\big)(S-S^*)^2\\
&-\Big(\mu-\theta_1g(U^*)-g(U^*)-\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)-\sigma_2^2\Big)(I-I^*)^2\\
&-[\theta_1(g(U^*)+\mu)+\theta_1\mu](S-S^*)(I-I^*)\\
&+\theta_1g(U^*)(S^{*^2}+I^{*^2})+2g(U^*)I^{*^2}+\Big[\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big](\frac{1}{2}+I^{*^2})\\
&+(\theta_1^2\sigma_1^2S^{*^2}+\sigma_2^2I^{*^2}).
\end{aligned} \tag{18}$$

Denote also

$$V_3(I)=c_3\left(I-I^*-I^*\ln\frac{I^*}{I}\right),$$

where c_3 is a positive constant whose value will be made precise later. It follows from the Ito's formula that

$$dV_3\big(I(t)\big) = \mathbf{L}V_3 dt + c_3\left(1-\frac{I^*}{I}\right)\sigma_2 I dW_2, \tag{19}$$

in which

$$\mathbf{L}V_3 = c_3(I-I^*)\big(\theta_1\alpha S f(U) - (g(U)+\mu)\big) + \frac{1}{2}c_3 I^*\sigma_2^2$$

Since, by the second equilibrium relation in (3), one has

$$\theta_1\alpha S^* f(U^*) - g(U^*) = \mu,$$

it follows that

$$\begin{aligned}
\mathbf{L}V_3 =& c_3(I-I^*)\Big(\theta_1\alpha S f(U) - \theta_1\alpha S^* f(U^*) - \big(g(U)-g(U^*)\big)\Big) + \frac{1}{2}c_3 I^*\sigma_2^2\\
=& c_3(I-I^*)\Big(\theta_1\alpha(S-S^*)f(U^*) - \theta_1\alpha S f(U^*) + \theta_1\alpha S f(U) - \big(g(U)-g(U^*)\big)\Big)\\
&+ \frac{1}{2}c_3 I^*\sigma_2^2\\
=& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + \frac{1}{2}c_3 I^*\sigma_2^2\\
&+ c_3(I-I^*)\Big(-\theta_1\alpha S f(U^*) + \theta_1\alpha S f(U) - \big(g(U)-g(U^*)\big)\Big).
\end{aligned}$$

Using again **(g.ii)** and **(g.iii)** along with (17) and **(f.i)**, **(f.ii)**, one has

$$\begin{aligned}
\mathbf{L}V_3 \le& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + \frac{1}{2}c_3 I^*\sigma_2^2\\
&+ c_3\theta_1\alpha S I f(U) + c_3\theta_1\alpha S I^* f(U^*) - c_3\theta_1\alpha S I f(U^*) + c_3 g(U^*) I\\
&+ c_3\big(\bar{g} - g(U^*)\big)I^*\\
\le& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + \frac{1}{2}c_3 I^*\sigma_2^2 + c_3 g(U^*)\left(\frac{1}{2} + (I-I^*)^2 + {I^*}^2\right)\\
&+ c_3\big(\bar{g} - g(U^*)\big)I^* + c_3\theta_1\alpha S I\big(1-f(U^*)\big) + c_3\theta_1\alpha S I^* f(U^*).
\end{aligned}$$

Since $I^* f(U^*) = \frac{\Lambda-\mu S^*}{\alpha S^*}$, we obtain using also (16) and the inequality

$$\frac{a}{b} \le (a-b)^2 + b^2 + \frac{1}{2b^2}, \quad \forall a, b \in \mathbf{R}, b \neq 0, \tag{20}$$

that

$$\begin{aligned}
\mathbf{L}V_3 \le& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + \frac{1}{2}c_3 I^*\sigma_2^2\\
&+ c_3 g(U^*)\left(\frac{1}{2} + (I-I^*)^2 + {I^*}^2\right) + c_3\big(\bar{g} - g(U^*)\big)I^*\\
&+ c_3\theta_1\alpha\big(1-f(U^*)\big)\left[(S-S^*)^2 + {S^*}^2 + (I-I^*)^2 + {I^*}^2\right] + c_3\theta_1 S\frac{\Lambda-\mu S^*}{S^*}\\
\le& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + \frac{1}{2}c_3 I^*\sigma_2^2\\
&+ c_3 g(U^*)\Big(\frac{1}{2} + (I-I^*)^2 + {I^*}^2\Big) + c_3\big(\bar{g} - g(U^*)\big)I^*\\
&+ c_3\theta_1\alpha\big(1-f(U^*)\big)\left[(S-S^*)^2 + {S^*}^2 + (I-I^*)^2 + {I^*}^2\right]\\
&+ c_3\theta_1\Lambda\Big((S-S^*)^2 + {S^*}^2 + \frac{1}{2{S^*}^2}\Big)\\
=& c_3\theta_1\alpha f(U^*)(S-S^*)(I-I^*) + c_3 g(U^*)\left(\frac{1}{2} + (I-I^*)^2 + {I^*}^2\right) + c_3\big(\bar{g} - g(U^*)\big)I^*\\
&+ \Big(c_3\theta_1\alpha\big(1-f(U^*)\big) + c_3\theta_1\Lambda\Big)(S-S^*)^2 + c_3\theta_1\alpha\big(1-f(U^*)\big)(I-I^*)^2\\
&+ \frac{1}{2}c_3 I^*\sigma_2^2 + c_3\theta_1\alpha\big(1-f(U^*)\big)\big({S^*}^2 + {I^*}^2\big) + c_3\theta_1\Lambda\left[{S^*}^2 + \frac{1}{2{S^*}^2}\right].
\end{aligned} \tag{21}$$

Finally, let us denote

$$V_4(U)=\frac{(U-U^*)^2}{2}.$$

It follows from Ito's formula that

$$dV_4\big(U(t)\big)=\mathbf{L}V_4dt+(U-U^*)\sigma_3UdW_3, \tag{22}$$

in which

$$\mathbf{L}V_4=(U-U^*)\left(\Gamma-eU-\frac{\delta IU}{K+U}\right)+\frac{1}{2}\sigma_3^2U^2$$

Having in view the third equilibrium relation in (3) and then using (12), it is seen that

$$\begin{aligned}\mathbf{L}V_4=&(U-U^*)\left(-e(U-U^*)+\frac{\delta I^*U^*}{K+U^*}-\frac{\delta IU}{K+U}\right)+\frac{1}{2}\sigma_3^2U^2\\ \le&-e(U-U^*)^2+(U-U^*)\left(\frac{\delta I^*U^*}{K+U^*}-\frac{\delta IU}{K+U}\right)+\frac{1}{2}\sigma_3^2U^2\\ \le&-e(U-U^*)^2+\sigma_3^2\big((U-U^*)^2+U^{*^2}\big)+\frac{\delta I^*U^*}{K+U^*}U+\delta U^*I\end{aligned}$$

Using now (17), this leads to

$$\begin{aligned}\mathbf{L}V_4\le&-e(U-U^*)^2+\sigma_3^2\big((U-U^*)^2+U^{*^2}\big)\\ &+\frac{\delta I^*U^*}{K+U^*}\left((U-U^*)^2+U^{*^2}+\frac{1}{2}\right)+\delta U^*\left((I-I^*)^2+I^{*^2}+\frac{1}{2}\right)\\ =&\Big[-e+\sigma_3^2+\frac{\delta I^*U^*}{K+U^*}\Big](U-U^*)^2+\delta U^*(I-I^*)^2\\ &+\sigma_3^2U^{*^2}+\frac{\delta I^*U^*}{K+U^*}\left(U^{*^2}+\frac{1}{2}\right)+\delta U^*\left(I^{*^2}+\frac{1}{2}\right).\end{aligned} \tag{23}$$

Denote

$$V_5=V_2+V_3+V_4,$$

that is

$$V_5=\frac{\big(\theta_1(S-S^*)+I-I^*\big)^2}{2}+c_3\left(I-I^*-I^*\ln\frac{I^*}{I}\right)+\frac{(U-U^*)^2}{2}. \tag{24}$$

By (18), (21) and (23), it is seen that

$$\begin{aligned}\mathbf{L}V_5\le&\big[-\theta_1^2\mu+\theta_1g(U^*)+\theta_1^2\sigma_1^2+c_3\theta_1\alpha\big(1-f(U^*)\big)+c_3\theta_1\Lambda\big](S-S^*)^2\\ &+\Big\{-\mu+(1+\theta_1)g(U^*)+(\theta_1S^*+I^*)(\bar{g}-g(U^*))+c_3g(U^*)+\sigma_2^2\\ &\quad+c_3\theta_1\alpha\big[1-f(U^*)\big]+\delta U^*\Big\}(I-I^*)^2\\ &+\left(-e+\sigma_3^2+\frac{\delta I^*U^*}{K+U^*}\right)(U-U^*)^2\\ &\Big\{c_3\theta_1\alpha f(U^*)-\big[\theta_1(g(U^*)+\mu)+\theta_1\mu\big]\Big\}(S-S^*)(I-I^*)\\ &+\theta_1g(U^*)(S^{*^2}+I^{*^2})+2g(U^*)I^{*^2}+\Big[\big(\theta_1S^*+I^*\big)\big(\bar{g}-g(U^*)\big)\Big]\left(\frac{1}{2}+I^{*^2}\right)\\ &+(\theta_1^2\sigma_1^2S^{*^2}+\sigma_2^2I^{*^2})+c_3g(U^*)\left(\frac{1}{2}+I^{*^2}\right)+c_3\big(\bar{g}-g(U^*)\big)I^*\\ &+\frac{1}{2}c_3I^*\sigma_2^2+c_3\theta_1\alpha\big(1-f(U^*)\big)\big(S^{*^2}+I^{*^2}\big)+c_3\theta_1\Lambda\left(S^{*^2}+\frac{1}{2S^{*^2}}\right)\\ &+\sigma_3^2U^{*^2}+\frac{\delta I^*U^*}{K+U^*}\left(U^{*^2}+\frac{1}{2}\right)+\delta U^*\left(I^{*^2}+\frac{1}{2}\right).\end{aligned}$$

Let $c_3 = \frac{g(U^*)+2\mu}{\alpha f(U^*)}$. Then

$$\begin{aligned}\mathbf{L}V_5 \leq& \left[-\theta_1^2\mu + \theta_1 g(U^*) + \theta_1^2\sigma_1^2 + c_3\theta_1\alpha\big(1-f(U^*)\big) + c_3\theta_1\Lambda\right](S-S^*)^2\\ &+ \Bigg\{ -\mu + (1+\theta_1)g(U^*) + (\theta_1 S^* + I^*)(\bar{g} - g(U^*)) + c_3 g(U^*) + \sigma_2^2\\ &\quad + c_3\theta_1\alpha\big[1-f(U^*)\big] + \delta U^* \Bigg\}(I-I^*)^2\\ &+ \left(-e + \sigma_3^2 + \frac{\delta I^* U^*}{K+U^*}\right)(U-U^*)^2\\ &+ \Theta.\end{aligned}$$

By (15), (19) and (22), it follows that

$$\begin{aligned}dV_5 =& \mathbf{L}V_5 dt + \big(\theta_1(S-S^*) + I - I^*\big)(\theta_1\sigma_1 S dW_1 + \sigma_2 I dW_2)\\ &+ c_3\left(1 - \frac{I^*}{I}\right)\sigma_2 I dW_2 + (U-U^*)\sigma_3 U dW_3.\end{aligned} \tag{25}$$

Now, by integrating (25) from 0 to t and taking expectations, one sees that

$$\limsup_{t\to+\infty} \frac{1}{t} E\int_0^t \left[\big(S(s)-S^*\big)^2 + \big(I(s)-I^*\big)^2 + \big(U(s)-U^*\big)^2\right] ds \leq \frac{\Theta}{\rho}.$$

This completes the proof. □

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