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Off-label use of epidemic modelling: assessing peer influence on academic performance and dropout

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Abstract
This paper examines the influence of social-cognitive factors such as self-efficacy, locus of control and negative peer pressure on the academic performance and dropout intentions of undergraduate students. To this purpose, an epidemiological modelling framework is employed, of concern being a 4-dimensional model which bears resemblance to a SEI model with two stages of infectivity. Sufficient conditions for the existence and stability of the equilibria are determined in terms of threshold parameters defined ad-hoc, similar in scope, definition and interpretation to the basic reproduction number of an epidemic model. We perform further numerical simulations to support and also explore our theoretical findings. We observe that the model exhibits rich dynamical behavior, such as multiple positive equilibria, backward bifurcations (which means that resit and dropout can become mainstays even if the resit reproduction number is less than one), transcritical bifurcations and Hopf bifurcations leading to oscillatory solutions.

Keywords Epidemic modelling · Compartmental model · Student dropout · Locus of control · Self-efficacy · Peer influence · Dropout reproduction number · Resit reproduction number · Stability.

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1 Introduction

Among the factors that influence the academic performance of undergraduate students, three of the most prominent are locus of control (LOC), self-efficacy (S-E) and peer influence. Locus of control, defined in Rotter [14] as being the degree to which people believe that the outcomes of the events occurring in their lives are contingent on their own actions (internal control) or rather on the influence of external factors (external control), influences the persistence of students in reaching their academic goals and the way in which they react to course failure (Dollinger [7]). The reason for students with internal LOC performing better academically is their belief in the control they have over their own academic performance. This leads to self-motivation, which makes them more dedicated to their studies (Nowicki and Strickland [12]). In contrast, students with external LOC are less motivated, as they believe that they have little to no control over the grade they receive. Students with internal LOC are also more resilient, as they attribute course failure to a (remediable) lack of effort, while students with external LOC blame it on low ability or on matters outside their reach (Weiner [18]).

Self-efficacy, defined in Bandura [2] as the belief in one's own capacity to achieve a desired outcome, translates in an academic setting to the way in which students perceive their own abilities to fulfill academic tasks and achieve specific outcomes. Particularly, higher S-E is a necessary trait when confronted with course failure, as it translates into a higher commitment to resitting courses and persisting in one's academic studies.

Bandura's social cognitive theory [2] asserts that an individual acquire behaviors mainly by interacting with others, via social influence and external as well as internal reinforcement, rather than passively absorbing knowledge from ambient sources. Accordingly, especially when confronted with course failure, students' decisions on whether or not to further pursue their academic studies are influenced by their peers (Amdouni *et al.* [1], Hyman *et al.* [10]).

Compartmental models describing the transmission of communicable diseases have been found useful to describe a multitude of social interactions which essentially amount to a change of habit or attitude due to the influence of peers, such as the spread of alcoholism via mechanisms of imitation (Buonomo and Lacitignola [3]) or of eating disorders via college peer pressure (Gonzalez *et al.* [9]), the curtailment of the smoking habit (Sharomi and Gumel [15]) or of gang violence via effective programs in correctional centers (Nyabadza *et al.* [13]). The advantage of using such compartmental models is that the analytical and numerical tools used to study their behavior are already well-understood, owing to and drawing on an extensive body of work, and have the potential to yield results with significant predictive power, easily translatable into adequate public or academic policies.

This paper investigates the effects of LOC, S-E and peer pressure upon the academic performance and dropout intentions of college students, building upon prior work of Zhang *et al.* [19, 20]. To this purpose, an epidemiological modelling framework is employed, of interest being a 4-dimensional system of ODEs which resembles a SEI model with two stages of infectivity, although the similarity isn't perfect, since one of the compartments shares certain features of both the infective and the removed class. This model is then investigated from both a theoretical, stability-related, and a numerical viewpoint, threshold stability results of practical relevance being established.

A lower dimensional model keeping track only of passing, resit and makeup students has been investigated in Ma *et al.* [11] from the viewpoint of finding sufficient conditions for the stability of equilibria and investigating their bifurcation. An investigation of high school student dropout which primarily focuses on the effects on parental involvement but also on demographic effects and peer pressure has been performed in Amdouni *et al.* [1], stability and bifurcation results being presented, as well as extensive numerical simulations. The influence of certain academic and demographic variables upon the academic performance of Chinese students enrolled in a cooperative Bachelor's degree program in Pure and Applied Mathematics has been considered in Zhang *et al.* [20] via inferential and path analysis, as well as ODE modelling, the effects of negative peer influence being assessed via a stability analysis of the equilibria. The investigation started in Zhang *et al.* [20] has been continued in Zhang *et al.* [19], the bifurcation of the positive equilibria being characterized and concrete values for the model parameters being determined by means of processing real-world data from a Ghanaian university. See also Georgescu and Zhang [8] for further elaboration.

The remaining part of this paper is organized as follows. In Sect. 2, we introduce our 4-dimensional ODE compartmental model of interest along with several comments regarding related models treated in other references. In Sect. 3, we discuss the existence and stability of the equilibria in terms of threshold parameters related to the basic reproduction number employed in Mathematical Epidemiology. In Sect. 4, we numerically illustrate how the dynamics of the system is affected when changes in the parameters occur. Finally, Sect. 5 presents a discussion of our mathematical results from a more applied perspective, appropriate policy measures being suggested along with concluding remarks.

2 The model

To investigate and characterize the influence of S-E and LOC upon the academic performance and dropout intentions of undergraduate students, we shall employ a 4-dimensional compartmental model that keeps track of passing students (P), of students that have failed at least a course and have to take make-up examinations (M), students that failed make-up examinations and have to resit at least a course (R) and students who have dropped out, but are still a part of the community (D). In our model, we account for the (negative) influence of dropout student on all other students, with the caveat that the amount of influence depends on students' academic statuses, leading to the influence terms $\beta_0 PD$, $\beta_2 MD$ and $\beta_3 RD$ on passing, make-up and resit students, respectively, with possibly different coefficients, and of resit students on make-up students, leading to the influence term $\beta_1 MR$.

All the above-mentioned peer influence terms are modelled using bilinear (mass action) incidence, implicitly relying on homogeneous mixing of students and on the assumption that each additional student of an influencing group contributes linearly to the total peer pressure experienced by the influenced students. Typically, this choice allows for the easier use of the next generation method and of bifurcation techniques, at the cost of a certain loss of realism.

Although bilinear incidence allows for a straightforward interpretation of interaction rates, alternative formulations may sometimes be more accurate in social contexts. Saturating incidence functions can be used to model limited attention, resistance to persuasion or peer influence fatigue, while a normalized (frequency dependent) incidence can be employed to account for bounded persuasion rates in large populations. On the contrary, higher-powered incidence rates can be used to account for an increased rate of persuasion due to multiple exposures over a short time period (see, for instance, Buonomo and Lacitignola [3]). Such choices of incidence rates may in turn affect qualitative dynamics in a significant manner: saturation can suppress unbounded growth of influence terms, reduce the likelihood of a backward bifurcation or prevent the existence of multiple stable equilibria, whereas a normalized incidence may shift or even remove threshold phenomena.

We also account for dropout and course failure for reasons other than peer influence, as typified by the terms γR and κP, respectively. It is assumed that, due to a high internal LOC, certain make-up students pass all their make-up examinations and re-enter the passing compartment, as represented via the term σM, while a number of the resit students successfully pass all their failed courses and re-enter the passing compartment, as typified by the term ηR, due to their high S-E.

Our model, inspired by those discussed in Amdouni *et al.* [1] and Zhang *et al.* [20], can then be stated in the following form

$$\begin{aligned}
\dot{P} &= \Lambda + \eta R + \sigma M - \beta_0 P D - (\kappa + \mu) P, \\
\dot{M} &= \kappa P + \beta_0 P D - M(\beta_1 R + \beta_2 D) - (\sigma + \mu) M, \\
\dot{R} &= M(\beta_1 R + \beta_2 D) - \eta R - \beta_3 R D - (\gamma + \mu) R, \\
\dot{D} &= \beta_3 R D + \gamma R - \mu_1 D,
\end{aligned} \tag{1}$$

the significance of the model parameters being detailed in Table 1.

Note that our β's are composite parameters to be understood as products between contact rates and probabilities that bad academic habits are transmitted and, unlike those in Amdouni *et al.* [1] and Zhang *et al.* [20], our model does not rely upon a constant population assumption and does not undergo normalization (i.e., our variables are compartment sizes, not percentages of the total population). Apart from the different rates of enrollment and the subsequent lack of extraneous terms meant to ensure the conservation of the population size, our model also accounts for the negative influence of dropout students on passing students (not considered in either [1] or [20]), for dropout due to causes other than peer influence (not considered in [20]) and for the movement of students from the resit compartment back to the passing compartment (not considered in [1]), as already mentioned above.

Table 1 The significance of the parameters that appear in model (1)

Parameter	Description
$\frac{1}{\mu}$	Average number of years spent in college by a typical, non-dropout student.
μ_1	The rate at which dropouts leave the community.
κ	Exam failure rate due to causes other than peer influence.
σ	Passing rate of make-up examinations.
η	Passing rate of resit courses.
γ	Dropout rate due to causes other than peer influence
β_0	Average negative influence of dropout students on passing students.
β_1	Average negative influence of resit students on make-up students.
β_2	Average negative influence of dropout students on make-up students.
β_3	Average negative influence of dropout students on resit students.
Λ	Yearly enrollment rate

3 The existence and stability of the equilibria. The resit reproduction number

Due to the presence of the κP in the second equation of (1), there won't be any truly "ideal" (i.e., with passing students only) equilibrium state. Also, it certainly isn't realistic to assume that all students will pass all examinations, even when insulated from perturbing outside influence.

Subsequently, we shall refer to a state in which the dropout and resit compartments are empty as an *ideal state* (or *resit and dropout-free state*), to a state in which only the dropout compartment is empty as a *subideal state* (or *dropout-free state*) and to a state in which all compartments are nonempty as a *realistic* state. That is, an ideal state is a state with no serious academic failure, while a subideal state is a state in which some have to resit courses, but still no one drops out.

By denoting $\tilde{d} = \max(\mu, \mu_1)$, $N(t) = P(t) + M(t) + R(t) + D(t)$ and observing that $N'(t) \leq \Lambda - \tilde{d}N$, it is then easy to see that the set

$$\Omega = \left\{ (P, M, R, D) \in \mathbb{R}^4_+,\ P + M + R + D \leq \frac{\Lambda}{\tilde{d}} \right\}$$

is positively invariant and absorbing. This shall be from now on our feasibility set.

3.1 The existence and stability of the ideal equilibrium

We now characterize the existence and stability of the ideal equilibrium

$$E_0 = \left(\frac{\Lambda(\mu + \sigma)}{\mu(\kappa + \mu + \sigma)}, \frac{\Lambda \kappa}{\mu(\kappa + \mu + \sigma)}, 0, 0 \right). \tag{2}$$

To determine the stability of E_0, let us use the next generation method with R and D as infective compartments. This yields the expression of the involved reproduction number, hereinafter called resit reproduction number and denoted $\mathcal{R}_{0R}$, as being

$$\mathcal{R}_{0R} = \beta_1 M_0 \cdot \frac{1}{\eta+\gamma+\mu} + \frac{\gamma}{\eta+\gamma+\mu} \cdot \beta_2 M_0 \cdot \frac{1}{\mu_1}. \tag{3}$$

While the first term can be written as the product between the average number of "successful" contacts $\beta_1 M_0$ of a single resit individual and the average time $\frac{1}{\eta+\gamma+\mu}$ spent in the resit compartment, the second term can be understood as the product between the probability $\frac{\gamma}{\eta+\gamma+\mu}$ that a resit individual enters the dropout compartment, the average number of "successful" contacts $\beta_2 M_0$ of a single dropout individual and the average time $\frac{1}{\mu_1}$ spent in the dropout compartment. As seen in the above, $\mathcal{R}_{0R}$ can then be interpreted as the average number of make-up students influenced by a single member of the resit compartment to resit the course during his or her entire academic journey, which may include a stage in the dropout compartment. By expanding (3), one sees that

$$\mathcal{R}_{0R} = \frac{\Lambda\kappa}{\mu(\sigma+\mu+\kappa)(\eta+\gamma+\mu)} \left[\beta_1 + \beta_2 \frac{\gamma}{\mu_1}\right]. \tag{4}$$

Note that $\mathcal{R}_{0R}$ is increasing with respect to γ if $\beta_2(\eta+\mu) - \beta_1\mu_1 > 0$ and decreasing if the opposite inequality holds; if the rate μ_1 at which the dropout students leave the community is high, then a high dropout rate due to unrelated reasons γ actually decreases the resit reproduction number, since the dropouts leave the community before being able to influence a significant amount of students.

For $\Lambda = \mu$ and $\gamma = 0$ (the assumptions in Zhang *et al.* [20]), one sees that

$$\mathcal{R}_{0R} = \frac{\beta_1\kappa}{(\sigma+\mu+\kappa)(\eta+\gamma+\mu)},$$

which is consistent with the expression given in [20], while for $\Lambda = \mu$, $\eta = 0$, $\mu_1 = \mu$ (the assumptions in Amdouni *et al.* [1]), we obtain that

$$\mathcal{R}_{0R} = \frac{\beta_1 M_0}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu} \cdot \beta_2 M_0 \cdot \frac{1}{\mu_1},$$

which is, *mutatis mutandis*, the expression of $\mathcal{R}_{0R}$ given in [1]. In regard to (4), the term $\beta_0 PD$ quantifying the number of students who have to take make-up examinations due to the negative influence of dropout students has no contribution to $\mathcal{R}_{0R}$ since the latter characterizes an average amount of resit students, not of make-up students. We also note that the subsystem of (1) involving only P and M for $R = D = 0$, written as

$$\dot{P} = \Lambda + \sigma M - (\kappa + \mu)P,$$
$$\dot{M} = \kappa P - (\sigma + \mu)M \tag{5}$$

has (P_0, M_0) as a globally asymptotically stable, unique solution, so that assumption (A5) of van den Driessche and Watmough [16] holds. As a consequence of [16, Theorem 2], we obtain the following stability result.

Theorem 3.1 *The resit and dropout-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_{0R} < 1$ and unstable if $\mathcal{R}_{0R} > 1$.*

As it will be observed in Subsect. 3.3, a realistic equilibrium may exist even if $\mathcal{R}_{0R} < 1$, for a certain portion of the parameter space, and consequently the stability of E_0 is not global for that portion of the parameter space.

3.2 The existence and stability of the subideal equilibrium. The dropout reproduction number

Let us now discuss the existence and stability of the subideal equilibrium. Note first that due to the presence of the γR term in the fourth equation of (1), there will be no such equilibrium unless $\gamma = 0$.

Let us then suppose for the moment that $\gamma = 0$ (which modifies both the model and the associated reproduction numbers accordingly) and find the subsequent subideal equilibrium $E_1 = (P_1, M_1, R_1, 0)$. Note first that in this case the expression of the resit reproduction number given in (4) reduces to

$$\mathcal{R}_{0R} = \frac{\Lambda\kappa\beta_1}{\mu(\sigma + \mu + \kappa)(\eta + \mu)}.$$

From the third equation, one sees that

$$M_1 = \frac{\eta + \mu}{\beta_1},$$

(there is no dropout-free equilibrium if $\beta_1 = 0$) and via substitution one sees that

$$P_1 = \frac{1}{\kappa}\frac{\eta + \mu}{\beta_1}\left[\frac{\eta + \mu}{\eta + \kappa + \mu}(\sigma + \kappa + \mu)\left(\mathcal{R}_{0R} - 1\right) + \mu\right]$$
$$R_1 = \frac{\eta + \mu}{\eta + \kappa + \mu} \cdot \frac{\sigma + \kappa + \mu}{\beta_1}\left(\mathcal{R}_{0R} - 1\right).$$

It is then easily seen that the dropout-free equilibrium exists whenever $\mathcal{R}_{0R} > 1$. To determine the stability of E_1, let us use again the next generation method, this time with D alone as the infective compartment, which gives the expression of the involved reproduction number, hereinafter called the dropout reproduction number, as being

$$\mathcal{R}_{0D} = \frac{\beta_3 R_1}{\mu_1} = \frac{\beta_3(\eta + \mu)}{\eta + \kappa + \mu} \cdot \frac{\sigma + \kappa + \mu}{\beta_1\mu_1}\left(\mathcal{R}_{0R} - 1\right), \tag{6}$$

or, by expanding $\mathcal{R}_{0R}$,

$$\mathcal{R}_{0D} = \frac{\beta_3}{\mu_1}\frac{\Lambda\kappa\beta_1 - \mu(\sigma+\kappa+\mu)(\eta+\mu)}{\mu(\eta+\kappa+\mu)\beta_1}. \tag{7}$$

Note that the second expression given in (6) is similar to the one given in Zhang *et al.* [20], although the concrete expression of $\mathcal{R}_{0R}$ is different.

Let us now discuss the stability of E_1. To this purpose, we shall evaluate the eigenvalues of the Jacobian at E_1. One of its eigenvalues is

$$\lambda_1 = \beta_3 R_1 - \mu_1 = \mu_1(\mathcal{R}_{0D} - 1),$$

the remaining eigenvalues being the roots of the equation

$$\begin{vmatrix} -(\kappa+\mu)-\lambda & \sigma & \eta \\ \kappa & -\left(\frac{\mu_1\beta_1}{\beta_3}\mathcal{R}_{0D} + \mu + \sigma\right) - \lambda & -(\eta+\mu) \\ 0 & \frac{\mu_1\beta_1}{\beta_3}\mathcal{R}_{0D} & -\lambda \end{vmatrix} = 0. \tag{8}$$

Proceeding then as in Zhang *et al.* [20, Theorem 3.2], one observes that the Routh-Hurwitz conditions are satisfied and all eigenvalues of the Jacobian have negative real parts for $\mathcal{R}_{0D} < 1$, while $\lambda_1 > 0$ if $\mathcal{R}_{0D} > 1$. Consequently, the following stability result holds.

Theorem 3.2 *The dropout-free equilibrium E_1 is locally asymptotically stable if $\mathcal{R}_{0D} < 1$ and unstable if $\mathcal{R}_{0D} > 1$.*

3.3 The existence and stability of the realistic equilibrium

We now characterize the existence of the resit and dropout-persistent equilibrium (shortened from now on as RDPE and understood as being the realistic equilibrium)

$$E_2 = (p^*, m^*, r^*, d^*).$$

Note first that its coordinates need to satisfy the following equilibrium relations

$$\begin{aligned} \Lambda + \eta r^* + \sigma m^* - \beta_0 p^* d^* - (\kappa+\mu)p^* &= 0 \\ \kappa p^* + \beta_0 p^* d^* - m^*(\beta_1 r^* + \beta_2 d^*) - (\sigma+\mu)m^* &= 0 \\ m^*(\beta_1 r^* + \beta_2 d^*) - \beta_3 r^* d^* - (\eta+\mu+\gamma)r^* &= 0 \\ \beta_3 r^* d^* + \gamma r^* - \mu_1 d^* &= 0 \end{aligned} \tag{9}$$

From the last equation of (9), one sees that

$$r^* = \frac{\mu_1 d^*}{\beta_3 d^* + \gamma}. \tag{10}$$

Substituting (10) into the third equation of (9) leads to

$$m^* = \frac{\mu_1(\beta_3 d^* + \eta + \mu + \gamma)}{\beta_1\mu_1 + \beta_2(\beta_3 d^* + \gamma)}. \tag{11}$$

Using now the second equation of (9), one obtains

$$p^* = \frac{\mu_1(\beta_3 d^* + \eta + \mu + \gamma)((\beta_3 d^* + \gamma)(\beta_2 d^* + \sigma + \mu) + \beta_1\mu_1 d^*)}{(\kappa + \beta_0 d^*)(\beta_3 d^* + \gamma)(\beta_1\mu_1 + \beta_2(\beta_3 d^* + \gamma))}. \tag{12}$$

One then obtains that d^* is the root of

$$Az^4 + Bz^3 + Cz^2 + Dz + E = 0, \tag{13}$$

the full expressions of A, B, C, D and E being given in the Appendix.

From (10), (11) and (12), one sees that to each positive root of (13) there corresponds a realistic equilibrium. Unfortunately, due to the specific expressions of the coefficients A, B, C, D, E, a complete characterization for the existence of the realistic equilibrium seems out of reach. Under the most general conditions (i.e., none of the parameters is zero), there are at most four realistic equilibria. Let us still note that

$$E = \gamma\mu\mu_1(\mu + \kappa + \sigma)(\mu + \gamma + \eta)\left(1 - \mathcal{R}_{0R}\right).$$

One may then obtain the following existence result.

Theorem 3.3 *If all model parameters are nonzero and*

$$\mathcal{R}_{0R} > 1, \tag{14}$$

then there is at least a RDPE of (1).

However, this does not exclude the situation in which a realistic equilibrium (or more of those, even though some of them may coincide; note that E/A gives the product of all roots, real or complex) may exist even if $\mathcal{R}_{0R} < 1$. This is, however, natural, since the dropout for reasons unrelated to peer influence at rate γ has the potential to affect the stability of the ideal equilibrium, as the dropout students are not assumed to leave the community immediately.

In some sense, the parameters β_0, β_2, β_3, μ_1 and γ can be thought as being the most responsible for the multiplicity of the realistic equilibria. If any of them is 0, then the positive d^* to be determined becomes the root of a third-degree equation with uncertain sign of the leading term, potentially with a smaller number of solutions. However, μ_1 cannot realistically be 0, since the dropout students will eventually leave the community. Also, one may assume that β_0 is less that both β_2 and β_3, since the dropout students will have less of a bad influence on those who have their academic life in order.

The existence and multiplicity of realistic equilibria can be understood in terms of the (interaction-driven) failure reinforcement mechanisms embedded in the structure of the system (1), rather than as a purely algebraic consequence of the comparatively higher degree of the equation (13). In particular, the equilibrium relations (9) reveal that the compartments M, R and D are coupled through a hierarchy of failure-reinforcing resit-replenishing loops.

The primary loop, $R \to M \to R$, is driven by the interaction term $\beta_1 MR$, whereby resit students increase the likelihood that make-up students will themselves enter the resit compartment. This mechanism is explicitly captured via the resit reproduction number $\mathcal{R}_{0R}$ and governs whether resit can persist in the absence of dropout. However, once dropout is present, a secondary amplification loop $R \to D \to M \to R$ emerges, driven by the terms $\beta_2 MD$ and $\beta_3 RD$, and feeds back into the resit and make-up subsystem by sustaining M and replenishing R. In fact, the rate μ_1 at which dropouts leave the community acts like a memory eraser for academic failure: fast departure of dropout students collapses the secondary loop and is therefore likely to decrease the multiplicity of the equilibria.

From this perspective, each positive root of (13) corresponds to a distinct balance between recovery mechanisms, represented by passing make-up examinations σM and passing resit courses ηR, and influence-driven persistence of academic failure generated by negative peer interactions. When dropout students remain socially active for sufficiently long periods to exert enough negative influence, the secondary loop may sustain nonzero levels of academic failure even when the intrinsic resit dynamics would otherwise dwindle. This explains why realistic equilibria may exist even for subunitary values of $\mathcal{R}_{0R}$ and why several such equilibria may coexist for the same parameter set, as recovery mechanisms and peer influence loops may balance themselves out at several distinct intensity levels.

Since the interaction term $\beta_0 PD$ does not create new resit entries directly, but rather enlarges the make-up pool M and indirectly strengthens in the process the $M \to R$ and $M \to D$ pathways, it is seen that β_0 does not control whether academic failure spreads (in particular, this is why the resit reproduction number $\mathcal{R}_{0R}$ does not depend upon β_0), but rather how large the basins of attraction of realistic equilibria become. In practical terms, this means that similar institutional settings may evolve toward very different long term outcomes depending on the initial exposure of students to resit and dropout.

In order to obtain further information about the existence of the realistic equilibria, we shall now consider several particular cases. Note that the parameters not explicitly mentioned to be 0 are tacitly assumed to be nonzero.

The case $\gamma = 0$

Assume that $\gamma = 0$, that is, there is no dropout due to causes other than peer influence. One then sees that d^* is the root of the reduced equation

$$A_1 z^3 + B_1 z^2 + C_1 z + D_1 = 0, \tag{15}$$

where

$$A_1 = \beta_0 \beta_2 \beta_3^2 \mu_1$$

$$
\begin{aligned}
B_1 &= (\beta_0\beta_1{\mu_1}^2 - \Lambda\beta_0\beta_2\beta_3 + \beta_2\beta_3\kappa\mu_1 + \beta_0\beta_2\mu\mu_1 + \beta_0\beta_3\mu\mu_1 + \beta_2\beta_3\mu\mu_1)\beta_3 \\
C_1 &= \beta_3^2\mu^2\mu_1 - \Lambda\beta_2\beta_3^2\kappa + \beta_1\beta_3\kappa{\mu_1}^2 + \beta_0\beta_1\mu{\mu_1}^2 + \beta_0\beta_3\mu^2\mu_1 + \beta_1\beta_3\mu{\mu_1}^2 \\
&\quad + \beta_2\beta_3\mu^2\mu_1 + \beta_3^2\kappa\mu\mu_1 + \beta_3^2\mu\mu_1\sigma - \Lambda\beta_0\beta_1\beta_3\mu_1 + \beta_0\beta_3\eta\mu\mu_1 + \beta_2\beta_3\eta\mu\mu_1 \\
&\quad + \beta_2\beta_3\kappa\mu\mu_1 \\
D_1 &= (\beta_3\mu^3 + \beta_3\mu^2\sigma + \beta_3\eta\mu^2 + \beta_3\kappa\mu^2 + \beta_1\mu^2\mu_1 + \beta_3\eta\kappa\mu + \beta_1\eta\mu\mu_1 + \beta_1\kappa\mu\mu_1 \\
&\quad + \beta_3\eta\mu\sigma - \Lambda\beta_1\beta_3\kappa)\mu_1
\end{aligned}
$$

While $A_1 > 0$, the signs of B_1 and C_1 are uncertain. Particularly, it is seen that different values of the enrollment rate Λ can make B_1 and C_1 both positive or negative. Note, however, that

$$
D_1 = \beta_1\mu_1^2\mu(\eta + \kappa + \mu)\,(1 - \mathcal{R}_{0D})\,.
$$

This implies that (15) always has a positive solution provided that $\mathcal{R}_{0D} > 1$. Combined with (10), (11) and (12), this implies that there is always at least a RDPE provided that $\mathcal{R}_{0D} > 1$. Consequently, the following existence result holds.

Theorem 3.4 *If $\gamma = 0$ and $\mathcal{R}_{0D} > 1$, then there is at least a RDPE of* (1).

The reason why the above result if formulated in terms of $\mathcal{R}_{0D}$, rather than in terms of $\mathcal{R}_{0R}$, as the "generic" result, Theorem 3.3 is that the free term D_1 in (15), essential for the statement of Theorem 3.4, corresponds to D in (13), rather than to the free term E, used to derive Theorem 3.3. Note that the sufficient condition $\mathcal{R}_{0D} > 1$ is strictly stronger than condition (14), necessary when γ is nonzero, and one may consequently conclude that dropout for reasons unrelated to peer influence promotes the existence of a realistic equilibrium, which is perhaps not unexpected. Also, if $B_1 < 0$ and $C_1 < 0$ (which happens, for instance, when the enrollment rate Λ is large enough), then the positive equilibrium obtained in Theorem 3.4 is unique, by an easy analysis of the monotonicity of the function defined by the left-hand side of (15).

The case $\gamma = \beta_0 = 0$

The case $\beta_0 = 0$, that is, dropout students have no influence on passing students, does not bring further insights all by itself. In this case, d^* is still the root of a cubic equation with rather complicated coefficients and only an existence result that is parallel to Theorem 3.3 is available, in spite of the particularization of β_0.

To obtain more meaningful insights, let us then suppose that $\gamma = \beta_0 = 0$, that is, there is no dropout due to causes other than peer influence and also dropout students have no influence on passing students. In this case, we are left with studying a system that is close to the normalized version of the one studied in Zhang *et al.* [20], but without the constant population assumption, since it features a different enrollment rate and does not feature any term that is linear in D in the first equation. One obtains that d^* is the root of

$$
A_2 z^2 + B_2 z + C_2 = 0, \tag{16}
$$

with

$$A_2 = \beta_2\beta_3^2\mu_1(\kappa+\mu), \quad C_2 = \beta_1\mu_1^2\mu(\eta+\kappa+\mu)\,(1-\mathcal{R}_{0D})$$
$$B_2 = \beta_1\beta_3(\mu+\kappa)\mu_1^2 + \mu_1\mu\beta_3\,(\beta_3(\mu+\sigma+\kappa)+\beta_2(\mu+\eta+\kappa)) - \Lambda\beta_2\beta_3^2\kappa.$$

Let us denote

$$\mathcal{R}_{0D}^* = 1 - \frac{B_2^2}{4\mu_1^3\mu\beta_1\beta_2\beta_3^2(\kappa+\mu)(\mu+\eta+\kappa)},$$
$$D = \mu\,(\beta_3(\mu+\sigma+\kappa)+\beta_2(\mu+\eta+\kappa))\,; \quad \mu_1^* = \frac{-D+\sqrt{D^2+4\kappa\Lambda\beta_1\beta_2\beta_3}}{2\beta_1(\mu+\kappa)}$$

We then obtain the following characterization result, parallel to Theorem 3.3 of Zhang *et al.* [20] and related to the considerations in the electronic supplementary material of Amdouni *et al.* [1], Section B.

Theorem 3.5 *The following statements regarding the existence and multiplicity of the RDPE of* (1) *hold.*

- **(i)** *If* $\mathcal{R}_{0D} > 1$, *then there is a unique RDPE of (1), given by* $I^* = (p^*, m^*, r^*, d^*)$, *where* $d^* = \dfrac{-B_2+\sqrt{B_2^2-4A_2C_2}}{2A_2}$.
- **(ii)** *If* $\mathcal{R}_{0D} = 1$ *and* $\mu_1 < \mu_1^*$, *then there is a unique RDPE of (1), given by* $I^* = (p^*, m^*, r^*, d^*)$, *where* $d^* = \dfrac{-B_2}{A_2}$.
- **(iii)** *If* $\mathcal{R}_{0D} = \mathcal{R}_{0D}^*$ *and* $\mu_1 < \mu_1^*$, *then there is a unique RDPE of (1), given by* $I^* = (p^*, m^*, r^*, d^*)$, *where* $d^* = \dfrac{-B_2}{2A_2}$.
- **(iv)** *If* $\mathcal{R}_{0D}^* < \mathcal{R}_{0D} < 1$ *and* $\mu_1 < \mu_1^*$, *then there are two RDPEs of (1), given by* $I_1^* = (p_1^*, m_1^*, r_1^*, d_1^*)$ *and* $I_2^* = (p_2^*, m_2^*, r_2^*, d_2^*)$, *where* $d_{1,2}^* = \dfrac{-B_2 \pm \sqrt{B_2^2-4A_2C_2}}{2A_2}$.
- **(v)** *If* $\mathcal{R}_{0D} \le 1$ *and* $\mu_1 \ge \mu_1^*$, *then there is no RDPE of (1).*
- **(vi)** *If* $\mathcal{R}_{0D} < \mathcal{R}_{0D}^*$, *then there is no RDPE of (1).*

Note that the related investigation in Amdouni *et al.* [1] is made in terms of $\mathcal{R}_{0D}$ only and it consequently does not delineate between several situations occurring when $\mathcal{R}_{0D}$ is subunitary.

The particular cases $\gamma = 0$ and $\gamma = \beta_0 = 0$ were analyzed in order to obtain sharper analytical insights under simplified conditions and are intended as illustrative situations that help us understand specific dynamics within the model, rather than as being fully representative for the dynamics of the complete model. For instance, the assumption $\gamma = 0$ suppresses dropout due to causes unrelated to peer influence, allowing us to focus solely on peer-related dynamics. Similarly, setting $\beta_0 = 0$ removes the negative influence of dropout students on passing students, simplifying the stability and multiplicity analysis. While these assumptions provide valuable insights, the dynamics of the full model should be considered for a comprehensive and reliable understanding of real-world scenarios.

The specific forms of statements (ii)-(v) in Theorem 3.5 suggests that a backward bifurcation may occur at $\mathcal{R}_{0D} = 1$ provided that suitable sufficient conditions are satisfied. To show that this is indeed the case, let us first align our notations to those of [4] by denoting $P = x_1$, $M = x_2$, $R = x_3$ and $D = x_4$. Consequently, (1) can then be restated as

$$\begin{aligned}
\frac{dx_1}{dt} &= \Lambda + \eta x_3 + \sigma x_2 - (\kappa + \mu)x_1 := h_1,\\
\frac{dx_2}{dt} &= \kappa x_1 - x_2(\beta_1 x_3 + \beta_2 x_4) - (\sigma + \mu)x_2 := h_2,\\
\frac{dx_3}{dt} &= x_2(\beta_1 x_3 + \beta_2 x_4) - \beta_3 x_3 x_4 - (\eta + \mu)x_3 := h_3,\\
\frac{dx_4}{dt} &= \beta_3 x_3 x_4 - \mu_1 x_4 := h_4,
\end{aligned} \tag{17}$$

or, in vector form, $\dfrac{dX}{dt} = H(X)$, with

$$X = (x_1, x_2, x_3, x_4)^T, \quad \text{and} \quad H = (h_1, h_2, h_3, h_4)^T.$$

Let β_3 be the bifurcation parameter, as we are primarily interested in discussing the influence of dropout students on the other students. Using (7), one sees that the critical value of β_3 is

$$\beta_3^* = \frac{\mu\mu_1(\eta + \kappa + \mu)\beta_1}{\Lambda\kappa\beta_1 - \mu(\sigma + \kappa + \mu)(\eta + \mu)}. \tag{18}$$

The Jacobian matrix at the dropout-free equilibrium E_1 evaluated for $\beta_3 = \beta_3^*$ is

$$J(E_1)|_{\beta_3=\beta_3^*} = \begin{pmatrix} -(\kappa + \mu) & \sigma & \eta & 0 \\ \kappa & -\left(\frac{\beta_1\mu_1}{\beta_3^*} + \mu + \sigma\right) & -(\eta + \mu) & -\frac{\beta_2(\eta+\mu)}{\beta_1} \\ 0 & \frac{\beta_1\mu_1}{\beta_3^*} & 0 & \frac{\beta_2(\eta+\mu)}{\beta_1} - \mu_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A right eigenvector $w = (w_1, w_2, w_3, w_4)^T$ and a left eigenvector $v = (v_1, v_2, v_3, v_4)^T$ of this matrix associated with the eigenvalue 0 are then given by

$$\begin{aligned}
w_1 &= \frac{\beta_3^*\left(\beta_1\mu_1 - \beta_2(\eta + \mu)\right)(\sigma - \eta)\,\mu - \beta_1^2\eta\mu_1^2}{{\beta_1}^2\mu_1\mu\,(\eta + \kappa + \mu)},\\
w_2 &= \frac{\beta_3^*\left(\beta_1\mu_1 - \beta_2(\eta + \mu)\right)}{{\beta_1}^2\mu_1}, \quad w_4 = 1,\\
w_3 &= \frac{-\beta_3^*\left(\beta_1\mu_1 - \beta_2(\eta + \mu)\right)\mu\,(\mu + \kappa + \sigma) - \beta_1^2\mu_1^2(\kappa + \mu)}{{\beta_1}^2\mu_1\mu\,(\eta + \kappa + \mu)},\\
v_1 &= 0,\ v_2 = 0,\ v_3 = 0,\ v_4 = 1.
\end{aligned} \tag{19}$$

To use Theorem 4.1 of Castillo-Chavez and Song [4], we need to determine the signs of

$$a = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 h_k}{\partial x_i \partial x_j}(E_1, \beta_3^*), \quad b = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 h_k}{\partial x_i \partial \beta_3^*}(E_1, \beta_3^*). \tag{20}$$

From (17) and (6), the only non-zero partial derivatives of h_4 in (20) are

$$\frac{\partial^2 h_4}{\partial x_3 \partial x_4}(E_1, \beta_3^*) = \frac{\partial^2 h_4}{\partial x_4 \partial x_3}(E_1, \beta_3^*) = \beta_3^*, \qquad \frac{\partial^2 h_4}{\partial x_4 \partial \beta_3^*}(E_1, \beta_3^*) = \frac{\mu_1}{\beta_3^*}. \tag{21}$$

Also, from (19), the only nonzero products in (20) are obtained for $k = 4$. It follows that

$$\begin{aligned} a &= v_4 w_3 w_4 \frac{\partial^2 h_4}{\partial x_3 \partial x_4}(E_1, \beta_3^*) + v_4 w_4 w_3 \frac{\partial^2 h_4}{\partial x_4 \partial x_3}(E_1, \beta_3^*) = 2 w_3 \beta_3^* \\ b &= v_4 w_4 \frac{\partial^2 h_4}{\partial x_4 \partial \beta_3^*}(E_1, \beta_3^*) = \frac{\mu_1}{\beta_3^*}. \end{aligned}$$

It is then seen that $b > 0$, while the sign of a is variable and can be discussed in terms of other parameters. By using Theorem 4.1 of Castillo-Chavez and Song [4] we are now able to characterize the backward bifurcation that occurs at $\mathcal{R}_{0D} = 1$.

Theorem 3.6 *The RDPE of* (1) *is locally asymptotically stable for* $\mathcal{R}_{0D} > 1$ *but close to 1 and a backward bifurcation occurs at* $\mathcal{R}_{0D} = 1$ *provided that one of the following equivalent conditions holds.*

(i) *The dropout rate remains under a critical value, with*

$$\mu_1 < \mu_1^{**} = \frac{\beta_2(\eta+\mu)(\eta+\kappa+\mu)(\sigma+\kappa+\mu)\mu^2}{\beta_1 \left\{(\kappa+\mu)[\Lambda\kappa\beta_1 - \mu(\sigma+\kappa+\mu)(\eta+\mu)] + \mu^2(\eta+\kappa+\mu)(\sigma+\kappa+\mu)\right\}}.$$

(ii) *The level of negative social influence of resit on make-up students remains under a critical value, with*

$$\beta_1 < \beta_1^{**} = \frac{b_1 + \sqrt{b_1^2 + 4a_1c_1}}{2a_1}$$

where

$$\begin{aligned} a_1 &= \Lambda\kappa(\kappa+\mu)\mu_1, \quad b_1 = \kappa\eta(\sigma+\kappa+\mu)\mu\mu_1 \\ c_1 &= -\beta_2(\eta+\mu)(\sigma+\kappa+\mu)(\kappa+\eta+\mu)\mu^2. \end{aligned}$$

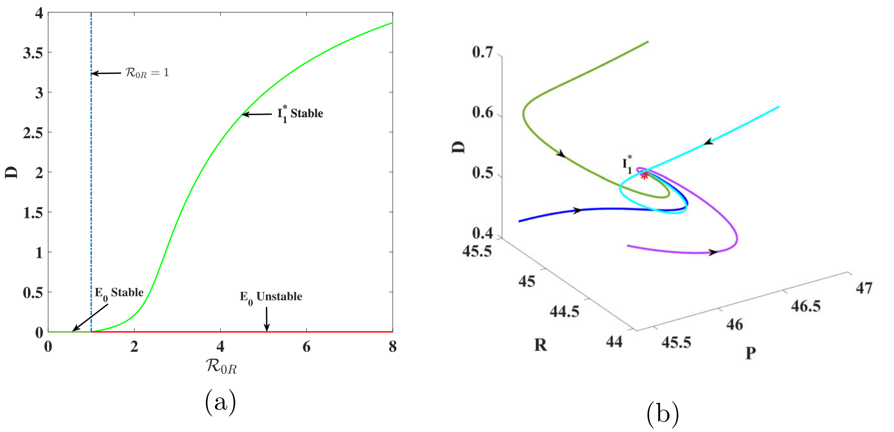


Fig. 1 (a) Depiction of a transcritical bifurcation; (b) Dynamics of solution trajectories at $\beta_1 = 0.01$ ($\mathcal{R}_{0R} = 2.3431$)

4 Numerical results

In this section, we numerically simulate the behavior of model (1), complementing the abstract results given above and outlining certain specific dynamical features. Note, however, that the choices of parameters are illustrative, rather than motivated by a concrete, particular situation.

4.1 Stability and bifurcation

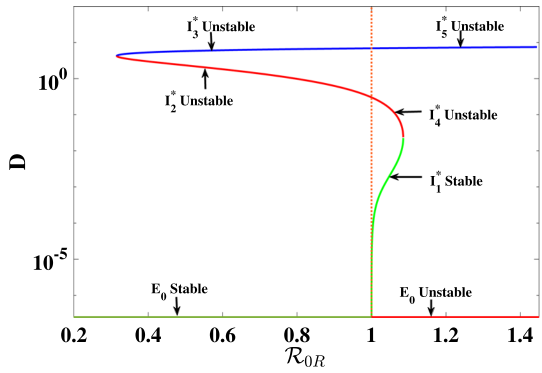
We start by exploring the nonlinear dynamical behaviour arising from the variation of model parameters, with a specific focus on further commenting upon stability properties and and their changes yielding bifurcations. Note that the scope of the numerical bifurcation results (bifurcations that occur at $\mathcal{R}_{0R} = 1$) is distinct from the scope of Theorem 3.6 (a bifurcation that may occur at $\mathcal{R}_{0D} = 1$).

For this purpose, consider $\beta_0 = 0.1$, $\beta_1 \in (0.004, 0.0341)$, $\beta_2 = 0.0134$, $\beta_3 = 0.02$, $\sigma = 0.51$, $\mu_1 = 1$, $\Lambda = 12$, $\mu = 0.1$, $\gamma = 0.001$, $\kappa = 0.5$, $\eta = 0.13$, so that $\mathcal{R}_{0R} \in (0.9391, 8)$. It can then be verified that in equation (13) $B > 0$, $C > 0$, while D changes its sign from positive to negative. Hence, according to Descartes' rule of signs, there exists no positive real root for $\mathcal{R}_{0R} < 1$ and a unique realistic equilibrium I_1^* for $\mathcal{R}_{0R} > 1$. Fig. 1 (a) subsequently shows the existence of a transcritical bifurcation as $\mathcal{R}_{0R}$ crosses unity. It is also observed that I_1^* is stable.

Further, we fix $\beta_1 = 0.01$ ($\mathcal{R}_{0R} = 2.3431$) and plot the trajectories corresponding to different initial conditions, as seen in Fig. 1 (b). We note that in each case the trajectories converge to $I_1^* = (46.2953, 23.7163, 45.2382, 0.475012)$, confirming the asymptotic stability of I_1^* numerically.

We next choose $\beta_0 = 0.2$, $\beta_1 \in (0.01527, 0.07)$, $\beta_2 = 0.034$, $\beta_3 = 0.5$, $\sigma = 0.41$, $\mu_1 = 0.8$, $\Lambda = 10$, $\mu = 0.1$, $\gamma = 0.001$, $\kappa = 0.5$, $\eta = 2.3$, so that $\mathcal{R}_{0R} \in$

Fig. 2 Bifurcation plot depicting the stability of multiple equilibria for $\mathcal{R}_{0R} < 1$ and $\mathcal{R}_{0R} > 1$, respectively



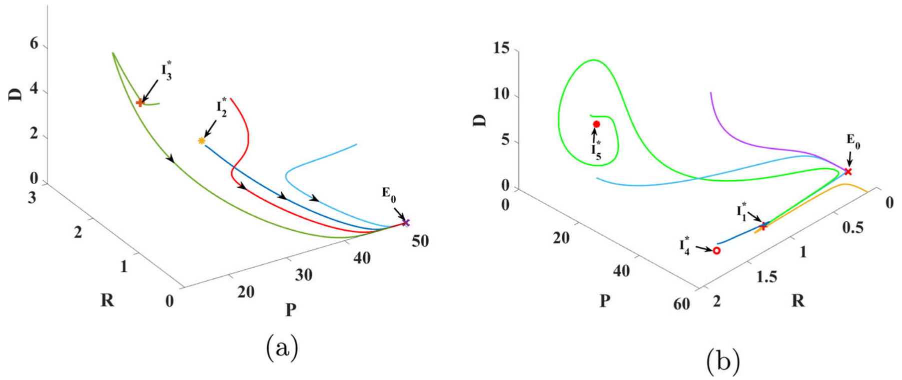


Fig. 3 (a) Phase portrait for $\beta_1 = 0.01994$ ($\mathcal{R}_{0R} = 0.4121$), (b) Phase portrait for $\beta_1 = 0.0515$ ($\mathcal{R}_{0R} = 1.0631$)

(0.3156, 1.4442). For these parameters, it can be verified that in equation (13) $B < 0$ and $C < 0$, while D changes its sign from positive to negative. According to Descartes' rule of signs, there are two realistic equilibria I_2^* and I_3^* for $\mathcal{R}_{0R} < 1$ and either three realistic equilibria (I_1^*, I_4^*, I_5^*) or a single one (I_5^*) for $\mathcal{R}_{0R} > 1$. Fig. 2 exhibits the existence of multiple roots for $\mathcal{R}_{0R} < 1$ and $\mathcal{R}_{0R} > 1$. It is observed that I_1^* is stable and $I_2^*, I_3^*, I_4^*, I_5^*$ are unstable for this range of β_1.

For fixed $\beta_1 = 0.01994$ ($\mathcal{R}_{0R} = 0.4121$), $I_2^* = (28.0244, 47.2207, 1.5989, 2.8945)$ and $I_3^* = (17.4151, 37.888, 1.59941, 5.38719)$ are unstable. Several trajectories corresponding to distinct initial conditions are plotted in Fig. 3 (a). The trajectories starting in the nearby region of I_2^* and I_3^* are moving away and approaching the ideal equilibrium $E_0 = (50.495, 49.505, 0, 0)$.

For another fixed value of β_1, namely $\beta_1 = 0.0515$ ($\mathcal{R}_{0R} = 1.0631$), $I_1^* = (52.3955, 46.5318, 1.04282, 0.00374316)$ is stable while the other realistic equilibria $I_4^* = (51.9793, 45.5902, 1.57077, 0.10746)$, $I_5^* = (12.8638, 29.4528, 1.59954, 7.01048)$ are unstable. This perhaps aligns to the generic idea that richer (in the susceptibles-like class P) equilibria have more chances to be stable. Several trajectories corresponding to distinct initial conditions are plotted in Fig. 3 (b). The trajectories

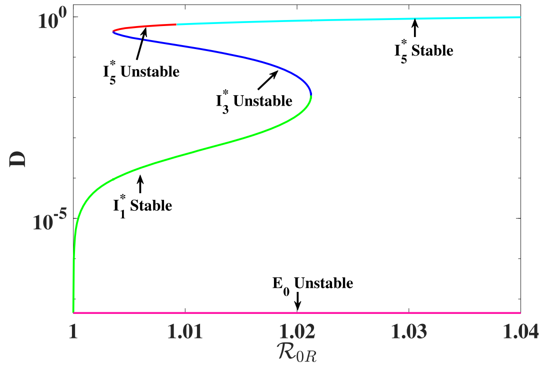


Fig. 4 Bifurcation plot depicting the stability of multiple equilibria for $\mathcal{R}_{0R} > 1$ and the occurrence of a Hopf bifurcation

starting in the nearby of I_4^* and I_5^* are moving away, approaching the realistic equilibrium I_1^*.

We now choose $\beta_0 = 0.01$, $\beta_1 \in (0.02, 0.025163)$, $\beta_2 = 0.0134$, $\beta_3 = 2$, $\sigma = 0.051$, $\mu_1 = 1$, $\Lambda = 12$, $\mu = 0.1$, $\gamma = 0.001$, $\kappa = 0.5$, $\eta = 2.13$, so that $\mathcal{R}_{0R} \in (0.8268, 1.04)$. It can be verified that that in equation (13) $B > 0$, $C < 0$, while D changes its sign from positive to negative. According to Descartes' rule of signs, there is no realistic equilibrium for $\mathcal{R}_{0R} < 1$ while for $\mathcal{R}_{0R} > 1$ a unique realistic equilibrium (I_1^*), three realistic equilibria (I_1^*, I_4^*, I_5^*) and again a unique realistic equilibrium (I_5^*) exist for this range of β_1.

In this regard, Fig. 4 depicts the existence of multiple roots for $\mathcal{R}_{0R} > 1$. It is observed that I_1^* is stable and I_4^* is unstable, while I_4^* switches stability from unstable to stable. At $\beta_1 = \beta_1^{crit} = 0.024412583043$ ($\mathcal{R}_{0R} = 1.00907$), $I_5^* = (28.6417, 84.4182, 0.499612, 0.644048)$ is unstable for $\beta_1 < \beta_1^{crit}$ and stable for $\beta_1 > \beta_1^{crit}$. The eigenvalues of the Jacobian at I_5^* are -1.5281, -0.7092, $\pm 0.0638i$.

As a result, a Hopf bifurcation occurs, the limit cycle around I_5^* being shown in Fig. 5 (a). The time series oscillation plots of P, R and D corresponding to the limit cycle at $\beta_1 = \beta_1^{crit}$ are shown in Fig. 5 (b). This is an important outcome of our numerical simulations, showing the oscillatory nature of the solutions, i.e. all population sizes continuously vary without settling to an equilibrium.

For fixed $\beta_1 = 0.024677$ ($\mathcal{R}_{0R} = 1.02$), $I_1^* = (28.4536, 83.0476, 0.499688, 0.799906)$ and $I_5^* = (29.2279, 90.2991, 0.437792, 0.00351875)$ are stable, while the other realistic equilibrium $I_4^* = (29.366, 89.8052, 0.492671, 0.0336133)$ is unstable. Several trajectories corresponding to distinct initial conditions, approaching either I_1^* or I_5^*, are plotted in Fig. 6.

4.2 A sensitivity analysis

In order to identify the impact of parameter variability upon the value of the resit reproduction number $\mathcal{R}_{0R}$ and find actionable insights for educational policy adjustments, we are led to performing a sensitivity analysis. To this purpose, we employ a one-at-a-time (OAT) parameter perturbation method, a robust and computationally efficient approach for identifying key driving parameters for nonlinear dynamics. Nine

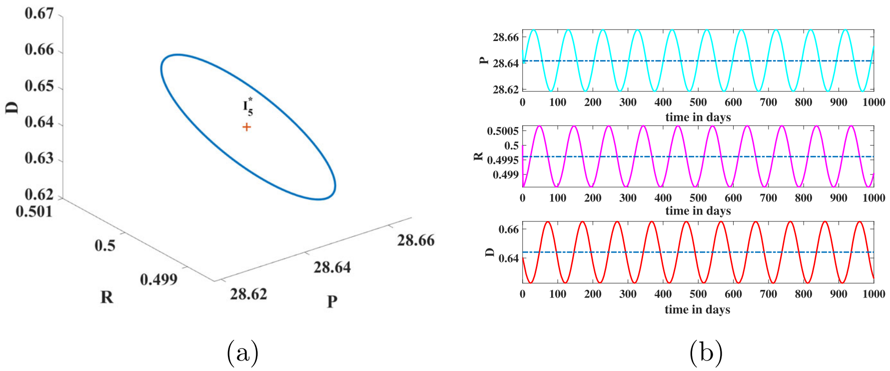
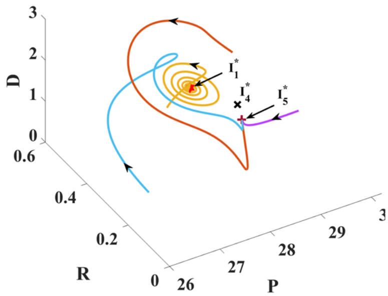


Fig. 5 (a) The existence of a periodic orbit around I_5^* at β_1^{crit}, (b) Oscillations of P, R and D corresponding to the periodic orbit

Fig. 6 Phase portrait for $\beta_1 = 0.024677$ ($\mathcal{R}_{0R} = 1.02$)



model parameters are accounted for, their baseline values $\Lambda = 12, \kappa = 0.25, \mu = 0.2, \sigma = 0.51, \eta = 0.13, \gamma = 0.001, \beta_1 = 0.01, \beta_2 = 0.0134, \mu_1 = 1$, being calibrated based via realistic (but hypothetical) academic scenarios.

To capture the impact of parameter variability, we introduce two standard metrics, namely the relative and absolute sensitivity, respectively, of a variable with respect to a parameter. The absolute sensitivity of a variable Q with respect to a parameter p is defined as $S_{abs} = \Delta Q/\Delta p$, where $\Delta p = p_{perturbed} - p_{base}$ (absolute change of the parameter p) and $\Delta Q = Q_{perturbed} - Q_{base}$ (absolute change of the variable Q), reflecting the raw change of the variable Q per unit change of the parameter p. While useful, the absolute sensitivity has certain shortcomings, including the facts that it is dependent on the units of the variable and parameter, respectively, and on their relative magnitudes. The relative sensitivity of a variable Q with respect to a parameter p is then defined as $S_{rel} = S_{abs} * (p_{base}/Q_{base})$, being a unit-agnostic normalized metric that quantifies the percentile change of the variable Q per percentile change in the parameter p, enabling direct cross-parameter comparisons of influence strength. For further sensitivity considerations, we shall from now on use relative sensitivity as our metric of interest.

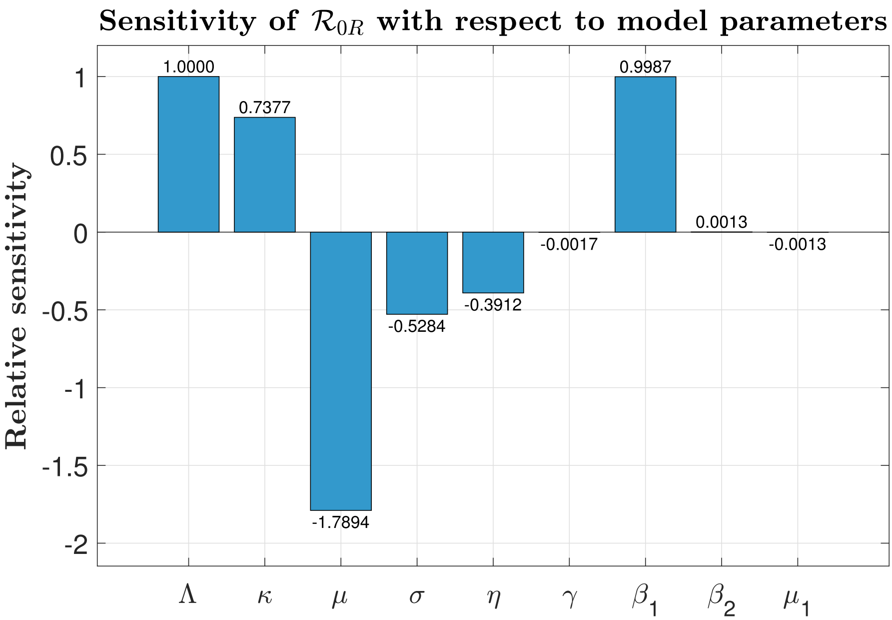


Fig. 7 Relative sensitivities of the resit reproduction number ($\mathcal{R}_{0R}$) with respect to nine key model parameters (Λ, κ, μ, σ, η, γ, β_1, β_2, μ_1).

For each parameter, we apply a 1% positive perturbation ($p_{perturbed} = p_{base} * 1.01$), while fixing all other parameters at their baseline values. A 1% perturbation is appropriate for sensitivity determinations and avoids distorting the nonlinear behavior of the model. The subsequent sensitivity analysis, whose numerical outcomes are presented in Fig. 7, yields three critical findings, which are in fact aligned with prior theoretical conclusions and numerical results.

Direction of parameter Influence

It is observed that Λ, κ, β_1, and β_2 exhibit positive relative sensitivities, which means that increases in these parameters raise the value of $\mathcal{R}_{0R}$, as they either expand the pool of at-risk students (Λ, κ) or amplify negative peer influence (β_1, β_2). It is also seen that μ, σ, η, γ and μ_1 exhibit negative relative sensitivities, which means that increases in the values of these parameters reduce the value $\mathcal{R}_{0R}$, as they either accelerate student attrition (μ, μ_1, γ) or improve academic recovery (σ, η).

Parameter importance ranking

As far as the parameter importance ranking is concerned, we can distinguish between highly influential, moderately influential and weakly influential parameters, respectively. The highly influential parameters are β_1 (the negative influence of resit students on make-up students), Λ (the yearly enrollment rate), and κ (exam failure rate due to causes other than peer influence), exhibiting the highest absolute values of relative

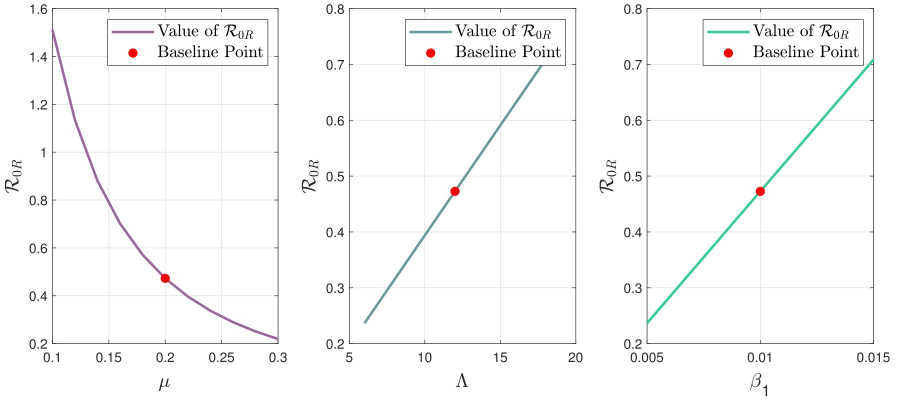


Fig. 8 Response curves of $\mathcal{R}_{0R}$ to the variation of the most influential parameters (β_1, Λ, μ)

sensitivity (typically $|S_{rel}| > 0.7$). This indicates that small changes in these parameters lead to substantial changes in $\mathcal{R}_{0R}$. It is also seen that μ (the inverse of the average number of years spent in college by a typical, non-dropout student) is also a highly influential parameter, but its variation is subject to academic and logistic constraints. The moderately influential parameters are σ (passing rate of make-up examinations) and η (passing rate of resit courses) show comparatively lesser relative sensitivities ($0.2 < |S_{rel}| < 0.5$), their influence on reducing academic failure being limited. The weakly influential parameters are: γ (dropout rate due to causes other than peer influence), β_2 (average negative influence of dropout students on make-up students), and μ_1 (the rate at which dropouts leave the community), which have relative sensitivities close to 0 ($|S_{rel}| < 0.1$). This finding is in fact consistent with the earlier remark that dropout students have limited long-term influence if they leave the community quickly.

Practical implications for policy design

The above sensitivity analysis identifies actionable levers to mitigate resit and dropout behavior. A possible course of action is to prioritize reducing β_1 via interventions targeting negative influences on resit students (e.g., peer mentoring programs, academic counseling) that can effectively lower the value of $\mathcal{R}_{0R}$ due to its high sensitivity with respect to β_1. A distinct avenue is to enhance academic recovery support by improving make-up (σ) and resit (η) passing rates via targeted tutoring or flexible assessment policies. This is projected to directly reduce $\mathcal{R}_{0R}$ by increasing student re-entry rate into the passing compartment.

Another possible approach is to manage enrollment and failure rates. While Λ and κ are highly influential, adjusting enrollment (Λ) may be encumbered by logistic constraints. Instead, reducing exam failure rate due to causes other than peer influence (κ) through curriculum adjustments or pre-enrollment support can provide a more feasible pathway to lowering the value of $\mathcal{R}_{0R}$.

For a more precise assessment of the necessary policy adjustments, we have pictured in Fig. 8 the response curves of $\mathcal{R}_{0R}$ to the variation of the three most influential parameters, as identified from Fig. 7 (β_1, Λ, μ), over the range of 50%-150% of their respective baseline values. Each subplot shows the nonlinear relationship between a single parameter and $\mathcal{R}_{0R}$, with the red dot marking the baseline value of the parameter and the corresponding baseline value of $\mathcal{R}_{0R}$, respectively. In particular, these representations confirm the direction of influence (positive for β_1, Λ, μ) and illustrate how $\mathcal{R}_{0R}$ responds to gradual changes in key parameters.

However, it is to be noted that these approaches share the inherent limitations of the OAT method which suggests them, as they do not capture interactions between parameters (e.g., the combined effect of β_1 and σ) or account for extreme parameter ranges. For a more comprehensive assessment, global sensitivity analysis methods (for instance, Sobol indices) could be applied, although this would increase computational complexity.

5 Discussion and concluding remarks

In this paper, we attempt to investigate how self-efficacy, locus of control and negative peer influence impact the academic performance and dropout intentions of undergraduate students, using an analytic approach that involves the analysis of a 4-dimensional compartmental model that mimics a disease propagation model. In fact, our model is (sort of) conceptually similar to a SEI model with two stages for the infective class, although the similarity isn't perfect, as the dropout compartment shares certain features with both the infective and the recovered classes of an epidemic model, which signals a departure from the epidemiological modelling framework. That is, dropout students are not removed from the academic environment immediately, but continue to exert influence on other students for some time. This feature turns out to be central to the qualitative behavior of the model.

In our model, self-efficacy is assumed to influence the ability to pass resit courses, while locus of control is assumed to influence the ability to pass make-up examinations. The model accounts for the negative influence of dropout students on all other students and of the resit students on make-up students, assuming that the amount of influence depends on students' academic statuses. Dropout and course failure for reasons other than peer influence are also accounted for.

It is seen that the stability of the ideal equilibrium (free of both resit and dropout students) and of the subideal equilibrium (free of dropout students) can be characterized in terms of two threshold parameters $\mathcal{R}_{0R}$ and $\mathcal{R}_{0D}$ called resit and dropout reproduction numbers, respectively, related in their scopes to the the famed basic reproduction number of Mathematical Epidemiology, admitting a clear interpretation in terms of peer-driven transmission of academic failure, and defined via a version of the next generation approach. Also, it is shown that if $\mathcal{R}_{0R} > 1$, then there is at least a realistic (positive) equilibrium, but a complete characterization of the existence and stability of those equilibria seems to be out of reach, highlighting the limited predictive power of reproduction numbers once dropout-mediated peer influence is taken into account, several particular cases being investigated instead.

The occurrence of a backward bifurcation at $\mathcal{R}_{0D} = 1$ is observed provided that either the dropout rate or the level of negative influence of resit students on make-up students remain under certain critical values. This further underscores the destabilizing role of dropout students and means that eradicating dropout as a long-term phenomenon is a difficult endeavour, as simply decreasing $\mathcal{R}_{0D}$ below 1 may not suffice, since even in such a situation dropout and resit behavior may remain endemic if dropout students exert sufficient negative influence before leaving the community. This finding has important policy implications: interventions aimed solely at reducing peer influence may be insufficient unless they are coupled with measures that lead to an early detection of disengagement and facilitate rapid reintegration into the academic environment, or at least limit the social impact of dropout behavior. In some sense, our mathematical results reinforce a broader educational message: preventing long term disengagement may be more effective than attempting to correct its downstream consequences, the central insight being that persistence of dropout students is not merely an outcome of academic failure, but actually an active driver of it.

Our supporting numerical explorations (intended to both illustrate and complement, as they describe bifurcation phenomena that are beyond the scope of our theoretical analysis too), witness a rich dynamical behavior. One notices that the model may exhibit multiple realistic equilibria for certain parameter ranges, indicating that starting with distinct initial conditions one can reach a different equilibrium. In fact, each realistic equilibrium can be thought as a self-sustaining academic regime. At the same time, highly performing regimes fueled by low dropout and high academic recovery could coexist with underperforming regimes, in which the dropout is high and the recovery mechanisms are overwhelmed, or with moderate dropout regimes dominated by peer influence. The initial exposure of students to dropout and resit influence has then the potential to play a significant role in deciding the long-term values of student dropout count, and this directly justifies why early intervention matters even when indicators look favorable (for instance, when $\mathcal{R}_{0R} < 1$).

To discuss the changes in the qualitative properties of the system, we have chosen to vary β_1, the average negative influence of resit students on make-up students, as this is a parameter that can be targeted for reduction via educational initiatives and counselling campaigns. Parallel investigations could have been performed in terms of β_0, β_2 and β_3 too. Further details (and perhaps finer-grained conclusions) were obtained by performing a sensitivity analysis of $\mathcal{R}_{0R}$, being noted that β_1 (the negative influence of resit students on make-up students), Λ (the yearly enrollment rate), and κ (exam failure rate due to causes other than peer influence), exhibit the highest absolute values of relative sensitivity, prompting to actionable levers to mitigate resit and dropout behavior.

Note that $\mathcal{R}_{0R}$ and $\mathcal{R}_{0D}$, as seen above, are not unique predictors of outcomes other than the local stability of the equilibria they're computed for. Particularly, they do not seem to be global stability predictors. Even in connection with the rate of dropouts leaving community being high enough, $\mathcal{R}_{0D} \leq 1$ ensures only the fact that there is no RDPE, rather than provides a global stability perspective.

We have also found numerical evidence of periodic oscillations beyond the scope of our theoretical analysis. Note that the numerically detected Hopf bifurcation emerges at $\mathcal{R}_{0R} = 1.00907$. We have not found any evidence of chaotic behaviour.

Not unlike many others, our modelling approach has certain drawbacks and limitations. First, the bilinear incidence used to model peer influence represents a "worst-case" influence scenario, potentially overestimating peer effects at high prevalence. However, alternative formulations such as a saturating or a normalized incidence could also be considered to account for limited attention, resistance to persuasion or to provide a more balanced view when dealing with large population sizes, respectively. Exploring alternative incidence functions, which can be done in future work, could therefore improve realism and may alter the existence, multiplicity, and stability of equilibria, as well as the occurrence of bifurcations observed in the present model.

From a conceptual viewpoint, our modelling approach is somewhat static, in the sense that it does not account for changes in learning habits or underlying beliefs; after coming back to the passing compartment former make-up and resit students are just as susceptible to negative outside influence as they originally were. That is, no student learns too much of anything from course failure.

The transmission of both good and bad habits is supported not only by the evidence presented in Bandura [2], but also by other influential works, for instance Vygotsky [17], Christakis and Fowler [5], these ideas being used to model (via a probabilistic approach) academic underachievement in Cortés *et al.* [6]. Our model takes a pessimistic view, in the sense that, being geared towards the use of Mathematical Epidemiology models, it accounts for negative influences only, while discarding any positive influence and possible help passing students (for instance) may have or give to other students. However, those positive influences, although realistically possible, do not fall within the reach of the next generation method.

The topic itself is, we believe, of significant social interest and the outcomes of our approach are presented as conclusions that are valid in the long term. However, educational administrations and policy makers are typically more interested in accurate short-term predictions (within 4-6 years), rather than in long-term assessments. Also, each academic year may introduce a variety of changes in enrollment, student demographics, internal policies or in general economic conditions, which makes it challenging to view the context as a continuation of the previous year's context. From this viewpoint, our model is just a crude approximation of the academic life's complexity.

Appendix

The full expressions of the coefficients A, B, C, D, E that appear in the formulation of (13) are:

$$
\begin{aligned}
A &= \beta_0\beta_2\beta_3^2\mu_1 \\
B &= -\Lambda\beta_0\beta_2\beta_3^2 + \beta_0\beta_1\beta_3\mu_1^2 + \beta_2\beta_3^2\kappa\mu_1 + \beta_0\beta_3^2\mu\mu_1 + \beta_2\beta_3^2\mu\mu_1 + \beta_0\beta_2\beta_3\mu\mu_1 \\
&\quad + 2\beta_0\beta_2\beta_3\gamma\mu_1 \\
C &= \beta_3^2\mu^2\mu_1 - \Lambda\beta_2\beta_3^2\kappa + \beta_0\beta_1\gamma\mu_1^2 + \beta_0\beta_1\mu\mu_1^2 + \beta_0\beta_3\mu^2\mu_1 \\
&\quad + \beta_1\beta_3\mu\mu_1^2 + \beta_2\beta_3\mu^2\mu_1 + \beta_3^2\kappa\mu\mu_1 + \beta_3^2\mu\mu_1\sigma + 2\beta_2\beta_3\gamma\kappa\mu_1 + \beta_0\beta_2\gamma^2\mu_1
\end{aligned}
$$

$$+ \beta_1\beta_3\kappa\mu_1^2 + \beta_0\beta_3\eta\mu\mu_1 + \beta_2\beta_3\eta\mu\mu_1 + \beta_0\beta_2\gamma\mu\mu_1 + 2\beta_0\beta_3\gamma\mu\mu_1 + 2\beta_2\beta_3\gamma\mu\mu_1$$
$$+ \beta_2\beta_3\kappa\mu\mu_1 - 2\Lambda\beta_0\beta_2\beta_3\gamma - \Lambda\beta_0\beta_1\beta_3\mu_1$$

and

$$\begin{aligned}
D = {} & -\Lambda\beta_0\beta_2\gamma^2 + \beta_1\gamma\kappa\mu_1^2 + \beta_2\gamma^2\kappa\mu_1 + \beta_1\eta\mu\mu_1^2 + \beta_3\eta\mu^2\mu_1 + \beta_0\gamma\mu^2\mu_1 + \beta_0\gamma^2\mu\mu_1 \\
& + \beta_1\gamma\mu\mu_1^2 + \beta_2\gamma\mu^2\mu_1 + \beta_2\gamma^2\mu\mu_1 + 2\beta_3\gamma\mu^2\mu_1 + \beta_1\kappa\mu\mu_1^2 + \beta_3\kappa\mu^2\mu_1 + \beta_3\mu^2\mu_1\sigma \\
& - 2\Lambda\beta_2\beta_3\gamma\kappa - \Lambda\beta_0\beta_1\gamma\mu_1 - \Lambda\beta_1\beta_3\kappa\mu_1 + \beta_0\eta\gamma\mu\mu_1 + \beta_2\eta\gamma\mu\mu_1 + \beta_3\eta\kappa\mu\mu_1 \\
& + \beta_2\gamma\kappa\mu\mu_1 + 2\beta_3\gamma\kappa\mu\mu_1 + \beta_3\eta\mu\mu_1\sigma + 2\beta_3\gamma\mu\mu_1\sigma + \beta_3\mu^3\mu_1 + \beta_1\mu^2\mu_1^2 \\
E = {} & \gamma\mu^3\mu_1 + \gamma^2\mu^2\mu_1 - \Lambda\beta_2\gamma^2\kappa + \eta\gamma\mu^2\mu_1 + \gamma\kappa\mu^2\mu_1 + \gamma^2\kappa\mu\mu_1 + \gamma\mu^2\mu_1\sigma \\
& + \gamma^2\mu\mu_1\sigma - \Lambda\beta_1\gamma\kappa\mu_1 + \eta\gamma\kappa\mu\mu_1 + \eta\gamma\mu\mu_1\sigma.
\end{aligned}$$

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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